Part VIII
Topic Study Groups

Mathematics Education at Preschool Level

Tamsin Meaney

Introduction: The Aim and the Focal Topics

At the present time, research into mathematics education in the early years is receiving much attention internationally. There is much debate about whether mathematics teaching/learning in the early years should be about supporting children to develop their own interests or to prepare them for school. Alongside this debate is interesting research which shows young children's capabilities on working abstractly with a range of mathematical topics, previously considered too advanced. This topic study group of ICME 12 aimed to provide a forum for exchanging insights in early mathematical learning. While much research has focused on children's learning of number, a growing body of work examines the learning of geometry, measurement and other mathematical topics in preschool. TSG 1 provided a forum for sharing this work and exploring how the learning of these aspects of mathematics in pre-school can be strengthened. It also supported discussion of the preschool teacher education across different countries.

Participants took part in four sessions. Three sessions (1, 2, and 3) were devoted to research and project presentations and the discussions based on these presentations. Session 4 was organised as a discussion to outline a general research agenda. This session finished with a proposal for the group to write a book.

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Presentations

Six presentations were made by researchers from 5 different countries. From Korea, Haekyung Hong discussed how after preschool mathematics classes had little academic gain for children even though many Korean children attended these classes. There were two presentations from Sweden where preschool children are expected to learn through play. In the first presentation by the research group Små Barn Matenatik, videos of children playing at one preschool were analysed using Alan Bishop's 6 mathematical activities. The second presentation by a Swedish graduate student, Laurence Delacour, discussed preschool teachers' adoption of a new curriculum from the perspective of the didactical contract. Oliver Thiel from Norway described a comparative study between Germany and Norway that looked at preschool teachers' competencies. Although most research on mathematics education in preschool tends to be about older children, Shiree Lee, New Zealand, presented research on very young children's exploration of space. The final presentation was by Brian Doig on a paper written with Connie Ompok on a cross-country investigation of games used to assess young children's mathematical knowledge.

Nosisi Feza, from South Africa, presented her poster about preschool teachers' knowledge of teaching mathematics and linked this to concerns about inequitable opportunities to learn.

Final Session

The final session was taken up with a general discussion about how mathematics education in preschools in different countries was conceptualised. In Sweden, 95 % of children attend preschool from the age of 12 months. They do not begin school until 7 years old and have an intermediate year, called preschool class, which still works with the preschool curriculum but acts as a bridge to school. However, in other countries a much smaller proportion of children attend preschools and school can begin as early as 4 years old (Ireland). Very few countries have a formal preschool curriculum and when they do there are differences between whether the focus is on the opportunities that preschools provide or on what children should learn. Preschool teacher education also differs with some countries requiring at least some staff at preschools to be university educated to other countries where staff have school qualification (Germany). Regardless of the education that preschool teachers have had, it seems that many governments are implementing professional development programs for teachers.

The active interest of governments in setting policies for early childhood sector makes this one of the most rapidly changing education fields in mathematics education. Consequently, one outcome of this topic group was the suggestion that a book should be written to document the current situation across the world.

Nosisi Feza from South Africa formulated the book proposal and our next step is to look for a publisher.

Framework for the Early Childhood Development of mathematics education across countries.

Introduction

- How is ECD mathematics education perceived in your country? What components of ECD are seen more important than others? What is the status of ECD provision generally and who is responsible for providing ECD facilities? What is the role of parents in the decision making? What drives ECD provision?

Historical Background

- The history of mathematics early childhood development in each country in terms of policies, national plans and challenges
- Reasons for change if any

Current Status

- Structural levels of ECD in your country e.g., ECD age range, beginning of formal schooling, preschool ages and structures etc.
- ECD policies
- Types of ECD facilities and their purpose
- Purpose of ECD provision generally
- Funding sources for ECD provisioning
- Departments that affiliate to ECD
- Access to ECD facilities in different settings
- ECD educator qualifications and training

Mathematics Education

- Visibility of mathematics education in Policies
- Curriculum with the focus to mathematics education
- Research and research funding on mathematics education in early years
- Monitoring systems for implementation of mathematics in ECD
- Regarding the current literature on early childhood education where is your country?
- What programmes and actions are taken towards ECD mathematics development in your country?
- What gaps do ECD mathematics practices have
- Diversity in the mathematics exposure from home to care and other ECD facilities
- What conclusions does the data make?
- What suggestions are conclusions making for policy, practice, and research of mathematics education in the ECD of your country

Mathematics Education at Tertiary Level and Access to Tertiary Level

Ansie Harding and Juha Oikkonen

Structure of TSG2 Sessions

TSG 2 had 4 sessions of 90 min each, themed as follows:

- Session 1: Teaching philosophies and professional development
- Session 2: Teaching practices
- Session 3: Student experiences/learning, also e-learning
- Session 4: Transition from school to university

The four sessions were all structured similarly. Presentations were classified as either long (15 min) or short (10 min). A session started with one long presentation followed by four or five short presentations (20 presentations in total). Each session closed with a discussion of 15 min. In addition three posters were discussed in the third session and displayed in the exhibition area.

Organizers Co-chairs: Ansie Harding (South Africa), Juha Oikkonen (Finland); Team Members Christopher Sangwin (UK), Sepideh Stewart (New Zealand), Miroslav Lovric (Canada), Sung-Ock Kim (Korea); Liaison IPC Member: Johann Engelbrecht (South Africa).

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General Comments

We are happy to report that TSG 2 ran smoothly and encountered no problems whatsoever. The team worked well together in organising the event before the time. Everyone stuck to deadlines and was forthcoming in suggestions and comments. During the conference itself the team members acted as chairs of the four sessions, respectively, and managed to create coherence amongst the attendees. The sessions were all well attended, drawing approximately 40 delegates per session. It was noticeable that many delegates seemed to develop a sense of belonging to TSG 2 and attended throughout. They were spontaneous in presenting questions and comments, especially during discussion sessions. Unfortunately co-chair Juha Oikkonen had to cancel attendance shortly before the conference on grounds of a medically related problem. He was extremely disappointed not to attend, having contributed in every respect to organizing TSG 2.

Comments Per Session

Session 1: Teaching Philosophies and Professional Development

This session kicked off with a presentation by the well-known twosome John and Annie Seldon, from the USA, a well-received presentation addressing the issue of student success in problem solving. This presentation was followed by four speakers giving an Iranian (Khakbaz Azimeh Sadat), Irish (David Wraith & Anne O'Shea) and Canadian (Miroslav Lovric) perspective, respectively, on related topics. The final presentation in this session was by Leigh Wood from Australia reporting on graduate skills necessary for successful transition from university to the professional environment.

- Annie Selden & John Selden: A Belief Affecting University Student Success in Mathematical Problem Solving and Proving
- Khakbaz Azimeh Sadat: How do Iranian Graduate Students Learn to Teach Collegiate Mathematics as Future Mathematics Professors?
- David Wraith & Anne O'Shea: The use of problem-solving techniques as a learning tool in university mathematics courses
- Miroslav Lovric: Learning Mathematics in an Interdisciplinary Science Program
- Leigh Wood: Preparing our graduates for the workforce

Session 2: Teaching Practices

This session started with a team of young but extremely competent educators from Finland describing an effective system introduced into tutorial sessions. Their enthusiasm added to the success of the session. The subsequent presentations described teaching practices from a variety of countries and a variety of perspectives, providing ample material for discussion.

- Terhi Hautala, Tiina Romu, Thomas Vikberg, Johanna Ramo: The Extreme Apprenticeship Method in Teaching Mathematics at University Level
- Olof Viirman: The Teaching of Functions as a Discursive Practice? University Mathematics Teaching from a Commognitive Standpoint
- Tolga Kabaca: Teaching the Cycloids by the use of Dynamic Software: Abstraction Process of Hypocycloid and Epicycloids Curves
- Liu Jiao & Yao Jing: The Application of Problem-based Learning in Higher Vocational Mathematics Teaching
- Rad Dimitric: Feedback from students' exams. A case study.

Session 3: Student Experiences/Learning, also E-Learning

The first presentation in this session was by Sepideh Stewart from New Zealand speaking on reactions of students to a particular approach to Linear Algebra. The presentation was informative and eloquently presented. Presentations focussed on how students learn and their experiences in doing so. Only one presentation was given on e-learning, perhaps surprisingly so as online learning is topical worldwide.

- Sepideh Stewart: Student Reactions to an Approach to Linear Algebra Emphasising Embodiment and Language
- Ann O'Shea, Sinead Breen, Kirsten Pfeiffer: An Evaluation of the Impact of Non-Standard Tasks on Undergraduate Learning
- Jeremy Zelkowski: Student Accountability & Instructor Variability: A research study in a terminal, required, applications focused calculus course.
- James Musyoka, Joyce Otieno, David Stern: Using e-learning to engage Mathematics and Statistics Students in a Kenyan University
- Ciriaco Ragual & Ester Ogena: Difficulties and Coping Mechanisms in Solving Mathematics Problems
- Diez-Palomar Javier: Family math education: New trends and possibilities for in the realm of mathematics at tertiary level (Poster)

- Haitham Solh: Strategies for Effective Teaching and Learning in Collegiate Mathematics Service Courses for Diverse Students (Poster)
- Ildar Safuanov: Design of a system of teaching elements of group theory (poster)

Session 4: Transition from School to University

Transition from school to university is a general problem as became apparent during this session. Ansie Harding gave the first presentation describing the problem faced in South Africa in this regard. Other presentations described transition problems experienced elsewhere in the world.

- Ansie Harding: On the horns of a dilemma: The transition from school to university in South African
- Randall Pyke: Initiatives at Simon Fraser University in First Year Mathematics and in the Transition from High School to University
- Lee Ji hyun: The Secondary-Tertiary Transition of the Axiomatic Method
- Hoda Ashjari: Recognising Texts in Undergraduate Mathematics Education

Conclusion

The four sessions were well-attended and enjoyed by all those who attended. New ties were established and collaboration possibilities were communicated.

Activities and Programs for Gifted Students

Peter Taylor and Roza Leikin

Introduction: The Aim and the Focal Topics

The aim of TSG-3 at ICME-12 was to gather educational researchers, research mathematicians, mathematics teachers, teacher educators, designers and other congress participants for the international exchange of ideas related to identifying and nourishing mathematically gifted students. The focal topics presented at the TSG-3 included but were not restricted to theoretical models of giftedness, the relationship between creativity and giftedness and the empirical research that will contribute to the development of our understanding in the field. Participants discussed effective research methodologies and research innovations (e.g., brain research) in the field of mathematical giftedness; the findings of qualitative and quantitative studies related to high mathematical promise, its realization, and the relationship between mathematical creativity and mathematical talent. Additional attention was given to the profiles of the gifted child: their range of interests, ambitions and motivations, social behaviour, how and at what age their giftedness is discovered or developed.

Educators who participated in TSG-3 discussed instructional design directed at teaching the gifted as well as development of appropriate didactical principles. The discussions were focused on the ways that lead students to discover and realize their

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mathematical talents, and the ways of developing mathematical innovation at high level. The participants discussed mathematical activities that are challenging, free of routine, inquiry-based, and rich in authentic mathematical problem solving; types of mathematics suitable for challenging gifted students; creation of mathematics challenges; out-of-school ways of fostering giftedness, e.g., mathematics clubs, mathematical shows and competitions.

Last but not least we paid attention to teacher education aimed at mathematics teaching that encourages mathematical promise and promotes mathematical talents, including issues of the psychology of teaching talented students, socio-cultural and affective characteristics of the mathematically gifted, and the types of mathematics and pedagogy suitable for educating teachers of gifted students.

Participants took part in four sessions. Three sessions (1, 2, and 4) were devoted to research and project presentations and the discussions based on these presentations. Session 3 was organised with round table presentations. In what follows we present main topics of the sessions and some examples of the studies and projects presented at the TSG-3 at ICME-12.

Examples and Main Insights

Opening the Discussion

Session 1 was devoted to introduction to the central topics of the TSG. Three lectures, by Linda Sheffiled, Roza Leikin and Alexander Soifer, opened three main reviews of the TSG: international projects for realisations of students' mathematical potential with special emphasis on high mathematical potential (REF), systematic research on characterisation of mathematically gifted students, and mathematics for mathematically gifted.

Linda Shefield's talk "Mathematically Gifted, Talented, or Promising: What Difference Does It Make?" stressed the importance of the developmental perspective of mathematical abilities and the importance of providing each and every student with oportunities to realise these abilities. Based on the position that science, technology, engineering, and mathematics (STEM) are critical to the economy, security, and future of the world, Linda Sheffield argued that we need students who will become adults who understand the complexities of a technological world, who ask the essential questions to safeguard that world, and who will become the leaders, researchers and innovators in the STEM fields of the future. According to Sheffield, too often, in the United States, these students go unrecognized, unmotivated, and under-developed at a time when they are most vital. Sheffield discussed in her presentation whether the way we historically define these future STEM leaders and innovators has an effect upon their growth and development. This talk served as a starting point to the discussion of the international project devoted to the realization of students' intellectual potential related to STEM.

Roza Leikin stressed the importance of conducting systematic and well-designed research on the characteristics of mathematically gifted students. She presented large-scale Multidimensional Examination of Mathematical Giftedness that she conducts with colleagues from the research group in the University of Haifa (Mark Leikin, Ilana Waisman, Shelley Shaul). The presentation was devoted to brain activity (using ERP- Event-Related Potentials—methodology) associated with solving mathematical problems that require transition from a geometrical object to a symbolic representation of its property. Some 43 right-handed male students with varying levels of general giftedness (Gifted-G, Non-gifted-NG) and of mathematical expertise (Excelling-E, Non-excelling-NE) took part in the study. The researchers aimed to investigate the differences in brain activation among four groups of participants (G-E, G-NE, NG-E, and NG-NE). The findings demonstrated different patterns of brain activity associated with problem solving among the four experimental groups. In educational practice the results suggest that different groups of the study population need specific instructional approaches to realize fully their intellectual potential.

Alexander Soifer claimed that mathematics cannot be taught, it can only we learned by our students while doing it. According to Soifer, the classroom ought to be a laboratory where students actually touch the subject, overcome difficulties, which we sometimes call problem solving. "What kind of problems?"—asked the author, and answered: "here comes Combinatorial Geometry!" It offers an abundance of problems that sound like a "regular" school geometry, but require for their solutions synthesis of ideas from geometry, algebra number theory, and trigonometry and thus they are rich, challenging and insightful, and thus appropriate for the education of mathematically talented individuals.

When the three presenters finished their presentations it became clear that the contrast between the presentations enlightened the importance and openness of the following questions: Who are the mathematically gifted? Can giftedness be developed or rather is it realized? How do different perspectives on giftedness determine research and practice in the education of the mathematically gifted? and What kinds of mathematics problems are most appropriate to mathematically gifted?

International Experiences and Projects for Gifted

The second session was devoted to the projects of different kinds directed at educational activities with mathematically advanced students.

Mark Saul described activities of the Center for Mathematical Talent (CMT) at the Courant Institute of Mathematical Sciences (New York University) which was organized in the fall of 2010. Its mandate is to identify and support mathematically talented students in and around the New York City area—especially those from backgrounds where such services have traditionally been weak. The goal at the CMT is to create institutions, materials, and practices that will unlock and nurture

these abilities in students, and will have an impact both on their lives as individuals and on the society in which they live.

Ildar Safuyanov reported on the experiences of fostering creativity of pupils in Russia. While the creative approach is understood by the authors and his colleagues as certain abilities and readiness of a person for creating something new, the purpose of educational process at school is the education of a person who would use a creativity approach for solving scientific or practical problems and for thinking independently. According to Safuyanov, differentiated teaching is an effective way of promoting creativity in conditions. Ildar Safuyanov discussed and compared different types of differentiated teaching and provided the audience with examples of internal differentiation by level of mathematical tasks.

Abraham Arcavi presented the Math-by-Mail project which is an online, interactive, extracurricular enrichment program in recreational mathematics conducted by mathematics educators from the Weizmann Institute of Science in Israel (leaders- Yossi Elran, Michal Elran, Naama Bar-On). Participants of the Math-by-Mail project are engaged in a multi-sense learning experience involving many skills such as comprehension, solving enquiry based problems and correspondence with mathematicians. The lecture demonstrated the scope of the program, its pedagogical and technological characteristics and its benefits for the talented math student.

Viktor Freiman from the University of Moncton, Canada, shared his innovative experience of designing and conducting professional learning communities with inclusive practices for students who "already know". In his project, mathematically gifted and talented students contribute to the virtual community. Same research findings demonstrated the effectiveness of the suggested approach as well as its complexity.

Duangnamol Tama reported on the project named "The Development and Promotion of Science and Technology Talented Project (DPST)". The project is supported by the the Thailand government. Thus national education focuses its efforts and policies on the national development of science, mathematics, and technology through the promotion of high caliber students in these areas.

At the end of this session the participants were exposed to the variety of approaches and variety of ideas directed at promotion of the mathematically gifted. Further discussion between the participants of the session was directed at answering the questions: Which features of the programs for mathematically gifted are culturally dependent and which of them are intercultural? Can successful projects from one country be applied in another country with a different cultural heritage? Do inclusive programs suit needs of the gifted?

Didactical Approaches and International Perspectives

At Session 3 participants of the TSG-3 were exposed to different didactical approaches and international perspectives on the education of mathematically advanced students. This was a round tables session. The authors were provided with

an opportunity to present their papers several times to different people who were interested in their presentations. The groups changed each 10 min and each participant had an opportunity to learn about several works presented at this session. These works included:

- The program of making students create math problems: One of the methods of developing students' abilities to think and express by Nobuo Itoh from Japan,
- The role of student motivation in developing and assessing the acquisition of higher-order thinking skills, by Vincent Matsko, USA
- How the mathematically gifted and talented senior primary school students in Hong Kong understand mathematics, by Wai Lui Ka, Hong Kong
- The research on the mode of motivating the gifted students, by Wang He Nan, Beijing
- Enhancing mathematical research in high school, by Laura Morera, Spain
- Mathematical creativity and attachment theory: an interdisciplinary approach for studying the development of mathematical creativity of preschool children with a precarious childhood, by Melanie Münz, Germany.
- Problem modification as an indicator of deep understanding, by Mihaela Singer Florence, Romania
- Little University of Mathematics, by Laura Freija, Latvia
- Effects of Modified Moore Method on Elementary Number Theory for Gifted High School Students: An Exploratory Study, by Hee Kyoung Cho, Korea,
- Korean Middle School Student's Spatial Ability and Mathematical Performance: Comparison between Gifted Students and General Students by Sungsun Park, Korea

These presentations ended up with multiple questions about the research conducted by different participants and the practices implemented in different countries. The need for the better connections between theory and practice become more and more clear. Following this session we ask: What research approaches can inform us in the best way? How does research methodology depend on definition of gifted chosen in the study? How research and practice can be interwoven to advance theories of mathematical giftedness and advance effectiveness of the practical projects for mathematically gifted students.

Characteristics of Mathematically Gifted Students

The fourth session of the TSG focused on characterization of mathematically gifted students.

BoMi Shin from South Korea reported on a study that provided probability tasks to mathematically gifted students to investigate analogical reasoning as it emerges during the problem-solving process of students. Atsushi Tamura from Japan presented a case study about a gifted high school student in which he identified 5

prominent characteristics in thinking processes by investigating how he devised mathematical proof. Furthermore, this study found that sharing the thinking process of the gifted in the classroom had a good effect on both the class and the gifted himself.

Amaral Nuno and Susana Carreira from Portugal described analysis of creativity in the problem solving processes presented by eight students (from grades 5 and 6, aged between 10 and 11) who have participated in and reached the final phase of a Mathematical Competition. They suggested ways for evaluation of students' creativity in mathematical problem solving in a situation that includes a competitive factor and takes place beyond the mathematics classroom, which is often seen as restrictive for the development of mathematical creativity.

Brandl Matthias from Germany (in collaboration with Christian Barthel) suggested that there are two ways of selecting promising students for the purpose of fostering (in mathematics): whereas the standard procedure is to offer additional courses or material for volunteers or those chosen by the teacher, the other and perhaps more elitist—but with respect to quantitative aspects easier—way is to select the students with the best marks. Brandl argued that from a psychological perspective these ways represent two opposite sides of the causality between giftedness and assessment. One result of this investigation is the finding of strong correlations between the profiles of mathematical interests of specific subgroups that fulfill the characteristics which define mathematical giftedness.

The lecture by Marianne Nolte discussed relationships between "High IQ and High Mathematical Talent!". The findings followed from the long-term PriMa-Project in the University of Hamburg. This project is a research project and a project for fostering mathematically talented children. To detect among them mathematically especially talented children demands a highly comprehensive search for talents. Marianne Nolte stressed the complexity of the evaluation of mathematical talent and stressed that search for talent poses the risk that children may be classified wrongly as especially talented or that children's talents are not recognised.

In conclusion the following questions were raised by the group: Do we know more than Krutetskii after we perform studies on characteristics of students with high mathematical abilities? How do researchers choose their research paradigm? How do research methodologies correspond to the students' age or to a specific characteristic of giftedness that is examined? How studies on students thinking can/should inform educational practices?

The work of the group demonstrated how much is done in the field of the education of mathematically advanced students but moreover it stressed how much should be done in order to get a better understanding of the phenomena of mathematical giftedness and the effective ways of realization of mathematical potential in all students including mathematically talented ones.

Acknowledgments We would like to thank all the participants for their interest in the topic, all the contributors for their interesting presentations and hard work at this TSG, Team members—Viktor Freiman, Linda Sheffield, Mihaela Singer, Bo Mi Shin—for fruitful collaboration in preparation and conducting this TSG.

Activities and Programs for Students with Special Needs

Jean-Philippe Drouhard

Scope and Aims of the TSG 4

Around the world, a considerable number of primary and secondary teachers are involved in teaching mathematics to special educational needs learners ("SEN-L") and a fair proportion of teacher educators are involved in preparing these teachers. But both, teachers and educators, very often are working under somewhat isolated circumstances. They are isolated geographically—it is not always easy to identify others working with SEN-L regionally, let alone nationally or internationally. And they are also isolated in terms of particular focus—specialists working with blind students, for example, may have little professional contact, if any, with specialists in the education of deaf students and those of Down's syndrome. Professional groups tend to be based more on the nature of the special needs of the students rather than on the learning of mathematics. This means that in the dialogue amongst educators concerned with SEN-L, mathematics education is hardly ever at centre stage. On the other hand, mathematics education researchers and teachers seldom have the specific knowledge about SEN-L. Mathematics educators do consider what mathematics for all should be, but the "all" rarely include SEN-L. Issues related to the mathematics education of students with special educational needs are currently under represented in the research community. What seems to be lacking is a *community* of mathematics educators dedicated to exploring this domain. Hence, there is a need to create common references and shared resources (in particular in

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the case of inclusive education). In short, there is a strong need for a common culture of mathematics education for students with special educational needs.

At the end of the meeting it was agreed to create some kind of common Internet platform in order to communicate about how to start such a web of research and shared experience on special educational needs learners. A website (at the moment under construction) has been opened: https://sites.google.com/site/m4senl/ Contact: maths4senl@gmail.com.

What could Mathematics Education gain from the establishment of such common references and resources? First, mathematics education could become more significant in the lives of many students. There is a large number of young people and adult students for whom mathematics teaching may be "secondary" because the focus of their education is elsewhere. Second, insights developed in research with SEN-S could benefit mainstream mathematics teaching, through a re-analyses of assumptions about how mathematics is learned and what specific assessments tell us about students' abilities. Third, SEN-S may show unexpected dissociations between different aspects of mathematical knowledge. It is possible to find, for example, exceptional computational skills with little understanding of their conceptual basis in autistic. Finally, the discussion of different sorts of curricula with different resources appropriate for mathematics teaching while keeping mathematics as the focus of the discussion could lead to more diversified approaches to mathematics education.

Abstracts of the Communications and Posters Presented Within the TSG Meeting

Renato MARCONE, Miriam GODOY PENTEADO[1]: *A blind student at the university: Challenges for mathematics teachers.*

This presentation is based on the story of Mara, a student who became blind during a mathematics undergraduate course. The information for this case were obtained from interviews with Mara, her mother, university staff, colleagues and her teachers. As no blind student had ever before been at the mathematics faculty in question, the case of Mara took everyone by surprise. The first reaction from teachers was that Mara should take another subject—mathematics would be too difficult. However, given that Mara did not change her mind, the university staff had to define actions that would allow her to continue studying. In the article are presented more details of teachers' approaches. This case gives evidence of the challenge to be faced and possibilities that can be considered for teaching mathematics for students with special needs at the university.

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Solange FERNANDES[2], Lulu HEALY: *Representations of three-dimensional forms constructed by blind students: Relations between "seeing" and the "knowing"*.

The aim of this paper is to analyse how blind learners manage the conflicts between "seeing" and "knowing" in relation to two-dimensional representations of two geometric solids (a cube and a square-based pyramid). It seeks to locate elements within their interactions which make up the repertoires of "knowing" of those who do not see with their eyes, treating the processes involved in such interactions as acts of perception, with their origins in the body, and which serve a mediating role between environment, culture and brain.

Juliane LEUDERS[3]: *Auditory representations for blind and sighted students*.
Research into special education teacher education and professional development is sparse. This study set out to investigate factors that support special education teachers' ability to teach students with special needs fraction ideas. Working with three teachers in high school settings, the year long investigation into teacher professional development identified a number of key factors that contribute to student misconceptions and what teachers can do to mediate their learning difficulties.

Teresa ASSUDE,[4] Jean-Philippe DROUHARD: *Mathematics teaching situations with deaf or hard of hearing pupils*.
This article aims to study some mathematics teaching situations which are proposed to the deaf or hard-of-hearing pupils in primary classroom for school inclusion (specialized classroom). We analyse some situations and identify some pupils' difficulties. Then we discuss the problem of the specificity or not of these teaching situations.

Rumiati RUMIATI, Robert J. WRIGHT[5]: *Research on number knowledge of students with Down syndrome: An experience from Indonesia*.
This chapter presents the results of a small scale research study on the number knowledge of students with Down syndrome in Indonesia. Five students with Down Syndrome and ages ranging from 7 to 19 years, from a special education school in Yogyakarta city were interviewed to document their abilities in identifying numerals, solving number problems involving the use of unscreened and screened collections of counters, and solving one-digit and two-digit number problems in horizontal format. The approach and the schedule of assessment tasks in the interview were adapted from that used in Mathematics Recovery. The interviews were conducted individually and videotaped in order to capture subtle clues related to students' abilities. The number knowledge of the five students with Down syndrome is described, compared and discussed.

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KOTAGIRI Tadato[6]: *Mathematical achievement and creativity inherent in children with special needs*.
The assessment of children's mathematical learning achievement entails recognition of the child's human rights to learn Basic Mathematics: (1) to be able to fulfil his/her potential, and more importantly, (2) to be prepared for creative participation in his/her community, both in work and in other activities. Nonetheless, because many children with Special Needs face severe difficulties in obtaining the Basic Mathematical understanding and skills which they both deserve and need, they are effectively being denied their basic educational rights. This paper, based on years of using a clinical approach to remedial education, provides evidence of such children's remarkable possibilities for the achievement of Basic Mathematics, in particular exposing instances of significant creative response.

Marjolijn PELTENBURG, Marja VAN DEN HEUVEL-PANHUIZEN, Alexander ROBITZSCH[7]: *Yes, I got them all? Special education students' ability to solve ICT-based combinatorics problems*.
This present study is aimed at revealing special education students' mathematical potential by means of a dynamic ICT-based assessment. The topic of investigation is elementary combinatorics, which is generally not taught in primary special education. Six combinatorics problems on finding all possible combinations of a number of different types of clothing items were presented on screen. Data were collected on students' performance in solving these items. The performances of students in regular education served as a reference. The total sample consisted of 84 students (8- to 13-year-olds) from special education and 76 students (7- to 11-year-olds) from regular education. Their mathematics ability ranged from halfway grade 2 to halfway grade 5. The results showed that special education students are able to solve combinatorics problems equally successful as regular education students.

Pamela PAEK[8]: *Longitudinal analyses of students with special education needs in the United States on high-stakes mathematics assessments*.
This paper analyzes one state's large-scale assessment (LSA) mathematics data over eight years in the United States, to identify patterns of progress and attrition rates for students with special education needs (SEN-S). A previous study (Paek and Domaleski 2011) showed that SEN-S tended to have slower growth and lower mathematics achievement compared to general education students (GE-S) across grades and years. However, the majority of SEN-S had missing data across years, indicating that any longitudinal reports of SEN-S' achievement and growth are not generalizable. Findings indicate that the majority of SEN-S do not have LSA data for a single year, change the types of assessment forms they take from year-to-year, and are not promoted to the next grade level as often as GE-S. These results reveal

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why a significant amount of SEN-S' data is missing, and how assumptions about data to measure achievement and growth for SEN-S are currently not tenable.

Eugenie KESTEL, Helen FORGASZ[9]: *An investigation of a targeted intervention program delivered by personal Video-conferencing for primary and middle school students with mathematical learning difficulties*.
This paper describes an ongoing study investigating the effectiveness of an individual, conceptual instruction based, tuition program delivered by Personal VideoConferencing (PVC) for upper primary and middle school students with Mathematical Learning Difficulties (MLDs). The experimental intervention targets number sense and fluency with basic facts in mathematics. The effect of using a personal videoconferencing delivery modality on the mathematics anxiety levels experienced by students with MLDs is also investigated.

Rebecca SEAH: *Mathematics professional development for special educators: Lessons learned from the field.*[10]
Research into special education teacher education and professional development is sparse. This study set out to investigate factors that support special education teachers' ability to teach students with special needs fraction ideas. Working with three teachers in high school settings, the year long investigation into teacher professional development identified a number of key factors that contribute to student misconceptions and what teachers can do to mediate their learning difficulties.

Leticia Pardo[11] *Special Education in Xalapa, Mexico: A brief history*.
The main focus of this work is to discuss briefly the history of Special Education services in Xalapa, capital city of the Mexican state named Veracruz. After 31 years serving this government office has experienced three phases of evolution: Integrated groups, Complementary Aid and Educational Integration. We recall some of the main characteristics of every one of these periods of time to explain the way that children with special needs were detected and how they were helped. One conclusion is that the philosophical base of Special Education has evolved from a kind of medical point of view to one based in social aspects.

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Mathematics Education in and for Work

Geoff Wake and Keiko Yasukawa

TSG Report

In considering the meaning of 'mathematics education in and for work', we viewed 'mathematics' as being inclusive of the formal academic discipline of mathematics as well as the range of practices in which mathematics is embedded. Thus we saw 'education' to be inclusive of formal, informal and non-formal learning, that is, in educational settings (e.g. adult community education, vocational and further education) as well as in the community and workplaces. Important to the work of our group is the consideration of learning as both an individual and collective endeavour. In addition we viewed 'work' to be inclusive of paid work and unpaid work such as work in the home, and activist work in community and social settings.
In the design of this Topic Study Group (TSG), focal topics chosen included empirical, theoretical and methodological issues related to questions such as:

- How is mathematics embedded in work practices; what is this mathematics like and how is it learned?
- What mathematics do people learn in preparation for work?
- How is mathematics/numeracy valued for and in employment in different societies?

Organizers Co-chairs: Geoff Wake (UK), Keiko Yasukawa (Australia); Team Members: Corinnes Hahn (France), Ok-Kyeoung Kim (Korea), Tine Wedege (Sweden), Rudolf Straesser (Germany); Liaison IPC Member: Morten Blomhøj (Denmark).

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- How does the mathematics taught and learned for work differ/match the mathematics used in work?
- How does the mathematics learning in and for work meet people's mathematical needs in other domains of their lives?

The presentations and discussions at the meetings of the TSG touched on these questions in intersecting ways. The number of papers formally submitted to the group was relatively low and raised concerns during our meetings about the evident lack of research and other activity associated with a fundamentally important aspect of mathematics education. We expand on the views of the group in relation to this at the end of this report.

Our common pattern of working in our meetings was to have a formal presentation of papers that had been submitted to stimulate discussion which after pursuing issues raised directly by the paper explored the themes and questions identified above.

The first paper presented was Ok-Kyeong Kim's 'Pharmacists and Mathematics'. Ok-Kyeong's study examined how two pharmacists recorded the mathematics that was embedded in their everyday practices as pharmacists. Although the pharmacists did not identify much mathematics in their work, when asked to keep a journal to record the use of mathematical thinking or skills, they began to notice their use of different mathematical concepts such as ratios, proportions, measurement and percentages. What was invisible to the pharmacists themselves at the commencement of the research project slowly emerged and gained visibility, stimulated by their recording of their everyday work practices. The paper raised important questions about the difference between invisibility and absence of mathematics in work, as well as the tensions in researching 'mathematics' in workplaces: is it mathematical practice, or is it pharmaceutical practice, and who has power in the naming of this practice?

Following Ok-Kyeong's presentation, TSG participants engaged in discussions about Jaime Carvalho e Silva's paper 'The Mathematics Teaching in Vocational Schools in Portugal'. Jaime reported on an initiative taken in Portugal of potential envy by mathematics educators in many other countries. The initiative has led to a nationally agreed set of mathematics modules for a wide range of vocational courses studied in the final three years of schooling. The modules cover a wide range of topic areas ensuring that there are suitable mathematical modules for each vocational course. Modelling and statistics feature strongly, and efforts are being made to incorporate 'realistic' examples and activities. Jaime reported that the focus now is on evaluating the efficacy of these modules from a range of perspectives including those of teachers, students and workers who have studied these them. The paper and ensuing discussions highlighted the ongoing question about how should we teach mathematics in vocational courses—as separate subjects or 'invisibly' as embedded content within the specialist vocational subjects.

Invisibility of mathematics in workplace practices featured again in the presentation of Keiko Yasukawa, Stephen Black and Tony Brown's paper, 'Mathematics Education for the Worker, for the Employer, and/or for the Global

Marketplace?—An Exploratory Study of a Complex Question'. The paper was based on a work in progress on the authors' investigation of what has been described as a 'crisis' of low levels of workers' literacy and numeracy levels in Australia that, according to policy makers and industry groups, are the cause of less than desirable productivity, especially in manufacturing. Keiko presented the researchers' preliminary findings from one factory where despite everyone (production workers and their managers) acknowledging that the workers' literacy and numeracy skills are very poor in relation to any normative measures, there is no impact on productivity or quality. As in Ok-Kyeung's study, the workers generally undervalued the mathematics involved in their work, arguably because so much of the mathematics was deeply embedded in the software systems they were using (for example, the computer aided design package used for modeling 3-dimensional objects). Their study did however point to an area of numeracy and literacy need that was (unsurprisingly) not identified by industry and employer groups: the literacy and numeracy practices required by workers, such as low-paid production workers, to critically interpret and negotiate to improve their working conditions.

The final paper presentation was Geoff Wake's paper, 'Seeking principles of design of general mathematics curricula informed by research of use of mathematics in workplace contexts'. Geoff's paper addresses the important question of how the mathematics curriculum can support students' transition from one mathematical (eg formal learning in school) context to another (eg informal learning in the workplace). Drawing on his previous studies of ways in which mathematics is often 'black-boxed', that is deeply embedded and invisible within workplace artifacts or procedures, and on learning as identity work among students in transition from school to work, Geoff articulated design principles for a general mathematics curriculum. These principles include viewing mathematics as not just an object of study, but as a practice that facilitates communication within, membership of, and transformation of a community of practice. Geoff's paper emphasized the value of using research on workplace practices to inform and transform general mathematics curriculum into one that affords students with authentic experiences of learning and becoming users and producers of mathematics.

A presentation of a poster by Minoru Ito based on his and his colleagues Tadashi Aoki and Akihiko Shimano on 'Partnership Program of Mathematics and Science Education in Japan' shifted the focus of the TSG members to a different kind of study. Minoru and his colleagues were involved in a partnership program between his university and a city in Japan to engage university academics and students to design and facilitate engaging mathematical experiences for students in the city's schools. This was an innovative and visionary project to address concerns both about growing disengagement of school students in mathematics and the expected demand of increased mathematical and technological knowledge that these same students are likely to face in their future to address the complex economic and environmental challenges in their society.

Lisa Bjorklund Boistrup and Marie Jacobson's poster presentation took a different but equally big picture view of mathematics education in and for work, in their discussion of the project led by Tine Wedege, 'Adults' mathematics: In work

and for school'. Their project was still in its early stages, but aims to uncover the relationship between the mathematics containing competencies that adults encounter in their workplaces with the mathematics learning demands that students face in their vocational studies.

The presentations in this TSG represented studies being conducted in several European countries—the UK, Portugal and Sweden, the USA, Japan and Australia, about a range of workplace and educational contexts—pharmacies, factories, high schools, vocational schools, nursing and caring work, and transport and garages, with each raising salient issues. The value of understanding mathematics as a social practice was shared by many of the presenters and discussion participants. That there was a tension between learning mathematics as part of a workplace practice and learning mathematics more explicitly in order to be able to critique and perhaps transform existing practices was acknowledged, as well as its corollary, which is the question of who should teach mathematics in vocational preparation courses—the vocational specialist or a mathematics specialist?

The TSG presentations and discussions also highlighted the many theoretical resources that are informing research being undertaken to understand mathematics education in and for work. Along with the presenters' own prior research, the work of other colleagues in workplace mathematics research including Hoyles and Noss, Wedege and Zevenbergen were drawn upon by several presenters. Socio-cultural theories of learning including Vygotsky's/L'eontev's/Engestrom's activity theory, Lave's situated cognition theory and Wenger's ideas of community of practice featured in several of the papers, reflecting the need to account for the collective nature of mathematical practices in workplaces.

In the same way that Geoff Wake's paper highlighted the importance of workplace research informing general mathematics curriculum design, research in vocational and workplace mathematics education should perhaps be more strongly informing what happens in mathematics learning at earlier stages of schooling. A final discussion of the group focused on these and related issues. Members of the group expressed their concerns at the relative lack of interest of the ICME community in this area of research given the important role that mathematics education plays in preparing young people for future work and critical citizenship. It was resolved that the co-chairs would be pro-active in raising the profile of the issues that emerged during discussions of the group and would seek to explore the possibility of a future ICME survey group providing an overview of the state of play of mathematics education in and for work across a range of cultural settings around the world.

Final Timetable

Tuesday, July 10 Session 1—10.30–12.00
10.30–10.45 Introductions and opening remarks: Geoff Wake and Keiko Yasukawa
10.45–11.20 Presenter: Ok-Kyeong Kim—Pharmacists and Mathematics, Discussant: Jaime Silva

11.20–11.55 Presenter: Olda Covian-Chavez—Mathematics applications in Topography: What elements for the training? (*not presented*), Discussant: Geoff Wake
11.55–12.00 Closing remarks
Wednesday, July 11 Session 2—10.30–12.00
10.30–10.45 Introductions and recap of previous day: Geoff Wake and Keiko Yasukawa
10.45–11.20 Presenter: Jaime Silva—The mathematics teaching in Vocational Schools in Portugal, Discussant: Geoff Wake
11.20–11.55 Presenter: Keiko Yasukawa—Mathematics Education for the Worker, for the Employer, and/or for the Global Marketplace?—An Exploratory Study of a Complex Question, Discussant: Ok-Kyeong Kim
11:55–12:00 Closing Remarks
Friday, July 13 Session 3—11.00–12.30
10.30–10.45 Introductions and recap of previous day: Geoff Wake and Keiko Yasukawa
10.45–11.20 Presenter: Geoff Wake- Seeking principles of design of general mathematics curricula informed by research of use of mathematics in workplace contexts, Discussant: Keiko Yasukawa
11.20–11.55 Overall threads and observations: Rudolf Strasser
11.55–12.00 Closing remarks
Saturday, July 14 Session 4—10.30–12.00
Poster presentations
Presenter: Minoru Ito—Partnership Program of Mathematics and Science Education in Japan
Presenter: Lisa Bjorklund Boistrup—Adults' mathematics: In work and for school
General discussions and future.

Mathematical Literacy

Mogens Niss

Introduction

The actual design and implementation of the structure and organisation of the four TSG sessions was carried out by *Mogens Niss*, with the assistance of *GwiSoo Na*, *Eduardo Mancera*, and *Michèle Artigue*. Unfortunately, Eduardo Mancera was eventually unable to attend the Congress and the TSG.

This TSG was included for the first time in the history of the ICMEs. Hence, there was no established ICME tradition to build on concerning this topic. Moreover, generally speaking, the very notion of mathematical literacy is not well-defined, especially as several related concepts, such as numeracy, quantitative literacy, mathematical proficiency, and mathematical competencies, are in general use as well. Against this background it was decided to devote a fair proportion of the session time to coming to grips with the notions of mathematical literacy and its "relatives".

The presentations given in the four sessions of TSG 6 were partly commissioned papers, partly contributed ones. As is often the case with TSGs, the attendance to this TSG was not completely stable, but varied across the four sessions, the average attendance being about twenty participants per session.

The themes of the four sessions were chosen as a reflection of perceived intellectual and scholarly needs, and of the papers contributed by participants. The main theme of the opening session was the *Notions and interpretations of mathematical literacy*, whilst *The role and impact of mathematical literacy in national and*

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international studies was the main theme chosen for Session 2. Session 3 was primarily devoted to the theme *The role, use and implementation of mathematical literacy in educational systems and institutions.* The main theme of the fourth and final session was *Mathematical Literacy and teachers*. In order to ensure discussion of the presentations, each session was concluded by questions or comments, in Sessions 1 and 4 assisted by a round-table.

Major Points from the Four Sessions

Session 1: Tuesday, 10th July, 10:30–12:00

To set the stage for the work of TSG 6, *Mogens Niss* gave a 30-minute introduction focusing on the notions of mathematical literacy and its relatives. He began by observing that mathematics educators have always insisted that knowledge, skills and insights pertaining to elementary mathematics go far beyond facts, rules and procedures. This was already the case with the First International Mathematics Study (FIMS), conducted by the IEA and published in 1967, which spoke about five "cognitive behaviours". Later on, organisations such as NCTM and OECD-PISA, and several individual researchers, made an effort to identify aspects of the "add-ons" involved, suggesting various terms for the enterprise. The term Mathematical Literacy was used at least as early as in 1944, but the first attempt at a definition seems to have been made in the first OECD-PISA framework in 1999, with minor modifications in subsequent frameworks. Other related terms are numeracy, quantitative literacy, mathematical proficiency, and mathematical competence (competencies). Niss asked whether these terms are just different names for the same thing, or each term stands for something independent. He concluded that when it comes to mathematical literacy, numeracy and quantitative literacy, many people use them interchangeably, even though it is actually possible to attach distinct specific meanings to these terms. Given that people tend not to stick to definitions, he proposed to use Mathematical Literacy as the overarching term for the common underlying idea of promoting mathematical empowerment by making mathematics functional in extra-mathematical contexts. In contrast, the terms mathematical proficiency and mathematical competencies refer to a much wider spectrum of mathematical mastery, pertaining also to intra-mathematical contexts.

Next, two 15-minute presentations on aspects of the range and scope of mathematical literacy were given. In the first one, *Steve Thornton* (with John Hogan) (Australia), suggested to utilise a notion of "slow mathematics"—inspired by the notion of "slow food" in contrast to fast food—as a metaphor for quality mathematics education and for mathematical literacy. Slow mathematics is meant to capture what working mathematically is actually about, going against the "one-size-fits-all" idea typical of traditional curricula. Thornton proposed that working mathematically should be made *the* curriculum, whereas content should be of

secondary importance. He illustrated his ideas by two examples, one on sums occurring in the dice game of Yahtzee, and one on cyclones and tides. *Karen François* (Belgium) (co-author not present) discussed the relationship between mathematical literacy and statistical literacy from a theoretical perspective, based on a literature review. François concluded that whilst there are indeed clear similarities and links between mathematical and statistical literacy there are also significant differences (e.g. statistical literacy focuses on decision making under uncertainty), as reflected in the sociological fact that statistical literacy and statistics education have developed into independent notions and fields of study. In other words statistical literacy should not simply be perceived as a special sub-field of mathematical literacy.

The session ended with a round-table in which *Nitsa Movshovitz-Hadar* (Israel), the speakers and the members of the audience discussed the range and scope of the concept of mathematical literacy. Movshovitz-Hadar made the point that mathematical literacy should encompass insights into the reality of current mathematical developments and described a project in Israel in which secondary school students were exposed to contemporary "mathematical snapshots" once every two weeks.

Session 2: July 11th Wednesday, 10:30–12:00

The session opened by a 30-minute invited presentation by *Ross Turner* (Australia) (in charge of implementing the mathematics part of the OECD-PISA study for several cycles). After considering the genesis and meanings of the notions of mathematical and scientific literacy, numeracy and quantitative literacy in various reports, Turner zoomed in on the ways in which the notion of mathematical literacy was developed in different PISA cycles, right from the beginning. The concept of mathematical literacy in PISA has always given rise to some tension within the group of participating countries. The key tension can be phrased as one between seeing mathematics as a superset, having mathematical literacy is a smaller part, or seeing mathematical literacy as the overarching domain, with mathematics as a subset. The tension is both a conceptual one, reflected in the ways in which different versions of the PISA framework draws upon mathematical competencies and overarching content areas ("big ideas"), and a political one, reflected in the fear voiced in some quarters, that PISA, by focusing on contextualised mathematics, would not provide an adequate coverage of school mathematics curricula, as only relatively low level mathematics seems to be needed to solve PISA problems. Nevertheless, several PISA items could be solved by a tiny minority of students only. Another problem is that the word "literacy" does not exist in many languages, making translation difficult. These tensions gave rise to a strong pressure on the OECD, and then on those in charge of PISA, to change the focus of PISA 2012 towards a more traditional view of mathematics as being constituted by well-known content areas, without directly forbidding the use of the term mathematical literacy. Turner concluded by mentioning the promising work done by some of the PISA

mathematics experts on the impact of mathematical competencies on the intrinsic demands of PISA items, and on these demands as predictors of observed item difficulty.

In the first of three 15-minute presentations, *Jeff Evans* (UK) offered a comparative analysis of the definition of numeracy in PIAAC (Project for International Assessment of Adult Competencies) and the definition of mathematical literacy in PISA, 2006. He found the PISA definition somewhat broader and more "humanistic" than that in PIAAC. Finally, Evans pointed to the criticism, raised by some, of the unidimensionality of the performance levels in both surveys.

Next, *Kees Hoogland* (The Netherlands) reported on a randomized, controlled, comparative study of 38,000 Dutch students solving image-rich, respectively word-based, numeracy problems. The aim of the study was to test the hypothesis that replacing word problems with image-rich problems would have a significant positive effect on students' result, and even more so with vocational students. In the study, 24 pairs of mathematically equivalent numeracy problems were constructed such that each pair contained a language-rich version and an image-rich version of the "same" problem. Each student was randomly given 12 problems of each type. The study was found to provide a fair degree of confirmation of the hypothesis stated.

The final presentation was given by *Yukihiko Namikawa* (Japan), who described a national project in Japan which first focused on scientific literacy and then moved on to mathematical literacy, focusing on citizenship. A key part of this project was the publication "Mathematical Literacy for All Japanese", containing chapters on the nature of mathematics, on the central objects and concepts of mathematics, on mathematical methods and mathematical competencies, on mathematical topics, and, finally, on the relationship of mathematics with humanity and science. Following a report published in 2008 by the Central Council for Education, a new comprehensive, national standards curriculum emphasising mathematical literacy for all is being phased in, challenging the education of teachers at all levels.

Session 3: Friday 13th July, 15:00–16:30

This session contained a variety of short presentations. *John Hogan* (with Steve Thornton) (Australia), after having proposed to define "being mathematically literate" as more or less the same as "being numerate", went on to suggest that this cannot be developed or observed in the mathematics classroom alone, it has to go across the curriculum. To illustrate how this can be pursued, Hogan briefly outlined some settings in the arts, English, health and physical education and science, corresponding to early, middle and later years, respectively. He finally sketched a numeracy framework developed for diagnostic, analytic and practical purposes.

Yelena Baishanski (USA, with co-author not present) spoke about achieving literacy through articulated reasoning in remedial mathematics courses for US community college students (i.e. La Guardia CC, New York). The project involved

activities on simple applied arithmetico-algebraic problems, arising out of "current compelling issues" meant to be engaging and meaningful to students, on which they can develop and practice their own skills in reasoning and written communication about reasoning, so as to develop confidence in their own powers of deduction.

In the next presentation, *Jenna Tague* (USA, with co-authors not present), dealt with two linked topics: the so-called STEM (Science, Technology, Engineering and Mathematics education) reform in the USA and a related development project at Ohio State University, reconceptualising engineering courses by focusing on mathematical literacy. More specifically, Tague proposed to devise a mathematical literacy framework within a "STEM for engineering students" context, taking inspiration from the Danish KOM project on mathematical competencies.

Based on the observation that many interpretations of mathematical literacy give a crucial role to mathematical modelling, *Abolfazi Refiepour Garabi* (Iran) presented two related empirical studies, one of Iranian mathematics textbooks and one of teachers' views about application and modelling problems in their classrooms. Mathematical modelling and applications were introduced in Iranian textbooks in 2008/2009. Comparing with Australian textbooks, the author finds that measured by the number of real world modelling problems, these textbooks tend to have a larger emphasis on mathematical literacy than do Iranian textbooks. Iranian mathematics teachers experience difficulties in using applications and modelling problems in their classrooms, especially because they don't have access to adequate sources for modelling tasks.

The final presentation was given by *Luis Rico Romero* (Spain, with co-authors not present). He presented a study in progress on Spanish in-service secondary teachers' assessment of mathematical competences. In the Spanish curriculum of 2006, the notion of competence, including mathematical competence, is given a key role at all educational levels, and also in the related system of performance indicators. The study focuses on teachers' understanding of, and intended methods with regard to, competency assessment in mathematics. The components of a workshop on this topic for teachers were outlined.

Session 4: Saturday 14th July, 10:30–12:00

In the first presentation, *Cigdem Arslan* (with *Günes Yavuz*), Turkey, reported on a research study on the mathematical literacy self-efficacy of prospective mathematics teacher students (PTs) in different programmes in a Turkish university. The study was conducted by way of a 25 item questionnaire, where each item was to be answered in a five-point Likert scale format. The study found that PTs indicate an above-medium level of mathematical literacy self-efficacy, and that there were no significant differences between their mean scores with respect to their year in university, between male and female students, or with respect to their choice of programme.

The presentation by *Lyn Webb* (South Africa, with co-authors not present) was based on the fact that mathematical literacy was introduced in South Africa as a mathematics option for prospective teachers as an alternative to "usual" mathematics. This led to the establishment of mathematical literacy programmes at some higher education institutions. The degrees obtained are rather different from those of traditional mathematics programmes. Two mathematics programmes are offered at two universities in KwaZulu Natal. After comparing the programmes with respect to their overall design, Webb concluded from the study that a balanced mix of types of knowledge, particularly disciplinary, pedagogical, practical and situational learning, is essential for teacher training qualification, and that content knowledge is not sufficient.

The final presentation was delivered jointly by *Dave Tout* (Australia) and *Iddo Gal* (Israel). They set out by contrasting internal views of educational goals (learning the trade of the discipline) with external views of educational outcomes focusing on real-world functional demands ("literacies"/"competencies"). Different surveys of students (e.g. TIMSS, PISA) and adults (e.g. ALL, PIAAC) have been conducted to shape educational policies and to design interventions. Mathematical literacy and numeracy are of the same nature, but mathematical literacy sits (mainly) in student and school contexts and numeracy in adult world contexts. This is reflected in PIAAC's definition of numeracy, focusing on the mathematical demands of a range of situations in adult life and on associated facets of numerate behaviour. The presentation went on to highlight various results from PIAAC and other adult numeracy surveys, and concluded by calling attention to three kinds of challenges to mathematical literacy/numeracy: Conceptual challenges ("what is it?"), educational challenges ("how can we develop it?"), and systemic challenges ("where is it (to be) located?").

This conclusion provided a handy lead-on to the final part of the session, a combined round-table and discussion amongst participants. Members of the round-table, moderated by *Mogens Niss*, were *GwiSoo Na*, *Yukihiko Namikawa* and *Ross Turner*. The round-table and the audience focused on important points for future work on mathematical literacy, such as examining the relationship between mathematical literacy and mathematical knowledge and skills, and finding ways to develop teaching and learning of mathematical literacy so as to ensure that all students (and adults) get something out of their mathematical education of subjective and objective value.

Teaching and Learning of Number Systems and Arithmetic (Focusing Especially on Primary Education)

Joana Brocardo and Geoffrey B. Saxe

Aims, Themes and Organization

Aims and Themes

The group's focus is on individuals' elementary mathematical representations and understandings with a special interest in the way these aspects of cognition develop through activities in and out of school. The mathematical domains of concern include whole numbers, integers, and rational numbers as well as representations related to each of these domains.

A related interest of the group is socio-cultural analyses. These analyses would include the ways that mathematics (including mathematical argumentation, representations, problem solving, teaching-learning interactions) is constituted in everyday practices as well as the interplay between developing mathematical understanding and representations in and out of school.

The group encourages cross-disciplinary contributions, including (but not limited to) participation by educational researchers, mathematics educators, developmental psychologists, and cultural anthropologists.

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Organization

TSG 7 received 29 submissions. We decided to emphasize discussion, articulating oral presentation and its discussion with poster presentations.

Two members of the organizing team and one external reviewer reviewed each paper. From the reviews and interactions by email among the members of the Organizing Team, an agreement was reached on a final list of presentations and posters, leading to 10 oral presentations and 17 posters. Due to cancellations only 10 posters were presented in two slots with 5 in each one. This turned the poster sessions of the group into an interactive session, in which each poster was presented by the author(s) and then discussed with all the participants.

The participants in the group came from 15 different countries of North and South America, Asia, Africa and Europe.

Papers, Posters, and Discussion Topics that Emerged in the Sessions

The presentations and the discussion varied markedly, reflecting diverse orientations and focal interests in teaching and learning about number systems and operations. Though diverse, the papers and posters conformed to four general themes.

The first theme was formalization of mathematical ideas, mathematical contexts, and models. The presentations and the discussion highlighted potentialities and barriers to the learning and teaching of number system and operations.

The second theme engaged participants with elementary mathematical representations and understandings that individuals construct. The presentations included case studies that illustrate the development of representation and understandings through activities in and out of school.

A third concerned kinds of numbers that are the focus of teaching and learning. These papers focused on teaching and learning of whole, fractional and decimal numbers. Papers and posters presented and analyzed processes whereby students overcome their misunderstandings and difficulties.

Finally the group discussed examples of everyday practices in school that can promote understanding in the domain of number and operations as well as the interplay between developing mathematical understanding and representations in and out of school. This discussion included examples and ideas related with mathematical argumentation, representations, problem solving and teaching-learning interactions.

The schematic contained in Fig. 1 illustrates the principal focus of TSG7 on number systems and operations themes, the concern for understanding processes of teaching and learning related to the focus, and the paper presentations, posters, and discussion that emerged on the four themes.

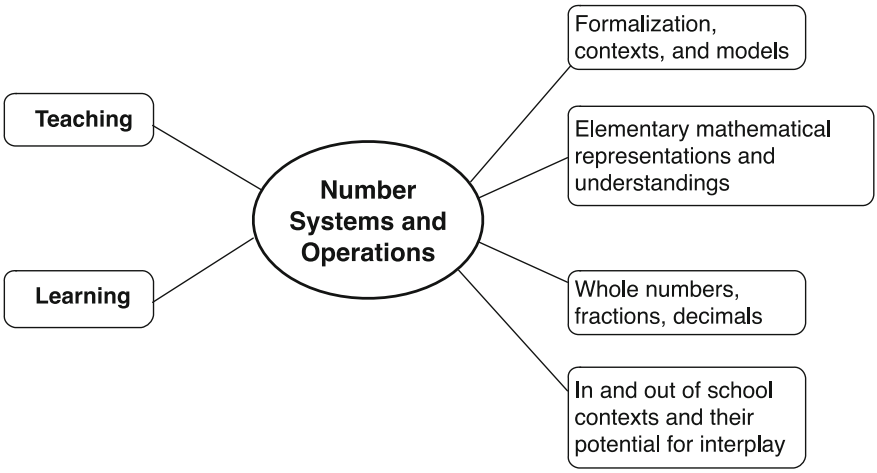

Fig. 1 Schematic of themes and presentation topics in TSG 7 for the 2012 meetings

Measurement—Focusing Especially on Primary Education

JeongSuk Pang and Kees Buijs

Preparation

Measurement, as well as related topics of geometry, forms an important mathematics domain on the level of both primary school and pre-vocational secondary school in many countries. At this level it relates primarily to quantifying certain aspects of real world physical objects such as the length, area, capacity, weight/mass, temperature or volume of objects, and to the reconstruction and application of the current measuring systems in a country (metrical or non-metrical). It also includes the use of measuring instruments such as the folding ruler and measuring tape, the measuring jug and the kitchen scale. Related geometrical topics include understanding of and working with the concept of scale, and the reconstruction and application of formulas for the area of a rectangle, triangle and other geometrical figures.

TSG-8 addressed researchers, curriculum developers, and reflective practitioners (teachers) working in the field of measurement and related geometry on the level of primary school. It aimed at providing a forum for generating discussion, exchanging insights, and establishing a state of the art sketch of the domain, including indications for the status of measurement as a foundation for advanced mathematics domains.

The TSG-8 organizing team called for papers dealing with various aspects of measurement such as theoretical perspectives on mathematical growth of students' thinking related to measurement, the development of measurement sense in students, connections between measurement and related domains such as number

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sense and decimal numbers, curriculum development and implementation related to measurement, instructional approaches to foster students' development related to measurement, and culturally defined tools and practices for measurement and cultural supports for the learning and teaching of measurement.

Each of the 15 proposals which we had received was carefully and rigorously evaluated by three reviewers from the TSG-8 organizing team members with the support of K. Subramaniam. Having further discussed the initially accepted proposals amongst the TSG-8 team members, four papers were accepted for long oral presentation (30 min of presentation and 10 min of discussion) and eight papers for short presentation (15 min of presentation and 5 min of discussion). The remaining three papers were recommended for poster presentations during the general poster sessions of the ICME-12. Due to a cancellation, the final program of TSG-8 consisted of four long oral presentations and seven short ones.

We organized the accepted papers into four 90-munite sessions as follows:

- Session 1: Students' difficulties and teaching methods (July 10th),
- Session 2: Curricular materials and teaching methods (July 11th),
- Session 3: Delving into students' understanding (July 13th),
- Session 4: Measurement instrument and its use (July 14th)

Implementation

Session 1: Students' Difficulties and Teaching Methods

The first session was chaired by the co-chairs of TSG-8. At the beginning of the session, JeongSuk Pang from Korea welcomed all participants and introduced the organizing team members. Kees Buijs from Netherlands then delivered introductory remarks, showing a series of pictures taken in Seoul and related them to demonstrate measurement in a daily life.

Three papers were presented in this session (one long presentation and two short presentations) and vivid discussion was followed. First, Yah Hui Tan and Meng Hua Chua from Singapore investigated students' difficulties in learning the concepts of length and mass, and examined how teachers' use of an adapted version of the Kolb's *experiential learning cycle* was helpful to address their students' difficulties. They addressed the importance of using various measurement tools to assess students' understanding and misconceptions of measurement concepts.

Second, JeongSuk Pang, JeongWon Kim, and HyeJeong Kim from Korea identified key instructional elements in teaching measurement by comparing and contrasting two sets of measurement teaching practices which were recognized as good instruction in Korea. This presentation raised an issue on what counts as effective *measurement* instruction.

Third, Wayne Hawkins from Australia presented four primary teachers' pedagogical content knowledge in teaching measurement to students in Years 3 and 4. By exploring teachers' knowledge of mathematics along with knowledge of students and teaching, Wayne helped the audience understand the complex nature of pedagogical content knowledge and provoked a discussion on the dynamic nature of such knowledge.

Session 2: Curricular Materials and Teaching Methods

The second session was chaired by the TSG-8 organizing team member, Olimpia Figueras from Mexico. Three papers were presented in this session (one long presentation and two short presentations) and insightful issues were discussed afterwards. First, JeongSuk Pang, SuKyoung Kim, and InYoung Choi from Korea reported a comparative analysis of the statements in two Korean elementary mathematics textbook series in terms of two coding criteria: degree of guidance and key learning elements of the measurement domain. This presentation suggested the need of re-conceptualizing key learning elements of measurement as well as the possibility of developing a new coding system for textbook analysis. Several participants showed their interest in using this coding system in analyzing their textbooks.

Second, Silke Ruwisch from Germany presented third grade students' understanding of capacity and proposed the need for explicit comparison and measurement actions with many different containers before building up mental representation.

Third, Jeenath Rahaman from India presented different ways in which multiplicative thinking was involved in the measurement of area. She shared some tasks that had prompted students to use multiplicative thinking in finding the area of given figures. This also gave the participants an opportunity to reflect on the importance of designing tasks to explore the connection between multiplicative thinking and measurement of area.

Session 3: Delving into Students' Understanding

The third session was chaired by the TSG-8 organizing team member, Silke Ruwisch from Germany. Three papers were presented in this session (one long presentation and two short presentations) and thought-provoking issues were raised. First, Kees Buijs from Netherlands reported gaps between the informal and formal knowledge of 13–14 years old pre-vocational students, and suggested some ways to bridge such gaps. This presentation provided unique information mainly because of the characteristics of the students who had participated in this study. Despite their reasonable knowledge of measurement units and basic measurement sense, the difficulties that

students had in solving more theoretical measurement problems were striking. As such, this presentation addressed a core issue in designing a measurement curriculum.

Second, Oyunaa Purevdorj from Mongolia presented second grade students' difficulties in understanding the given word problem, drawing a rectangle, and finding out the perimeter of a rectangle, and attributed the causes of such difficulties to the ways curriculum and textbook were designed, and the ways that teachers taught them in the country. This presentation helped participants understand the close relationship among curricular documents, teaching methods, and students' learning outcomes.

Third, Andrea McDonough from Australia reported on a design experiment to teach lower primary students about the measurement of mass. By illustrating multiple tasks and hands-on lessons in which students were expected to focus on the key measurement understandings of comparison and unit, Andrea prompted the audiences to grasp how to maximize the opportunity to learn the measurement of mass.

Session 4: Measurement Instrument and Its Use

The final session was chaired by the TSG-8 organizing team member, Andrea McDonough from Australia. Two papers were presented in this session (one long presentation and one short presentation) and general discussion was followed. First, K. Subramaniam from India presented measurement units and modes in the Indian context. He illustrated unique informal measurement units and multiple modes of quantification that are still being used in the Indian context. The presentation raised issues of how to design the school mathematics curriculum to incorporate students' practical knowledge of measurement and measurement sense.

Second, Bona Kang from USA reported four emerging sociomathematical norms regarding linear measurement and then the students' meaningful shift to use rigid tools. As such, she suggested the positive impact of social processes on the students' use of informal tools in measurement. This presentation raised an issue of a reflexive relationship between social and cognitive processes in measurement activity.

The final session was closed by two co-chairs. They appreciated all the participants who presented their studies, engaged in a rich discussion, and provided comments throughout the four sessions.

Reflection

The adequate number of papers presented in each session enabled TSG-8 to have an opportunity for participants to present their results, share ideas, and discuss issues within an affordable time frame. On the one hand, such an opportunity was effective

in comparison to other TSGs because they had to run parallel sessions at the same time to provide more opportunities to present papers but had difficulties in sharing participants' ideas as a whole group. On the other hand, it was surprising that not many papers were submitted to TSG-8, even though measurement and related geometry are considered as an essential part of the mathematics curriculum especially at primary level in many countries. One reason might be a lack of attention to this domain. Another reason might be that a number of proposals were submitted to other TSGs by drawing more attention to the genre of research rather than the content domain of mathematics.

Generally speaking, TSG-8 had regular attendants who were ready to bring up rich discussion within a permissive atmosphere throughout the four sessions. Despite the relatively small number of papers presented in this group, a number of important issues came up and participants agreed the necessity of further international comparative studies in the domain of measurement. We hope that the topic study group dealing with measurement continues to serve a well-recognized group of the congress.

Teaching and Learning of Algebra

Rakhi Banerjee and Luis Puig

Overview

Topic Study Group 9 aimed to bring together researchers, developers and teachers who investigate and develop theoretical accounts of the teaching and learning of algebra. The group sought both empirically grounded contributions focussing on the learning and teaching of algebra in diverse classrooms settings, the evolution of algebraic reasoning from elementary through university schooling as well as theoretical contributions throwing light on the complexities involved in teaching and learning of algebra. Prospective contributors were requested to address one or more of the following themes: early algebra, use of ICT in algebra classrooms, proof and proving in algebra, problem solving, semiotics, designing of algebra curriculum.

Organization

We received 44 contributions for the TSG. Based on the review of these papers (each paper was reviewed by two members of the organizing team), 25 of these contributions were chosen for oral presentations and the rest were recommended for

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poster presentations. 16 of the oral presentations were short presentations (10 min for presentation and 5 min for discussion) and 9 were long presentations (20 min for presentation and 10 min discussion). For reasons of optimizing the available time and in order to fit in all the contributions, the group was divided into two subgroups and presentations were made simultaneously in the two sub-groups. The co-chairs of the team often helped in identifying the underlying theme in different presentations across the two sub-groups. Participants were requested to read up the articles to be presented in a session beforehand to be able to participate better. Some time was kept daily for the whole group to meet and discuss issues arising from the presentations or points which participants wanted to raise. More time was allotted for whole group activity on the first and the last day.

The participants were largely those who had contributed to the group and brought in perspectives from all over the world. The presentations touched upon students' understanding of different aspects of algebra, theoretical perspectives to make sense of students' work and help them learn better, teachers' understanding of the algebra they teach and professional development initiatives to help them focus on the important aspects of algebra. Pattern generalization and early algebraic thinking was an issue of discussion in various presentations. Problem solving and reasoning, proving, understanding of functions were explored in a few presentations. There were a couple of reports on algebra in particular culture/communities and curriculum/instruction status in a country. A few presentations focused on the use of computer aided tools for instruction or evaluation. An area which did not get any attention was how semiotics helps us understand students' developing knowledge of symbols, process of signification and communication.

Implementation

Session 1: July 10, Tuesday, 10:30–12:00 (Room no. 308a and 309)

On the first day, 45 min were kept for whole group discussion and only 4 presentations were scheduled for oral communication (2 long presentations and 2 short ones). The participants were reminded of the themes that the TSG would focus on and were given a general overview of the nature of the submissions received. They were further informed about the modalities of the conduct of the TSG.

The four presentations covered varied themes. One of the presentations focused on understanding of properties of operations with respect to fractions, operations on fractions, ability to think relationally and to perceive structure in expressions and their importance for learning algebra and developing algebraic thinking. Another one looked at the usefulness of variation theory as a means of improving teaching and learning and discussed how teachers went about designing lessons in the area of

rational expressions using the theory. A third presentation looked at pre-service teachers' ability to engage in inductive reasoning and generalization in problem solving contexts. The fourth presentation focused on professional development of teachers in the area of functions that helped them develop and design activities that promote algebraic thinking among students.

Thus, we listened to an interesting set of ideas in terms of design of tasks, theoretical frameworks on the first day. They highlighted strengths and limitations of teachers' and students' thinking and reasoning while working on the tasks and provide valuable insights for designing of programmes with teachers and students in the future.

Session 2: July 11, Wednesday, 10:30–12:00, Room no. 308a and 309

Eight presentations were scheduled for the second day, four of them were long presentations and four short ones, divided equally between the two rooms.

All the presentations in one of the rooms dealt with pattern generalization. One of them discussed strategies used by students in secondary school for generalizing two patterns. Another one looked at difference in performance among students categorized by their abilities in mathematics on pattern generalization tasks as well as the strategies used for working on the tasks. One study compared competence of students in two countries: Hongkong and United States, in pattern generalization task. A last paper explored young Australian indigenous students' engagement with generalization of contextual growing patterns and ways in which cultural gestures help them in accomplishing the task.

The studies highlighted many categories of patterns and strategies for generalizing them and the participants discussed issues arising out of pattern generalizing tasks in their own countries and classrooms and made suggestions towards improving students' abilities to generalize, nature of tasks and instructions for students etc.

Presentations in the second room were not in a single topic. The long one dealt with algebraic proof in secondary education. In this presentation findings of a teaching experiment were reported on how the understanding of the generality of algebraic proof emerged when students study operative proofs. The study started from the knowledge from previous research that even students who are able to construct proofs using symbolic algebra rely on checking with numerical examples as a "proof". Students that followed the experimental teaching, that included the use of operative proofs, start noticing the generality of operation and then they start appreciating algebraic proofs.

The short ones dealt with very different issues. One presented a proposal to describe the structure of algebraic competence by using linear structure models. The second one dealt with algebraic reasoning in early algebra as generalized arithmetic,

examining elementary school students' understanding of the properties of whole numbers operations. The reflection of students on the properties of operations with whole numbers is a way to teach and learn algebraic reasoning in early ages. In the study, it was found that students had capabilities in generalizing the properties of numbers and operations, but they had not developed such capabilities, because school practices have not provided enough opportunities and experience in order to develop them. This was showed by the fact that student were able to generalize the commutative law, but neither the associative nor the distributive laws.

The final one dealt with a research study on the ability of secondary students to translate statements between symbolic algebra and vernacular language and vice-versa. In this study, students performed better when translating from symbolic algebra expressions to vernacular language, and most errors when translating the other way round were attributed by the authors to "peculiar features of algebraic language".

Session 3: July 13, Friday, 15:00–16:30, Room no. 308a and 309

Eight more presentations were scheduled for this day, in a manner similar to Session 2.

Two presentations in one of the rooms highlighted students' capacities to reason algebraically in various situations. One of the presentations shared a teaching experiment aimed to promote the development of algebraic thinking among grade 4 students in the context of identifying numerical relations and patterns and thus deriving generalizations. Another one talked about an online game which focused on developing students' (grade 6) abilities to solve contextual problems dealing with covariation and functional relations and thus enter the domain of algebraic thinking. The other two focused on curricular issues. The third presentation analysed the differences in the treatment of the concept of function in two different kinds of middle school curricula used in the United States. The last presentation dealt with the status of algebra instruction, and in particular instruction of equations, in China where the author examined the textbooks, students' learning and teachers' instruction to come to understand the issue.

The four presentations in the other room focused on varied themes in algebra education. The first presentation briefed on a part of a larger study trying to understand the relationship between students' understanding of fractions as quantities and their abilities to form equations which require such multiplicative understanding. The second presentation reported on students' understanding of function concept among nursing students after they had worked in a context-based, collaborative instructional module. The third reported freshmen students' ability to use their algebra pre-requisite skills while working on calculus problems (Applied calculus optimization problem). The last presentation explored elementary school

students' non-formal algebraic reasoning while solving word problems, especially by focusing on the mathematical structure or attending to the relation between quantities in the problem.

Session 4: July 14, Saturday, 10:30–12:00, Room no. 308a and 309

We had scheduled five presentations on the last day, however one of the presenters did not show up, so we ended with four presentations, two long and two short. The first paper in one of the rooms presented a theoretical framework to account for the difference in performance of students who have been taught problem solving through a particular heuristic of drawing a diagram and its impact on their ability to use letter-symbols later in learning algebra. The other presentation highlighted the use of geometrical method in a dynamic environment while solving quadratic equations.

The short presentation in the other room analyzed secondary school students' structure sense, while they had to reproduce rational expressions involving identities. The long one addressed the use of ICT for diagnostic and differentiation purposes, by presenting an online set of resources to diagnose students' knowledge on algebra, and to provide teachers with appropriate resources for managing a differentiated algebra curriculum to meet students' different needs.

A wide range of issues thus got addressed through the presentations and led to fruitful and engaging discussions. These highlighted the abilities and limitations of children's/students' understanding in various conditions; teachers' understanding and role in developing algebraic thinking; the effects of curriculum, textbooks, tasks and technology in promoting students' understanding and teachers' abilities to teach effectively. Participants often related their own experiences within their countries. An interesting pattern that emerged from the presentations and discussions was the way Asian countries emphasise and inculcate the use of symbols and symbolic writing from an early age, whereas this is a much delayed activity in US and many parts of Europe. Thus, the research studies also looked for evidences of having achieved this competence and/or ways to strengthen it. The western countries look for emerging symbols and idiosyncratic use of symbols among children to elucidate their reasoning and thinking processes. This eventually leads them to develop a better understanding of symbols and systematic use of them at a later stage.

Conclusion

We did of course face some difficulties in organizing the TSG. The organizing team worked quite well before the conference in giving inputs and reviewing the proposal submissions in time. However, the actual organization was not very simple. The breaking into subgroups although helped us logistically, we lost on listening to each speaker and had to satisfy with the summaries presented by them during the whole group discussion. This would have been more fruitful had everyone read the papers before the session, which was rarely the case. Unfortunately, often due to limited capacities of participants to express in the English language, summaries or discussions could not be taken beyond a point and those who had facility with this language were the ones who got heard more. Some more time or some other ways of organizing the TSG may prove to be more fruitful. Since all the presentations in every TSG were scheduled well in advance and all participants knew the exact listing of presentations by speakers, participants moved from one to another TSG. Thus, the group kept changing each day making it difficult to engage in themes, issues and concerns of a particular TSG.

Teaching and Learning Geometry

Colette Laborde

Aims, Themes and Organization of the Topic Study Group

Aims and Themes

This group provided a forum for discussion of the teaching and learning of geometry, with a focus especially on the middle and secondary school and university levels. The focus of the group was on theoretical, empirical, or developmental issues related to

- Curriculum studies of new curriculum implementation, challenges and issues, discussion of specific issues such as place and role of transformations
- An application of geometry on the real world and other subjects,
- The use of instrumentation such as computers in teaching and learning of geometry,
- Explanation, argumentation and proof in geometry education
- Spatial abilities and geometric reasoning
- Teacher preparation in geometry education.

The issues were addressed from the historical and epistemological, cognitive and semiotic, educational points of view related to students' difficulties and related to the design of teaching and curricula.

TSG 10 received 40 submissions. We decided to subdivide the group into 2 subgroups during 3 slots of the group and to organize a poster session during one slot.

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Organization

Each paper was reviewed by two members of the organizing team who gave an evaluation and suggestions for the writing of the full paper. From the reviews and interactions by email among the members of the Organizing Team, an agreement was reached on a final list of presentations and posters, leading to 3 long oral presentations, 17 shorter presentations and 20 posters. Finally, due to cancellations, 3 long oral presentations and 14 presentations took place. Only 4 posters were displayed at the poster session. Most of the poster presenters left their posters in the main poster session of the congress. This turned the poster session of the group into a very interactive and vivid session with a small number of papers, in which each poster was presented by the author(s) and then discussed with all the participants.

The presenters in the group came from 12 different countries of North and South America, Asia and Europe.

Content of the Group

Range of the Themes Addressed in the Group

Several themes dealing with various mathematical contents were addressed in the group (Table 1).

A Multifaceted Approach of Geometry

As visible in the previous table, geometry was approached from various points of view. It should be noted that these points of view are not independent but intertwined. For example, the notion of "geometric transformation" was addressed by several presentations focusing on various themes: curriculum design, students' learning or teachers' knowledge. Some key issues arose from the range of themes addressed by the group:

- the notion of shape and generally of representation in geometry teaching and learning with an extension to the use of Dynamic Geometry environments
- the link between geometry and the real world
- the notion of transformation
- teacher education

The notion of "shape" as a corner stone of school geometry was investigated by Usiskin in his long presentation: "(1) a "figure"—we study many different shapes in geometry; (2) a "type of figure", as in the declaration that an object is triangular-shaped; and (3) a "property of a set of similar figures", as in the statement that two

Table 1 The addressed themes and contents

Theme	Mathematical content	School level
Mathematical analysis of the domain	Shapes and relationships with functions, graphical representations	Secondary, University
Curriculum and textbooks	Plane geometry, transformations	Secondary
Problem solving	Combinatorial problems	Secondary, College, University
Reasoning and proving	3D and 2D configurations	Middle school
Modeling the real world	Mirror and line reflection, trigonometry	Elementary, Middle school, Secondary
Use of tools and technology	Centroids in 2D and 3D geometry, geometrical relationships, tessellations and transformations	Primary, Middle school, Secondary
Introduction to axiomatic system	Geometry of the sphere	College, University
Students' solving strategies	Area of trapezoids	Upper elementary, Early secondary, Secondary, College, University
Students' recognition of shapes	Solids	Primary, Middle school
Reading and writing	3D geometry	Upper secondary
Teacher education	Transformations, measurement	Pre and in-service teacher education

figures are congruent if they have the same size and shape, or two figures are similar if they have the same shape." Usiskin investigated how the notion of shape has been extended in school geometry with four components of present school geometry: coordinate geometry, transformations, applications of geometry, dynamic geometry software environments. An important claim of Usiskin is that whereas geometry is usually considered as studying abstractions of real objects, "geometry studies real figures as well as abstract ones".

This extension of the notion of shape can be linked to the notion of diagram or representation of geometric objects in 2D or 3D. The issue of representation was involved in several contributions.

In 3D, there is a larger variety of representations than in 2D: real models, 2D representations in various perspectives, computer representations. Ludwig and Steinwandel carried out an investigation on 242 10 to 15 year-old students who had to identify the shape of faces and to give the number of faces, edges and vertices of Platonic and Archimedean solids represented by either models, or computer animations or diagrams. In his long presentation, Ludwig showed that students benefit more from real models. The assistance by computer animations and by pictures was

not so fruitful in tasks where the students need mental rotation to solve the task. Lavador used the Bruners' classification to design a teacher guide about measurement of solids, starting from enactive representations to move to images and iconic representations that lead then to symbolic representations.

The chosen representations in geometry problem solving (be it in 2D or 3D) may help or hinder a constructive reasoning for 12–15 year old students (Jones, Fujita and Kunimune); for the same problem depending on the diagram students may recognize or not the configuration for applying a known theorem. In his long presentation, Jones showed some examples in 2D and 3D and stressed the existence of prototypical representations that may turn into obstacles for recognizing the same property in other representations. Students' difficulties in interpreting diagrams seem to prevail across the world and are mentioned in contributions from Germany, Japan, and England. Jones concluded that "questions remain about how different mathematical representations influence students' decision making, conjecture production, and proof construction processes in the classroom, and how can such representations can be utilized by teachers to develop students' productive reasoning process." This is exactly the question also addressed in Kageyama's contribution that studies how students recognize analytical and logical properties of figures in construction tasks and use figural properties as justifying tools.

The link between geometry and the real world underlies several contributions and was even the focus of a few presentations. The issue seems to be more complex than expected. In some cases, referring to the real world can be very helpful for students (Ludwig). Whereas for Usiskin, although geometry is usually considered as studying abstractions of real objects, "geometry studies real figures as well as abstract ones", Boehm, Pospiech, Narciss and Körndle claimed that mathematics is an abstract world and they investigated what might be the potential confusions regarding a physical phenomenon after having experienced mathematics and physics lessons on this topic. Their study dealt with a very relevant phenomenon the mirror image in geometrical optics, as very often reflection is introduced in mathematics as modeling the mirror image. Their empirical data showed that we must pay attention to the fact that reality itself is not taught but a model of the reality and we must take into account the role of the used model in the teaching. It may happen that they do not go hand in hand as for reflection and mirror image and students may build inadequate knowledge. The results of the empirical study showed that students learn better when the scientific model is split into different science areas and when they are introduced to a multi-perspective modeling encompassing all model parts.

The link between real objects and theoretical objects of geometry was also viewed from the perspective of physical manipulations: real models for solid geometry (Ludwig, Suarez) but also strings, scissors, geoboard at elementary school (Faggiano). Faggiano stressed the fact that the manipulation by children contributes to the construction of meaning to geometric objects and relations only if they are involved in suitable tasks designed by the teacher.

Representations of geometric objects in Dynamic Geometry Environments are of a new nature and largely extending the range of manipulations and thought

operations. Surprisingly a relative small number of contributions addressed this issue. Mammana (Ferrarelo and Pennisi) asked students to generalize properties from 2D to 3D by using two Dynamic Geometry environments (Cabri II plus and Cabri 3D). Their observations showed how the computer environments helped students not only to verify their conjectures but also to prove them. The same idea of combining exploring and generalizing was also investigated by Withney, Kartal and Zawojewsky with collegiate students using Lenart spheres for constructing an axiomatic system of spherical geometry. Faggiano combined the use of dynamic geometry and manipulatives at elementary school and concluded to the benefit of such combination. Lindamann carried out an investigation on the provocative question: "Which learning environment, DGE or traditional one produces a greater learning in a college geometry course?". No significant difference was found between the results of both kinds of learning environments. However as noted by Lindamann, students using technology gained other skills related to technology.

Transformations was a theme addressed by many contributions at least from two perspectives, a curricular perspective and from the perspective of pre- or in-service teacher education. La Ferla et al. compared the Common Core standards in the United States and the Turkish curricula and showed that the teaching of transformations is reinforced by the Common Core standards and becomes more aligned with the Turkish curriculum. Innovative teaching introducing pre-service or in-service teachers not only to transformations, but also to their use in solving geometry problems was reported by several contributions. Saego reported by means of very relevant examples about a professional development and its rich materials guiding teachers to move beyond conceptualizing similarity as a numerical relationship between two discrete figures to instead understand a precise conception of similar figures from a transformations-based perspective. Xhevdet Thaqi compared curricula of Spain and Kosovo and investigated "how do prospective teachers understand, learn and present each component of geometric transformations, if there is any differences between two different countries." The study concluded that of importance among student teachers is the concept image of transformation as displacement and change of place.

Teacher education was part of several presentations, be it the focus of the paper or joint to another issue such as the teaching and learning of transformations. As stressed by Somayajulu, teacher knowledge is especially fragile in geometry as a subject. This is certainly a major motivation for improving teacher education in geometry.

Geometry as a source of problems was illustrated by some contributions: Soifer presented geometry combinatorial problems for advanced students, Manizade and Mason carried out a thorough analysis of possible solving strategies of calculating the area of a trapezoid and showed how solving this task may be done at various Van Hiele levels. Hak Ping Tam and Hsin Han Wang concluded their study about the presentation of Pythagoras theorem in Taiwan textbooks by claiming that this theorem is a good opportunity for making students aware of the fact that multiple proofs can be given for the same theorem.

In conclusion, the various presentations of the group illustrated very well how rich the field of geometry teaching and learning is and how it can be investigated from various points of view with some emerging key issues, namely the nature and the role of representations.

Teaching and Learning of Probability

Per Nilsson and Jun Li

Aims and Focus

Probability has strong roots in the curricula of many countries but is relatively new in others. And although probability has been introduced into the mainstream school mathematics curricula in many countries, research does not necessarily support a rapid inclusion into the curriculum because many problems in teaching and learning probability are still unsolved. For example, should probability be taught to all students? When should students be introduced to probability? What is probability literacy? How is probability literacy developed? What kind of knowledge do teachers need in order to teach probability in more concrete, meaningful and effective ways? How do we facilitate the development of such teaching knowledge? How could investigating students' conceptions of probability from various perspectives further inform our teaching? At ICME 12 in Seoul, Topic Study Group 11 provided a forum for presentations and discussion from an international view about the current state and important new trends in research and practice related to the teaching and learning of probability.

Traditionally, the teaching of probability concerns two different interpretations of probability: (1) a classical conception, where probability is based on combinatorics or formal mathematics, and (2) a frequency conception, where probability is

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based on empirical evidence and long-termed behaviour of random phenomena. The Topic Study Group (TSG) tried to look beyond these two interpretations and consider as the first focus how to teach probability concepts in ways that develop understanding and support the use of probability to make rational decisions in situations that affect peoples' lives and their work. It is important to note that the notion of probability as used in the Topic Study Group included aspects of chance, randomness, risk and its relationship to statistics.

The second focus was on teachers' knowledge for teaching probability. While teacher knowledge is critical for effective teaching of probability, very few studies deal with teacher knowledge and they (including the papers presented in the TSG) indicate that neither pre-service nor in-service teachers have enough knowledge for teaching probability. There is a growing global interest in learning what kind of knowledge teachers need to be able to effectively teach probability concepts and how to facilitate the development of such teacher knowledge. To promote more discussion and research in this area, the plenary panel discussion was narrowed to teacher knowledge for probability teaching.

The paper contributions were structured according to four general themes: *Curriculum Development and Policies, Research on Students' Thinking and Reasoning, Probability Literacy and Instructional Challenges, Teacher Knowledge in Probability Teaching*. They were presented in four sessions allotted to TSG 11.

The first three sessions began with an invited keynote speech: Ramesh Kapadia (United Kingdom), Manfred Borovcnik (Austria), Iddo Gal (Israel). The aim of these lectures was to sketch an overall picture of the TSG theme. A plenary panel was arranged for the last session that included all three keynote speakers and liaison, Gail Burrill, who were invited to reflect on the theme. Each session was closed by a summary by the session chair.

Session 1: Curriculum Development and Policies

Egan Chernoff, chaired the session., which began with opening remarks by co-chairs Per Nilsson and Jun Li, followed by an invited keynote speech by Ramesh Kapadia, and presentations by Jenny Gage (United Kingdom), Xianghui Wu (China).

Kapadia's address reviewed the main changes in the research related to probability education from the Piagetian-Fischbein era, the Kahneman_Tversky era to the current period. He summarized key research in the three eras and stressed the importance of developing new ideas from the past. He also provided an overview of curriculum development in England since the 1970s in the hope that some of the lessons can be applied elsewhere of the world. Based on the research and curriculum development, he suggested introducing probability at the elementary level, using a judicious mixture of subjective theory, a priori theory and frequentist theory of probability.

Gage presented an on-going project investigating mathematical modelling as a means for the learning of probability. She described school trials solving two

problems by students between 10 and 14 years of age. The results suggested that the modelling approach and using values from the tally (natural frequencies), not probability, seemed to help students grasp the essence of the analysis of a problem and enabled them to use tree diagrams and 2-way contingency tables successfully.

Wu's paper was based on the belief that learning by game-playing should be central in children and adolescents' education as it stimulates the learning processes of flexibility, enjoyment, and adaptability. He shared with us his teaching experiences using three carefully designed games in his Grade 9 class.

In summarizing the session, Chernoff pointed to how the three talks highlighted that terms like misconceptions and subjective probability require serious discussion in future research. He raised the question of whether the frequency interpretation of probability should be emphasized with more care. He called on the need to address the teaching of risk and suggested we may benefit from research relevant to other TSGs, such as mathematical applications and modelling in the teaching and learning of mathematics.

Session 2: Research on Students' Thinking and Reasoning

Per Nilsson was the chair of Session 2. The session began with an invited address by Manfred Borovcnik, followed by presentations from Judith Stanja (Germany) and Theodosia Prodromou (Australia).

In his talk, "Conditional probability- a review of mathematical, philosophical, and educational perspectives", Borovcnik argued that conditional probability is a key concept in learning and accepting probability and that objective probability alone may not really help to change people's private criteria for dealing with conditional probability problems. He suggested the subjective approach is much closer to how people think and can thus much better explain conditional probabilities. He analyzed the need for teaching strategies to make plausible that conditional probabilities have nothing to do with time and causes, and showed various strategies for solving the Monty Hall problem. Borovcnik also reflected on translating probabilistic questions into *absolute (natural) frequencies.* His conclusion was that a wider conception of probability might be useful.

Stanja shared her attempt to characterize children's (age 8–9) elementary stochastic thinking by taking the role of semiotic means into account. Some theoretical ideas from Duval were outlined to serve as a basis for her description and analysis of interview data. She particularly stressed the complementarity of artefact and sign in learning probability and assessing child's understanding.

Prodromou addressed issues regarding the possibilities and challenges of using a computer-based modelling approach in the teaching of probability to 15 year-old students. In her investigations she particularly focuses on how the modelling approach can be used for building links between variation, theoretical models, simulations, and probability. Her results suggest that the way students express the

relationship between signal and noise is of importance while building models from the observation of a real situation.

Summing up the session, Nilsson stressed the need to develop research methodologies in order to investigate the semiotic nature of teaching and learning probability. Approaching the teaching and learning through mathematical modelling seems timely. In Prodromou's study this was made in a computer-based learning environment. The session challenged research to develop real-world approaches for the teaching of probability through mathematical modelling.

Session 3: Probability Literacy and Instructional Challenges

Enriqueta Reston was the chair of Session 3. The session began with an invited address by Iddo Gal, followed by presentations from Hongshick Jang (Korea), Taek-Keun Oh and Kyeong-Hwa Lee (Korea).

Gal sketched an outline of probability literacy, its development, needs and connections to frameworks of adult competencies and mathematics curricula. He defined probability literacy by knowledge elements and dispositional elements and explained their relationships to both *internal* and *external* goals of probability education. To meet external demand better, he suggested teaching directly for probability literacy by increasing the use of tasks based on real-life problems in teaching and assessment, allowing time for subjective probability, and addressing dispositions and personal sentiments.

Jang suggested that empirical evidence involving the process of mathematical modelling in teaching is helpful to senior high school students' learning of probability. He presented his evidence both in terms of efficiency of teaching and motivation of students, but argued the necessity of mathematical formulation within the various types of uncertainty and the need to go beyond the conventional notion of mathematical modelling.

Oh and Lee addressed the teaching and learning of probability for gifted students. They found that learning through debate in solving probability tasks can be valuable for developing creativity of gifted Grade 11 students as the process stimulates flexibility, elaboration, and originality.

In summarizing the session, Reston reflected on whether there is any consensus on the meaning of probability literacy. Moreover, how does it relate to mathematics literacy? Statistical literacy? What are the overlaps? What are the gaps, if any? She also raised questions regarding what concrete actions and future directions will enable us to address instructional challenges in developing probability literacy among our students.

Session 4: Teacher Knowledge in Probability Teaching

Kyeong-Hwa Lee chaired the final session. After the presentations by Enriqueta Reston (Philippines), Per Nilsson (Sweden) and Egan Chernoff (Canada) the session ended with a panel debate on Teacher Knowledge in Probability Teaching.

Reston described a study exploring elementary mathematics teachers' conceptions of probability through inductive teaching and learning methods. As a background, she elaborated on the diversity of possible inductive teaching methods including, for instance, *inquiry teaching, problem-based teaching* and *investigations*.

Based on a survey study approach, Nilsson investigated correlations between Swedish teachers' content knowledge of probability and their level of education, teaching years and self-assessments of probability concepts. He found that the teachers have low confidence in understanding probability and have difficulties in applying the concepts in probability tasks.

Chernoff reported on research using the *attribute substitution model* to account for certain normatively incorrect responses of prospective teachers' understanding of random behaviour generated from a series of coin flips. His study considered individuals who, when presented a particular question, answer a different question instead. He argues that making connections between mathematics education and other domains of research will give mathematics education researchers new insights.

Before the Plenary Panel, Lee reviewed the meaning of knowledge for teaching given by Shulman in 1980s and Ball after 2000. During the panel time, Burrill, Kapadia and Borovcnvik shared with all participants their insights on this topic.

Burrill choose teachers' pedagogical content knowledge for teaching probability as her main point. She indicated that having deep understanding of content knowledge is crucial for teaching. Teachers' knowledge of students and their ways of thinking about probability are essential as well. She recommended the Common Core State Standards for mathematical practices as a frame for engaging students in probability tasks and highlighted key points for teaching probability to teachers. Kapadia addressed teachers' content knowledge and pedagogical content knowledge as well. To develop probabilistic understanding, he appealed for investigations of teachers' knowledge across different countries with shared instruments. Borovcnik examined seven sources from which teachers could obtain their knowledge. He called for enhanced teaching of probability at the university level and connecting that closely to pedagogical issues, for example, to provide well-organized textbooks, which highlight modeling and other important ideas and to discuss the origins of students' misconceptions and how to use these in teaching to build understanding. He also listed and commented on several journals, websites of statistical associations and e-platforms he thought could be used to support teachers' development of probabilistic reasoning.

Several papers were presented in poster form: Haneet Gandhi, India; Zhengwu Long, China; Robyn Ruttenberg-Rozen, Canada; Narita, Masahiro, Japan; Tânia M.

M. Campos, Rosana Nogueira de Lima and Verônica Yumi Kataoka, Brazil; Natsumi Sekiya, Japan; Franziska Wandtner, Goetz Kersting, Reinhard Oldenburg, Germany; Michimasa Kobayashi, Japan. The posters elicited further discussion on the organizing themes of the sessions.

Time for formal presentations and discussions is always very limited at an international conference. But we are convinced that the work of the group initiated discussions on critical areas in probability education, such as teachers' knowledge for teaching, that will attract further investigations and support collaboration among people who are interested in the teaching and learning of probability.

Teaching and Learning of Statistics

Dani Ben-Zvi and Katie Makar

TSG-12 Rationale

Being able to provide sound evidence-based arguments and critically evaluate data-based claims are important skills that all citizens should have. It is not surprising therefore that the study of statistics at all educational levels is gaining more students and drawing more attention than it has in the past. The study of statistics provides students with tools, ideas and dispositions to use in order to react intelligently to information in the world around them. Reflecting this need to improve students' ability to think statistically, statistical literacy and reasoning are becoming part of the mainstream school and university curriculum in many countries.

As a consequence, statistics education is a growing and becoming an exciting field of research and development. Statistics at school level is usually taught in the mathematics classroom in connection with learning probability. Topic Study Group 12 (TSG-12) included probabilistic aspects in learning statistics, whereas research with a specific focus on learning probability was discussed in TSG-11 of ICME-12.

Organizers Co-chairs: Dani Ben-Zvi (Israel), Katie Makar (Australia); Team Members: Lisbeth Cordani (Brazil), Arthur Bakker (The Netherlands), Jangsun Paek (Korea); Liaison IPC Member: Gail Burrill (USA).

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TSG-12 Meetings During ICME-12

This growing interest in statistics education was reflected in the popularity of this group and in the more than 40 papers accepted for presentation. The members of TSG-12 came from twenty different countries and varied significantly by experience, background and seniority. The presentations were divided into six themes of key issues in statistics education research: (a) theoretical issues in learning statistics; (b) integrating statistics with students' experiences; (c) the emergence of students' statistical reasoning; (d) teachers' statistical knowledge and learning and professional development of teachers; (e) learning technology in statistics education; and (f) learning statistics in school and at the tertiary level.

The four meetings of TSG-12 were organized to create a sense of community among all presenters and participants, who shared a common desire to improve statistics education by focusing on conceptual understanding rather than rote learning. To build and support this sense of community we asked participants to prepare for TSG-12 before they arrived in Seoul by reading all papers in advance, so we could discuss each other's work; the co-chairs kept informal correspondence with all participants before, during and after the conference; and finally, participants were asked to be involved every day of the program so we could get to know one another, develop collegial networks, welcome our emerging scholars and discuss the important work in statistics education research around the world.

Because of the large number of proposals we received, the time available only allowed for relatively short presentations by the authors. However, we felt it critical that all proposals be given time for presentation in some format. The four meetings were therefore organized to capitalize on community-building and discussions around our collective and individual research. Some of the sessions ran in parallel, some in roundtable format. While there was a poster session which is common for all TSGs, half of one TSG-12 session was dedicated to poster presentations so that the TSG-12 community could engage more directly with their authors and each other in a relaxed setting. Another highlight of the program was a panel of discussants on the final day to reflect as a community on the themes, presentations, issues raised and discussions over the four days.

The accepted papers were organized in the following ways:

- About twenty poster presentations to engage TSG-12 community discussions with diverse and thought-provoking studies;
- Eleven short presentations (5 + 5 min discussion) in a roundtable format organized into four themes to enrich understanding of the themes and allow for extended discussions around common interests;
- Twelve longer presentations and discussions (10 + 5 min discussion) to enhance the overarching themes of the short presentation and poster sessions;
- Four major long presentations (20 + 10 min discussion) to provoke initial discussions and stimulate final day reflections among the whole TSG-12 community. These papers were authored by Andreas Eichler and Markus Vogel

(Germany), Arthur Bakker, Xaviera van Mierlo and Sanne Akkerman (The Netherlands); Luis Saldanha and Michael McAllister (USA); and Dani Ben-Zvi and Keren Aridor-Berger (Israel).

TSG-12 Beyond the Conference

Informal feedback received after the conference was extremely positive. We felt at the end that much can be learned by integrating results from such a variety of research and practice in statistics education. This integration of theories, empirical evidence and instructional methods can eventually help students to develop their statistical thinking. These ongoing efforts to reform statistics instruction and content have the potential to both make the learning of statistics more engaging and prepare a generation of future citizens that deeply understand the rationale, perspective and key ideas of statistics. These are skills and knowledge that are crucial in the current age of information.

An informal set of proceedings was created to allow for immediate distribution of the TSG-12 papers among those within and beyond the TSG-12 members. The proceedings are available at: http://dbz.edtech.haifa.ac.il/publications/books. Many of the members of the community that came together for TSG-12 have remained in touch through a sharing of contact details and plans to meet again at the Ninth International Conference on Teaching Statistics (ICOTS-9) in 2014. Based on the TSG-12 papers, the book *Teaching and learning of statistics: International perspectives*, edited by Ben-Zvi and Makar, was published in 2014 by the Statistics Education Center, the University of Haifa, Israel.

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TSG-12 Resources

- TSG-12 Website: http://www.icme12.org/sub/tsg/tsgload.asp?tsgNo=12.
- TSG-12 Proceedings (eBook): http://dbz.edtech.haifa.ac.il/publications/books.

Teaching and Learning of Calculus

Victor Martinez-Luaces and Sunsook Noh

Aims

This Topic Study Group was a forum for discussions about the research and development in the teaching and learning of Calculus, both at upper secondary and tertiary level. Long and short presentations as well as the posters, showed advances, new trends, and an important work done in recent years on the teaching and learning processes of Calculus.

Organization

At ICME-12, TSG-13 had four one and a half hour timeslots and two general posters sessions. On the website of ICME-12 it is possible to access to all relevant documents including long presentations, short presentations and posters.

The accepted papers were organized as follows:

Organizers Co-chairs: Victor Martinez-Luaces (Uruguay), Sunsook Noh (Korea); Team Members: Margot Berger (South Africa), Francisco Cordero (Mexico), Greg Oates (New Zealand); Liaison IPC Member: Johann Engelbrecht (South Africa).

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- 4 papers were accepted for long presentations.
- A group of 13 papers were accepted for short presentations. Finally, only 1 of these papers was not presented in the group sessions.
- In each session 3 short presentations and a long one were delivered.
- Each session was devoted to an important topic in Calculus teaching and learning.
- Posters were presented in the general poster session which was common for all TSGs.

The structure for each of the four 90-min session included some brief opening remarks by the co-chairs of the committee, followed by a long presentation (20 min) and 3 short paper presentations (10 min each). After the long and short presentations of each session, the whole group had at least half an hour for questions, comments and general discussion.

The following paragraph provides details on the 4 oral sessions and the 2 poster presentations related to TSG-13.

Long and Short Presentations Delivered

Tuesday, July 10

This morning session was devoted to an important topic in Calculus teaching: the derivative concept. The long presentation was delivered by William Crombie, from U.S.A, who proposed an alternative architecture of Calculus, in order to allow the access to advanced concepts from an elementary standpoint to a larger group of learners. An example of this approach is given by the idea of "transition line" that can be used even before developing limits and derivatives.

After that, the first short presentation was given by Jungeun Park, from U.S.A., who studied the student's discourses on the derivative using a communicational approach to cognition. Particularly, she focused on students' descriptions about the derivative and the relationships among a function, the derivative function, and the derivative at a point.

The next speaker, Miguel Diaz, from Mexico, documented the understanding of the derivative and its meaning on the part of 12 teachers, who teach Calculus in a high school in Mexico, using for this purpose several questionnaires specifically designed.

Finally, Hyang Im Kang from Korea reported how 11th grade students went through in reinventing derivatives on their own via a context problem involving the concept of velocity.

Wednesday, July 11

This second morning session was devoted to modelling, applications and other topics and it started with Victor Martinez-Luaces, from Uruguay, who described teaching experiences with inverse problems—of both causation and specification types—and modelling in Engineering Calculus courses.

The first short presentation was delivered by Mohammad Pourkazemi from Iran. He showed how by giving applied examples of Economics and Management in each section of Calculus, it is possible to increase the interest in Mathematics among students.

Next speaker, Anne D'Arcy-Warmington, from Australia suggested a reversal of the order, showing Calculus applications first and then the rules as a consequence in a semi-modelling style approach.

Finally, Greg Oates from New Zealand reported on 11 contemporary studies selected from the last Delta conference, which presented direct applications to, or important implications for, current practice in the teaching of undergraduate Calculus.

Friday, July 13

The third session was devoted to several important concepts in Calculus, like integrals, series, etc. The long presentation was delivered by Anatoly Kouropatov, from Israel. In his paper, he discussed the idea of accumulation as a core concept for a high school integral Calculus curriculum.

Short presentations started with Maria Teresa Gonzalez, from Spain. In her paper, she described the growth of mathematical understanding in university students, engaged in mathematics classroom tasks about the concept of numerical series.

The second short talk was given by Rafael Martinez-Planell, from Puerto Rico. His paper focused on student graphical understanding of two variable functions. His study—which applies APOS and Semiotic Representation theories—was based on semi-structured interviews with 15 students.

This Friday session finished with Jennifer Czocher, from U.S.A., who investigated about topics in introductory differential equations and their relation with the knowledge that students are expected to retain from their Calculus courses.

Saturday, July 14

The last session of TSG 13 was about pre-Calculus and first Calculus courses, and started with the long presentation delivered by Dong-Joong Kim, from Korea. In his paper, Kim investigates characteristics of the limit concept through the simultaneous use of historical and experimental analyses.

David Bressoud, from U.S.A. was in charge of the first short presentation. He showed the preliminary report of results from a large-scale survey of Calculus I students in the United States. The analysis highlights students' mathematical background as well as aspects of instruction that contribute to successful programs.

Another large scale survey—in this case, carried out in China—was the starting point of the following talk delivered by Xuefen Gao. Her study, involving 256 college-level Calculus students and 3 teachers, investigated the problems and misunderstanding of concepts in Calculus and designed concept-based instruction to help students to understand concepts.

Finally, Jose Antonio Fernandez, from Spain presented results of an exploratory study performed with students of ages 16–17. He investigated the different uses that these students make of terms such as "to approach", "to tend toward", "to reach" and "to exceed", terms that describe some properties of the concept of finite limit.

Poster Sessions

10 posters corresponding to TSG 13 were presented in 2 general poster sessions.

In poster 13-1, Young Gon Bae, from Korea studied how university students matched graphs and functions. In the next poster (13-2) Rie Mizukami, studied the main changes in the Calculus content at senior high schools in Japan. The third poster (13-3) explained by Jacinto Eloy Puig, from Colombia, analyses the important interconnections between infinity and infinitesimal quantities. In the next one—13-4—Youngcook Jun, from Korea, explored how to use CAS to develop a step-by-step solver for Calculus learning. In poster 13-5, Kazuki Chida, from Japan, proposed how to obtain laws about trigonometric functions from a very simple differential equation, without any reference to either an angle or a triangle. The next poster—13-6—showed by Kanna Shoji, from Japan, is aimed for the development of teaching materials, in order to make the students understand the relation between real-life and mathematics. In poster 13-7, Allan Tarp, from Denmark explored Calculus roots in primary and middle school. The next poster, i.e., the 13-9, expounded by Abolfazi Gatabi, shows how Iranian students participate in classroom discussion about infinite and infinitesimal concepts. Poster 13-11, presented by Mikie Takahashi, from Japan, focuses on approximate value calculation and its relation with practical high school mathematics. Finally, in the last poster (13-13), Misfer AlSalouli, from Saudi Arabia, investigates mathematics high school teachers' conceptual knowledge regarding the topics on Calculus.

At the end of the second poster session, the authors had the opportunity for oral presentation of their posters, having the benefit of an audience related to the TSG.

Conclusions

Several issues related to teaching and learning of Calculus regularly appeared in the general discussions located at the end of the oral sessions. The main themes in those discussions were: technology, visualisation, problem-solving, modelling and applications, and assessment, among others. TSG-13 papers also featured learning theories, construction of Calculus concepts and ideas (limits, integrals, derivatives, etc.), roots of Calculus concepts and other important topics in Calculus teaching and learning.

Most of the papers (long and short presentations and posters) showed an interest for innovative approaches to different topics, in order to help students to improve their knowledge and comprehension of Calculus. In several cases, these innovations were directly related to the use of technology, whereas in others, they were more involved in teaching approaches, courses materials, or specific tasks to be carried out by students of different educative levels and careers.

It is hoped that this interesting discussions and interaction between teachers and researchers of different countries will stimulate innovative ideas that will progress the advancement of mathematics education—particularly, in Calculus teaching and learning—into the following years of this new century.

Reasoning, Proof and Proving in Mathematics Education

Viviane Durand-Guerrier

Overview

The work of TSG 14 intended to serve a dual role: presentation of the current state of the art in the topic "Reasoning, proof and proving in mathematics education" and expositions of outstanding recent contributions to it. The topic will be considered at all levels of education: elementary, secondary, university (including pre-service teacher education), and in-service teacher education. The Organizing Team of the Study Group had invited theoretical, empirical or developmental papers that address one or more of the following themes: Historical/Epistemological/logical issues; Curriculum and textbook aspect; Cognitive aspect; Teaching and teacher education aspect, so that any paper of relevance to the overall focus of the Study Group.

The role and importance assigned to argumentation and proof in the last decade has led to an enormous variety of approaches to research in this area. Historical, epistemological and logical issues, related to the nature of mathematical argumentation and proof and their functions in mathematics, represent one focus of this wide-ranging research. Focus on mathematical aspects, concerning the didactical transposition of mathematical proof patterns into classrooms, is another established approach, which sometimes makes use of empirical research. Most empirical research focuses on cognitive aspects, concerning students' processes of production of conjectures and construction of proofs. Other research addresses implications for the design of curricula, sometimes based on the analysis of students' thinking in arguing and proving and concerns about didactical transposition. Recent empirical

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research has looked at proof teaching in classroom contexts and considered implications for the curriculum. The social-cultural aspects revealed in these studies motivate a current branch of research which is offering new insights. Comparative studies, trying to come to a better understanding of cultural differences in student's arguing and in the teaching of proof can be seen as part of this new branch of research. In this respect, papers presented at ICMI study 19 on "Argumentation and Proof" illustrate this diversity. Differences concern the focus researchers take in their approach, as well in the methodological choices they make. This leads not only to different perspectives, but also to different terminology when we are talking about phenomena. Differences are not always immediately clear, as we sometimes use the same words but assign different meanings to them. On the other hand, different categories that we build from empirical research in order to describe students' processes, understandings and needs are rarely discussed conceptually across the research field. Conceptual and terminological work is helpful in that it allows us to progress as a community operating with a wide range of research approaches.

Eleven papers and seven posters have been presented during the four sessions. There were thirty-five non-presenting participants who attended at least one session. The papers were from: Hong-Kong (1), Japan (2), Japan and UK (1), Turkey (1), UK (1), USA (5). The posters were from: Canada (1), Colombia (1), France (1), Japan (1), Peru (1), USA (2). The non presenting participants came from: Denmark (1), France (2), Germany (2), Hong-Kong (1), Japan (5), Korea (10), Norway (1), Portugal (1), South Africa (1), Sweden (3), Thailand (2), UK (1), USA (4).

For each session the attendee ranged from forty to fifty participants. The composition of the attendee was representative of the diversity of the participants in the congress: mathematicians, didacticians, expert researchers as well as young researchers, teachers form primary school to university.

According to the topics addressed by the papers accepted we identified four main themes to which each paper and poster may be related:

- Theme 1: Conception of proof from different theoretical perspectives.
- Theme 2: Proof in the classroom: the role of the teacher.
- Theme 3: Evaluation of proofs.
- Theme 4: Curriculum and materials.

Each of the four 90-min sessions (July 2013 10th, 11th, 13th, and 14th) were devoted to one of these themes. The structure for each 90-min session included some brief opening remarks by the chair of the session; the presentations, 10-min for long presentation and 5 min for short presentations. The general discussion on the papers and posters took place at the end of each session.

Theme 1: Conception of Proof from Different Theoretical Perspective (10th July 2013)

In this session, three papers were presented, so that two related posters, presenting a variety of theoretical backgrounds.

Kotaro Komatsu (Japan), in line with a long tradition of considering Lakatos epistemology as relevant for mathematics education, proposed to consider *Lakatos' Heuristic Rules as A Framework for Proofs and Refutations in Mathematical Learning: Local Counterexample and Modification of Proof*. Ysuke Tsujiyama (Japan) paid interest to characteristization of proving process in school mathematics based on Toulmin's concept of field, while Michelle Zandieh, Kyeong Hah Roh, Jessica Knapp (USA) explore *Student Proving through the Lens of Conceptual Blending*.

In their posters, Paul Dawkins, Kyeong Hah Roh (USA) emphasized *the Roles of Metaphors for Developing Students' Logical Control in Proof-oriented Mathematics*, while Shiv Karunakaran (USA) considered *Examining the Structure of Proving of Experienced Mathematics Doctoral Students*.

The final discussion enlightened the diversity of the theoretical perspectives; questions were addressed from teachers to researchers on the relevance of their theoretical backgrounds for designing tasks aiming to develop reasoning, proof and proving in class.

Theme 2: Proof in the Classroom: the Role of the Teacher (11th July 2013)

In this session three papers and one poster were presented; various aspects of the delicate role of teachers in classroom concerning proof have been enlightened.

Annie and John Selden presented the paper from Milos Savic (USA) who considers the controversial question *Where is the Logic in Student-Constructed Proofs?* Andreas J. Stylianides and Gabriel J. Stylianides (U.K.) focused on "*The big hurdle we have to overcome is getting students out of the mode of thinking that math is just plug-in-and-move-on kind of thing*"*: Challenges in beginning to teach reasoning-and-proving*. Anna Marie Conner (USA) considered *Warrants as Indications of Reasoning Patterns in Secondary Mathematics Classes*.

In his poster, Medhat H. Rahim (Canada) proposed to consider *Description and Interpretation of Student-Teachers' Attempts to Construct Convincing Arguments and conjectures through Spatial Problem Solving Tasks.*

The discussion in the session, along with the content of presentation, put light on the difficulties for teachers to engage students in mathematical activity involving proof and proving; a main issue concerns the possibility of making students aware of the necessity for proof and proving. Taking in consideration that Geometry was the most represented mathematical domain in the papers and posters presented in the group, a question raised in the discussion: is this matter of fact unavoidable, or is it possible to work on proof in class in other mathematical domains. Finally participants agreed that although geometry is a relevant traditional domain for teaching reasoning, proof and proving in secondary school in many countries, there are also other relevant domains such as arithmetic, linear algebra, analysis etc., depending on the level.

Theme 3: Evaluation of Proofs (13th July 2013)

Two papers and two posters were devoted to evaluation of proofs or arguments. A third poster related to the theme 2 was also presented.

Yeşim İmamoğlu, Ayşenur Yontar Toğrol (Turkey) have presented *An Investigation of Senior Mathematics and Teaching Mathematics Students' Proof Evaluation Practices*. Yating Liu, Azita Manouchehri (USA) focused on means for *Nurturing High School Students' Understanding of Proof as a Convincing Way of Reasoning* and look for a theoretical framework.

In their posters, Shintaro Otsuka (Japan) paid interest on *Reasoning in Explaining False Statements: Focusing on Learner's Interpreting Propositions*, while Viviane Durand-Guerrier, Thomas Barrier, Faiza Chellougui, Rahim Kouki (France, Tunisia) provided *An Insight on University Mathematics Teaching Practices about Proofs involving Multiple Quantifiers*. Maria Nubia Soler Alavarez (Colombia) presented *Types of Rasoning used by Training Mathematics Teacher in a Class about Rational Numbers.*

Questions concerning validity were at the core of this session. The papers showed the variety of practice related to this question, opening a discussion on the distance between requirement addressed to students concerning rigor and ordinary teachers practices which generally do not fulfill these requirements. Finding an adequate balance between these two aspects in class is not easy.

Theme 4: Curriculum and Materials (14th July 2013)

In this session, three papers and one poster were presented, providing a variety of landscapes.

Mikio Miyazaki, Taro Fujita, Keith Jones (Japan, U.K.) presented material for *Introducing Proof in Lower Secondary School Geometry: A Learning Progression Based on Flow-chart Proving*. Yip-Cheung Chan (Hong-Kong) aim *Rebuilding The*

Harmony Between Figural and Conceptual Aspects For Reasoning, Proof and Proving in Dynamic Geometry Software. Ruthmae Sears (USA) investigates *The Impact of Subject-specific Curriculum Materials on the Teaching of Proof and Proof Schemes in High School Geometry Classrooms.* Estela Vallejo and Uldarico Malaspina (Peru) offered *A Look at the Justifications in the Basic Education in Peru: the National Curricular Design and some Texts used in the 1st Grade of Secondary level.*

The discussion on the presentations concerned the diversity of approach in curriculum and material, enlightening the interest of comparative studies on reasoning, proof and proving.

As closing remarks, the participants agreed that the discussion which took place at the end of each session were rich and concerned as well the implication for teaching, the theoretical assumptions, the role of logic, the specificity of geometry, the need for proof or proofs without words.

A common feeling was that, although *Reasoning Proof and Proving* have been studied for a long time, further international researches are needed.

Mathematical Problem Solving

Manuel Santos-Trigo and Zahra Gooya

Introduction

The program was designed to set up to organize, structure, and discuss the academic agenda of mathematical problem solving and its developments. The program included an open invitation to the mathematics education community to contribute and reflect on research and practicing issues that involve: (a) Addressing the origin, characterization, and foundation of mathematical problem solving, (b) discussing problem solving frameworks used to support research and curricula reforms in mathematical problem solving; (c) analyzing local and international research programs in mathematical problem solving; (d) discussing curriculum proposals that support the development of mathematical problem solving; (e) analyzing different ways to assess mathematical problem solving performances; (f) discussing the role played by the use of different digital tools in students' development of mathematical problem solving proficiency; (g) addressing programs that foster learners' development of problem solving approaches beyond school; and (h) identifying future developments of the field.

The international problem solving community responded to the invitation and sent more than 30 proposals, of those 18 were selected for presentation during the

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sessions, and 10 were assigned to the poster session. In this report, we inform about the subjects and themes that authors addressed in their written proposals, and the results and discussions that emerged during the authors' oral presentations held during the development of the sessions at the ICME conference. A pdf file that includes all authors' contributions can be retrieved from: http://www.matedu.cinvestav.mx/~santos/icme12/ICME12TSG15book.pdf.

An Overview

The authors' contributions addressed and discussed several issues that were identified in the open invitation letter they received and was available through the congress web-page. Here, we highlight common issues addressed in the contributions that include mathematical reflections on what problem solving entails, the variety of studies and methodologies used to frame research studies, the range of participants in those studies that involves elementary, secondary, high school students, in-service and practicing teachers, and university students, and a variety of theories used to support and develop problem solving research.

(a) Two contributions reviewed issues regarding what types of problems are relevant to discuss with students, and the importance for instructors to create an instructional environment in which students can actively be engaged in problem solving experiences. One example used to illustrate problem solving strategies and conjectures that emerged during the solution process was a variant of a task discussed by Polya (1954, pp. 43–52): *Into how many parts is space divided by 5 planes?* The discussion became important to identify ways to formulate and pursue conjectures in which a set of heuristics appears important during the entire solution process. The same theme "heuristic methods" is also addressed in another contribution to discuss examples where students have an opportunity to rely on strategies such as pattern recognition, working backwards, guessing and testing, looking for simpler problems, etc. to solve tasks set in different contexts. Both contributions offer ways to analyze tasks that can be useful to construct instructional paths to foster students' mathematical problem solving experiences.

(b) Eight contributions recognized the importance for learners to work on small groups to discuss and defend their ideas, listen to others, and communicate results. Two contributions emphasized students' social interactions as a way to enhance cognitive experiences. One proposes a teaching module to guide university students to comprehend and develop conceptual knowledge associated with a first differential equation course. In general, authors used a bricolage perspective that relies on several conceptual frameworks to support the study; another contribution builds up a local conceptual framework to guide practicing elementary teachers to develop problem-solving experiences through social interactions.

(c) Four contributions rely on statistical analyses to compare students' problem solving performances. For example, three studies emphasize the use of pre and post-tests to analyze and compare groups of students' problem solving achievements as a result of receiving differential problem solving instruction. For instance, one group explicitly addressed the importance of using analogical thinking in their approaches versus a group that followed a regular teaching approach. Other studies relied on the use of Case Study methodology in which the participants' problem solving behaviors are analyzed in detail. It is common in this process or use of task-based interviews, groups or class videos, or a combination of qualitative tools to gather data and to foster the development of problem solving approaches. In general, a tendency in six contributions was to rely on both the use of quantitative and qualitative tools to analyze learners' problem solving behaviours.

(d) It was observed that five contributions have explicitly relied on frameworks that extend problem-solving approaches such as models-and-modeling perspectives. The analyses of problem solving performance of students that consistently have shown high achievement in international assessments was also addressed in seven of the contributions. For example, a study focused on analyzing the extent to which some Korean students epistemological beliefs about mathematics are related to their problem solving behaviours. Similarly, another study analyzes how a problem-based learning (PBL) was implemented in China.

It must be noted that the use of mathematical competitions to promote learners' development of problem solving skills has been encouraged in different countries. For example, one study analyzes how a web-based mathematical problem competition became important for 13–14 years to engage in problem solving experiences that go beyond those that appear in regular classroom contexts. Yet, another contribution analyses how a set of didactic techniques based on the problem centred Japanese tradition is implemented in Swedish. In this particular study the author relies on the use of Anthropological Theory of Didactics which is a framework commonly used in the French mathematics education tradition.

(e) Problem solving activities also play an important role in teachers professional development programs and the education of prospective teachers. A contribution focuses on fostering both prospective and practicing teachers' competence to pose, formulate, and pursue questions or problems. The framework that authors used to support the problem posing experiences involves epistemic, cognitive, and mediation analysis of tasks and learners interaction and is called an Onto-Semiotic approach. Likewise, the implementation of problem solving activities has taken different directions and aims. For instance, one contribution emphasizes the second Polyas' proposed stage of problem solving "designing a plan or planning the solution" to improve colleges students abilities to solve arithmetic problems.

Remarks and Future Directions

Learning, constructing, or developing mathematical knowledge via problem solving activities continues to be an important goal in curriculum proposals and a central theme in research programs around the world. However, a salient feature of the group contributions is that there are multiple ways and a variety of interpretations of what a problem solving approach to learn mathematics entails, and ways to frame and implement curriculum proposals. To analyze and reflect on common aspects around problem solving approaches we must construct and activate an international community that continuously shares research programs and discusses problem-solving developments. This community must include active researchers whose academic agenda involves both theoretical and practicing themes in problem solving. And teachers who show clear interest in implementing problem solving approaches in their classrooms are key elements since they look for ideas to consistently frame their practices around problem solving activities. In particular, teachers' discussions focus on demanding actions and directions that will help reduce efficiently a long list of contents and to concentrate on problem solving activities to study key concepts deeply. What fundamental mathematical ideas and processes should be central in curriculum proposals that promote problem-solving approaches?

Another teachers' interest is to address the role of students' international assessments (PISA, TIMMS) in problem solving approaches. That is, to discuss the extent to which the mathematics and ways of reasoning involved in those international assessments is consistent with problem solving approaches. Another important issue that emerged during the group session is the role played by the use of different forms of digital technology in fostering learners' development of mathematical problem solving experiences. It was recognized that there is little information on the type of mathematical reasoning that students construct as a result of using several technologies, and how that reasoning expand or complement paper and pencil approaches. There was a consensus that it is urgent to include in the research and practicing agenda the extent to which theoretical and conceptual frameworks used in problem solving needs to be adjusted in order to explain and foster the students' development of mathematical learning in problem solving scenarios that enhance the systematic use of digital technology.

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Visualization in the Teaching and Learning of Mathematics

Gert Kadunz and Michal Yerushalmy

Report

The history of visualization within mathematics education is a long one. Since the beginning of the 1980s mathematics educators are interested in the practical challenges of teaching visualization, in visualization of mathematics as exhibits in school or aligned with educational psychology and are looking for theoretical frameworks.[1] Recall the earlier text of Norma Presmeg (cf. Presmeg 1986, 1994, 1997), Theodore Eisenberg's widely recognized paper "On the understanding the reluctance to visualize" (Eisenberg, 1994) and more recent analysis of visualization in mathematics education can be found in Arcavi (2003) or David (2012). Regardless of their focus these papers nearly all offer a common picture for which a mathematician's success owes a considerable amount to visualization skills (Heintz 2001). On the other hand the history of mathematics shows visualization to have been cut back and even avoided to a certain extent. In the time of Leonhard Euler the visual was also used as a means for proving or establishing the existence of a mathematical object, whereas the mathematicians of the 19th and 20th century

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[1] E.g. Jerome Bruner and his view on the use of images, Jean Piaget's learning theory or George Lakoff and his view on metaphors.

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reduced the use of visualization for gaining new ideas when solving problems. Heuristics was the task of visualization. We suspect that this gap between the two trends was one reason why dealing with visualization became a significant topic for researchers in mathematics education.

Beyond our specific domain, for the last two decades we have seen a growing interest in the use of images as a general cultural change. It was Thomas Mitchel's dictum that the linguistic turn is followed now by a "pictorial turn" (Mitchel 1994) or Gottfried Boehms (Boehm 1994) "iconic turn". Their concentration on visualization in cultural sciences is based on their interest in the field of visual arts and it is still increasing (Bachmann-Medick 2009). Other technology-enabled visualization developments such as medical imaging, which have introduced sophisticated methods for reconstructing and manipulating images, changed the public and scientific conventions in regard to what formerly was invisible. As happened with modern telescopes which allow us to see nearly infinite distant objects or microscopes which bring the infinitely small to our eye structures become visible and with this kind of visibility they become a part of the scientific debate. Visualization technology causes new paradigms to be developed as structures that could only speculated about are now subject of scientific debate. We may say that their ontological status has changed and in that regard images became a major epistemological factor.

Such new developments, caused substantial endeavour within cultural science into investigating the use of images from different perspectives. Mitchell (1987), Arnheim (1969) or Hessler and Mersch (2009) are examples. The introduction to "Logik des Bildlichen" (Hessler 2009), which we can translate as "The Logic of the Pictorial", focusses on the meaning of visual thinking. In this chapter they formulate several relevant questions on visualization which should be answered by a science of images. Among these questions we read: epistemology and images, the order of demonstrating or how to make thinking visible.

When we consider these short deliberations then we can recognize two positions. We have a long tradition of visualization within mathematics education which is based and supported by practical and theoretical practices. At the same time there are several recent developments within cultural science concerning visualization. Hence there is a need to find means of transmission and terms that would support the exchange of ideas and research questions between cultural science and mathematics education. A theory-based example of such means of transmission is relevant to a topic that our group explored in regard to the relevancy of the semiotic system. Here we mention the semiotics of Charles S. Peirce and more precisely, his idea of diagrammatic thinking which became a tool for investigating mathematical activities (Dörfler 2005; Hoffmann 2005).

The presentation of the visualization group at ICME12 can now be seen as a realization of the above mentioned views on visualization that reflect the diversity of challenges of visualization within mathematics education. Among these presentations we find theoretical deliberations concentrating on visual semiotics, presentations central to mathematics education visualization and curriculum attempting to use technology to bridge the gap between mathematicians and

mathematics education views, presentations concentrated on the use of new software and newer hardware to enhance visualization and on what might develop into new paradigm of the visualization science using brain imaging technology attempting to make the invisible visible. In the next few paragraphs we attempt to sketch the group work with illustrations from the many papers[2] presented.

As the first example we refer to Christoph Schreiber presenting his view on Peirce's semiotics "Semiotic Analysis of Collective Chat-Based Problem-Solving Processes". Schreiber illustrated the development of 'Semiotic Process Cards' based upon Charles Sanders Peirce's triadic sign relation. These cards were used as instruments for analyzing mathematical chat sessions. Within a certain teaching situation called 'Math Chat', students were asked to solve mathematics' problems while being restricted to the use of visible inscriptions only. The characteristics of this experimental setting was that pupils were required to document all their attempts at solving mathematical problems as visual inscriptions in written and graphical form. To develop a suitable instrument Schreiber combined an interactionist approach together with Perice's semiotic perspective. As a result Schreiber was able to describe the

Mathias Hattermann's text "Visualization—the Key Element for Expanding Geometrical Ideas to the 3D-Case" is an example of the group discussion in regard to the visual qualities of design of learning with technological tools. In his Hattermann described the activities of students at university level when using software for 3D-geometry (Cabri 3D). To do so he started with the presentation of two geometric constructions from plane geometry. Hattermann asked how do basic ideas in the context of plane geometry can foster or hinder similar constructions of 3D-geometry? It is the intimate relation between the tool used and the visible geometric diagrams or in other words the *instrumental genesis* of the software and the process of geometrical construction which is in the core of Hattermann's answer to his question. In this respect an experimental approach using the drag mode in 3D can help to find answers to describe the finding of a correct solution. The instrumental genesis of the utilized tool must be accomplished so that mental schemes can be used to extend basic ideas to the 3D-case.

The design and qualities of software was one component of the "Visual Math" curriculum design story that Michal Yerushalmy presented. The challenge was to establish technology-based setting that would motivate algebra students to argue, refute, and revise conjectures, and to study whether prominent visualization habits of mathematical reasoning can become part of the routine pedagogy of school mathematics. Beyond software Yerushalmy described why did the design of an organizational map was a major challenge in finding out how known algebra tasks may be redesigned into a sequence emphasizing quasi-empirical process of reasoning. The museum view was a leading image in the design of the VisualMath interactive eBooks in algebra, functions and calculus. Based on theoretical

[2] All papers presented within TSG 16 can be found at http://www.icme12.org/sub/tsg/tsg_last_view.asp?tsg_param=16.

framework of interactive diagrams that is based on visual-semiotic analysis, Yerushalmy design Interactive Diagrams that provide opportunities for the interactive text to present the curriculum's ideas to be the subject of the reader's inquiry.

Roza Leikin's "From a Visual to Symbolic Object in Algebra and Geometry: ERP[3] Study with Mathematically Excelling Male Adolescents" is in a sense the literal realization of our aforementioned hint "how to make the invisible visible". Leikin and coauthors performed a comparative analysis of brain activity associated with transition from visual objects to symbolic objects in algebra and geometry. The goal of this study was to examine differences in ERPs between gifted and non-gifted excelling in mathematics adolescents while solving mathematical tasks in algebra and geometry. One finding regarding the giftedness effect was that, relative to gifted participants, non-gifted participants produced greater brain activity. This finding is consistent with the neural efficiency hypothesis of intelligence, stating that brighter individuals display lower brain activation while performing cognitive tasks. Another finding indicates a significantly higher brain activity connected to geometry test compared to algebra test. Hence Leikin and assumes that geometric tasks increase the participants' working memory load by keeping the visual geometric object in working memory until the problem is solved.

In addition to the aforementioned view on the relation of the visual and mathematics these examples reflect a fruitful diversity of visualization too. In this respect visualization appears to be a vivid part of research within mathematics education.

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[3] An event-related potential (ERP) is the measured brain response that is the direct result of a specific sensory, cognitive, or motor event.

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Mathematical Applications and Modelling in the Teaching and Learning of Mathematics

Jill Brown and Toshikazu Ikeda

Introduction

Applications and modelling have been an important theme in mathematics education during the last 40 years; in particular, through ICMEs regular working/topic groups and lectures on applications and modelling, and the series of International Community on the Teaching of Mathematical Modelling and Applications (ICTMA) conferences, held biennially since 1983. Relations between the real world and mathematics are particularly topical. One reason for learning mathematics is to understand and make sense of the world. The mathematics education community was invited to submit proposals addressing one of six themes and related issues. The focus could be at any level of education including teacher education and the work of mathematicians in the field. It is not surprising therefore that this TSG attracted much attention, with 44 papers submitted. Papers were reviewed by two reviewers. Thirty-six papers were accepted for presentation, from 17 countries (Australia, Austria, Brazil, China, Cyprus, Germany, Israel, Japan, Korea, Mexico, Portugal, Singapore, South Africa, Sweden, Switzerland, UK, USA) and included several teacher authors. Authors received feedback from the co-chairs, and were given time to revise papers in response to this. Also 24 posters related to this TSG (from 10 countries) were presented. Accepted papers were assembled into groups

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for summary, presentation, and discussion. Given the large number of papers, two concurrent sessions were held with participants together initially and for the final discussion. Given space constraints, only presenting authors are named.

Goals and Curriculum

Paraic Treacy presented *The role of mathematical applications in the integration of mathematics and science*, using the authentic integration triangle to argue how secondary students in Ireland can be supported to apply their mathematical knowledge to authentic tasks particularly in science contexts. Also looking at secondary school curriculum, Karen Norwood discussed *Mathematics instruction using decision science and engineering tools (MINDSET): A multi-step problem solving and modelling course for high school students.* She reported on the development and implementations of a year long US curriculum using a problem solving modelling approach. Xiaoli Lu presented a *Comparative study on mathematics applications in mathematics textbooks* where selected current texts from China and the US were scrutinised for mathematical applications. They report, disappointingly 'most examples in textbooks are traditional mathematical problems without real world contexts'. Jean-Luc Dorier's report on *Modelling: a federating theme in the new curriculum for mathematics and sciences in Geneva compulsory education (age 4-15)* outlined a new curriculum with modelling as a central theme. However, the definition of modelling was modified from that of Niss, Blum and Galbraith (2007); so rather than modelling involving the extra-mathematical and mathematical domains, although two domains are required, the real world is not an essential one of these.

Jussara Araujo presented *Critical construction of mathematical models: An experience on the division of financial resources,* reporting on graduate mathematics education students engagement in a critical mathematical modelling task where 'fair criteria' had to be determined to allocate money. The task raised awareness amongst the participants of the importance of modelling. Jung-Ha An reported on *Developing mathematical modelling curriculum using difference equations.* Examples were shown to demonstrate the use of difference equations in the modelling process in a general mathematics education course. Also at tertiary, *Mathematical experiments: A new-designed course for non-mathematical undergraduates in Chinese universities* was the focus of Jinxing Xie who shared experiences in designing and teaching courses, for non-mathematics students, on applied mathematics through experiments, modelling and software use.

Teaching Material, Pedagogy, and Technology

At the primary level, Nicholas Mousoulides presented *Modelling as a bridge between real world problems and school mathematics*. He argued for a modelling approach, using engineering MEAs, as a rich source of situations that build on and extend students' existing mathematical learning. Takashi Kawakami presented *Necessity for modelling teaching corresponding to diversities: Experimental lessons based on dual modelling cycle framework for the 5th grade pupils*. He reported on a teaching experiment with students working on two related tasks. Focussed on teachers of Year 8–9 students, Janeen Lamb presented *Planning for building models of situations: What is involved*? Data from 8 participants in a project aimed at enhancing teachers' instructional practices were analysed. After completing a modelling activity using an applet, teachers planned how to implement the task in their classrooms. Two studies focused on Year 12 Japanese students: Masahiro Takizawa presented *Colors and Mathematics,* illustrating how the colour of an image can be used to teach functions and transformations, by adopting a modelling approach. The paper presents a teaching experiment with Year 12 students, using the 'Colors' software. Tetsushi Kawasaki presented *A study of mathematical modelling on Year 12 students' function education,* reporting the use of modelling in promoting the teaching and learning of two variable functions. The author reports results of a teaching experiment with 15 students.

Issic Leung presented *The effect of changing dimensions in illustrative examples in enhancing the modelling process,* arguing for a greater emphasis on illustrative examples (e.g., a sketch or diagram). Making greater sense of what is represented should subsequently support mathematical modelling. Also taking a theoretical stance, Vince Geiger presented, *On considering alternative frameworks for examining modelling and application activity: The role of texts and digital tools in the process of mathematical modelling,* discussing several modelling cycles and frameworks used in either engaging in modelling or by researchers in the field. He argued that models for teaching and learning can be applied to modelling situations. His focus is on the interplay between task, teacher, students and tools.

Experimental Research

Irit Peled presented *More than modelling skills: a task sequence that also promotes children's meta knowledge of modelling,* reporting on the development of meta-knowledge of modelling by Year 5-6 students as they worked on 10 tasks. Meta-knowledge included different ways of mathematising a given problem and hence different models for a single situation can be used appropriately. Focused on Year 5 students Maike Hagena and Rita Borromeo Ferri presented, *How do measurement sense and modelling competency influence each other? An intervention study about German middle class students dealing with length and weight.* Susanne Grunewald

presented *Acquirement of modelling competencies: First results of an empirical comparison of the effectiveness of two approaches to the development of (metacognitive) modelling competencies of students,* reporting use of modelling activities in measurement contexts with Year 8 students. Stanislaw Schukajlow and Andre Krug presented *Treating multiple solutions in the classroom and their influence on students' achievements and the affect–The preliminary results of the quasi-empirical study,* comparing Year 9 students' work on 'Pythagoras tasks' where no assumptions were required to those where different assumptions and hence multiple solutions were possible, hypothesising the latter leads to better achievement (modelling and intra-mathematical).

Jin Hyeong Park reported on *Conceptual understanding of mathematical knowledge through mathematical modelling in a spreadsheet environment.* Park sees modelling as representing real phenomena mathematically in order to understand the real world reporting a case study of 15 gifted Year 8 students engaged in an Iced Coffee Task. Findings included development of conceptual calculus understanding and ability to mathematise from their models back to the real world. Also focussing on spreadsheet use, Manfred Borovcnik reported *Applications of probability: The Limerick experiments* that is, responses of probability workshops participants (inservice secondary teachers), arguing that probability is best taught from a modelling and applications perspective, particularly where technology is used. Here any situation in a classroom is considered as being 'real world'.

Xueying Ji presented *A quasi-experimental study of high school students' mathematics modelling competence*, reporting modelling competence of Year 10–11 students in China. She found students did not realise the importance of validating their results or critically assessing their models. Milton Rosa presented *Ethnomodelling: A research concept on mathematical modelling,* arguing the application of techniques in ethnomathematics along with the tools of modelling allows us to see a different reality. Further, research should be from an etic and an emic perspective.

Assessment, Teacher Education, and Obstacles

Peter Frejd presented *Alternative modes of modelling assessment: A literature review,* reporting different assessment methods (i.e., written tests, projects) and viewpoints (atomic or holistic). Xenia-Rosemarie Reit and Matthias Ludwig's paper, *A cross-section study about modelling task solutions*, reported a study where 337 solutions to the *Restringing a tennis racket task* were analysed. Four main solution approaches were identified. Differences were found in terms of approach taken and progress on the solution path. Kaino Luckson presented *The nature of modelling activities and abilities of undergraduate students: some reflections on students' mathematics portfolios,* focusing on modelling tasks undertaken by pre-service teacher education students via distance education. Michael Besser reported on *Competency-oriented written feedback in every-day mathematics teaching: How to report on students'*

solutions of modelling tasks and how to assess the quality of these reports? This study looked at teacher feedback in the context of technical and modelling tasks, considering strengths/weaknesses of specialized written competency based feedback.

In the field of pre-service teacher education, Thomas Lingefjard presented *Learning mathematics through mathematical modelling,* arguing that by developing modelling tasks and then engaging in teaching scenarios conceptual understanding occurs. In addition, students came to understand that technology changes what is possible in developing modelling tasks. Dawn Ng presented, *Activating teacher critical moments through reflection on mathematical modelling facilitation* where the focus was on the teacher's role and in particular, on the teacher interpretation of student ideas and interventions. The interplay between listening and questioning was critical. Also focused on the role of the teacher was Peter Stender on *Modelling in mathematics education development of forms of intervention and their placement in the teacher education* and Dominik Leiss *Adaptive teacher interventions in mathematical modelling*. Both report studies where the balance between student autonomy and teacher interventions was critical.

A Final Word

There are many interpretations of the terms *mathematical modelling* and *applications*. Whilst diversity is desirable, it is helpful to have a common basis for our interpretations. TSG discussion contributes to a shared understanding and the majority of teachers and researchers, see the *real world* as a critical and essential component of modelling and applications. Following Niss et al. (2007), both mathematical modelling and applications are seen as connecting the mathematical world and the real world. These two worlds are distinct, with the later "describing the world outside mathematics" (p. 3). It is also important to distinguish between modelling and applications. The former begins in the real world and requires a modeller to mathematise the situation, that is, to translate the problem situation into a mathematical situation. In an application, this mathematising has already been done for the solver who works in the mathematical world.

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Analysis of Uses of Technology in the Teaching of Mathematics

Morten Misfeldt and Wei-Chi Yang

Overviews

This Topic Study Group aimed at providing a forum to discuss the current state of art of the presence of technology in diverse aspects of teaching mathematics conveying a deep analysis of its implications to the future. Technology was understood in a broad sense, encompassing the computers of all types including the hand-held technology, the software of all types, and the technology of communication that includes the electronic board and the Internet. The discussions served as opportunity for all interested in the use of technology in education environment, to understand its diverse aspects and to share the creative and outstanding contributions, with critical analysis of the different uses.

The Topic Study Group had 42 contributions and more than 80 participants. The topics addressed were diverse but evolve around the use of technology in the classroom practice, design and use of digital teaching materials, Technology in teacher education, Distance education and the use of learning management systems. The use of technology in the teaching of mathematics is an expanding and diverse field, and in the following we will summarize the status and consensus that became apparent through the work in the Topic Study Group. One way to gain an overview of the image of the field presented in the topic study group is to look at the different

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technologies involved. The technologies adopted and described in the TSG did mainly fall into 6 categories (1) handheld and pc based computer algebra systems, (2) dynamic geometry systems, (3) learning management systems and internet access, (4) domain specific visualizations and manipulatives, (5) video streaming, and (6) touch technology such as ipads and smartboards.

Another way to gain such overview is to present the discussions and concerns that were prevalent in the discussion and contributions. These concerns relate to (a) an increase in efficiency of mathematics instruction with the aid of technology—including technological support for development of specific mathematical concepts and competencies, (b) teacher training and teacher practice with technology, (c) the use of technology to support motivation and recruitment to mathematics, and (d) technological support for teaching processes—such as digital task assignment and marking.

Technologies Used for Mathematics Instruction

Computer Algebra Systems, Dynamic Geometry Systems and spreadsheets has been a part of mathematics instruction for decades, yet the mediations of the technologies as well as the research problems addressed by the community is still developing. The presentations in the TSG showed that these technologies are to some extend adopted in the mathematics education practices. The contributions relating to these technologies hence addressed issues relating to teachers adoption, the possibility to deploy such technologies for supporting low achievers, the teaching of specific mathematical concepts in new ways with technology, and the integration of these technologies into learning management systems. Learning management systems signifies a class of systems that is used to support and augment teaching practices. In relation to these systems a number of initiatives to augment their mathematical capabilities were presented. Apart from the integration of Computer Algebra Systems and Dynamic Geometry Software into learning management systems, the work related to the use of such systems related to the construction of multimodal learning environments including video and interactive manipulatives, within learning management systems. Technology that allows for the development of interactive visualizations and for sharing content were presented for several topics and educational levels ranging from primary school to university. Online task environments for students to train their skills with mathematical tasks were also presented. Online streaming of video was presented both as stand-alone and as a part of an online environment for teaching of mathematics. One project applying tablet pcs and interactive whiteboards was also presented. Hence a wide range of the applicable educational technologies where present in the Topic Study Group.

Problems Addressed in the TSG

The main problem addressed in the contributions was the potentials of using technology to enhance teaching of mathematics to become a more efficient enterprise. This problem was addressed in a multitude of ways in the topic study group. Interventions aiming at using technology (typically CAS and DGS) to teach mathematical topics in new ways where presented in the groups. It is difficult to summarize the role and influence of technology across these interventions since many factors other than the use of technology influences such interventions. Teacher training and teacher practice with technology conducted was addressed in several of the contributions. One motivation for a specific attendance to this area is that the teachers' choices and practices are, in many ways determining for the success of technology integration in the teaching of mathematics. Motivation and recruitment is an important theme underlying several of the contributions to the topic study group. New interactive illustrations or video presentations might not only make it possible for more students to grasp the abstract mathematical concepts, it might also make mathematics more appealing to larger groups of students.

Apart from addressing concerns, some of the reports in the Topic Study Group also demonstrated new technological developments, addressing the aim of improving mathematics instruction. Technology can automate aspects of the process of teaching mathematics, such as assigning and marking tasks. This is a development with many possible advantages and represents an area were the technological development currently is quite rapid. Another area where new technological developments were presented was the integration of mathematical tools such as CAS and DGS, into web 2.0 internet technology, in a way that supports collaboration and distance education.

A special issue of the electronic Journal of Mathematics and Technology is under preparation. This issue will include papers from the Topic Study Group.

Analysis of Uses of Technology in the Learning of Mathematics

Marcelo C. Borba and Hans-Georg Weigand

Introduction

In ICME12, the role of technology in mathematics education was divided into two distinct study groups: Analysis of uses of technology in the teaching (TSG 18) and learning (TSG 19) of mathematics. Of course, these two aspects of mathematics education are closely intertwined, but we tried to concentrate the TSG 19 discussions around the aspect of LEARNING with ICT (Information and Communications Technology).

The TSG 19 especially addressed the following issues in the learning of mathematics:

- the design of digital technology
- the design of learning environments
- large-scale and long-standing digital technology implementation projects
- assessing mathematics learning with and through digital technologies
- the interaction between ICT and learners of mathematics
- connectivity of ICT
- theoretical and empirical models for learning with ICT
- the implementation of curricula

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Outline of Contributions

All submitted papers were reviewed by three reviewers and 33 papers and one poster were finally accepted. For presentation, papers were grouped into four groups:

- Group A: E-learning, Interactive Textbooks, Games, Mobile Applications
- Group B: Theoretical Aspects
- Group C: Dynamic Geometry Systems (DGS), Calculators, CAS
- Group D: Topics in Mathematics

Each of the four 90-min sessions was devoted to one of these four groups of papers. The time available did not allow for formal presentations of every paper by their authors. Two papers from each group were selected for presentations by the authors. The remaining papers in that group were summarized by a member of the organizing committee, with opportunities for comments by the authors and for discussion of the papers by all participants. The structure for each 90-min session included some brief opening remarks by the co-chairs of the committee, followed by a 30-min period for summary and discussion of those papers not presented later in the session. Following this summary discussion, each of the two selected papers were presented by their authors (15 min each, with 10 min for presentation and 5 min for discussion). After the individual paper presentations, participants engaged in 15 min of roundtable discussions focused around questions of emergent issues raised by the papers considered in that session. At the conclusion of each session, the TSG 19 co-chairs had made some brief closing remarks.

Group A: E-learning, Interactive Textbooks, Games, Mobile Applications

- Gerry Stahl (College of Information Science, Drexel University, Philadelphia, USA): Designing a Learning Environment to Promote Math Discourse
- Robyn Jorgensen (Griffith University—Australia), Tim Lowrie (Charles Sturt University—Australia): Digital Games and Mathematical Learning: A summary paper

Gerry Stahl emphasized the fact that more and more teachers and students were learning online—with distance education, online masters programs, home schooling, online high schools, etc.—which makes the incorporation of virtual collaborative learning environments a natural trend. He presented a virtual GeoGebra learning environment that integrates synchronous and asynchronous media with an innovative multi-user version of a dynamic math visualization and exploration toolbox.

Jorgensen and Lowrie presented a summary of a three-year project that explored the possibilities of digital games to enhance mathematical learning. They especially found that using games in classrooms might have much more benefits than just learning mathematics.

Group B: Theoretical Aspects

- Abramovich Sergei (*State University of New York at Potsdam, USA*), Eun Kyeong Cho (*University of New Hampshire, USA*): Pre-teachers' learning of mathematics through technology-enabled problem posing
- Barbara Schmidt-Thieme (University of Hildesheim Germany), Hans-Georg Weigand (University of Wuerzburg, Germany): Choosing adequate Digital Representations,

Abramovich and Cho considered the potential of new technologies to turn a routine arithmetical problem into a challenging mathematical investigation. The authors suggested that an important didactic task for teachers will be to decide if technology-enabled problem posing results in a contextually, numerically, and pedagogically coherent problem. This influences the choice of the adequate software.

Schmidt-Thieme and Weigand presented examples of students' working with representations and posed some main future research questions concerning the use of representations in a technology-based environment, e.g.: Which criteria characterize an adequate representation of a problem's solution? Which different levels of argumentation, reasoning and proof are related to a special representation? Which criteria characterize a good (in the sense of giving some feedback about learners' competencies) documentation of a solution of a problem?

Group C: Dynamic Geometry Systems (DGS), Calculators, CAS

- Arthur B. Powell, Loretta Dicker (Rutgers University, USA): Toward Collaborative Learning with Dynamic Geometry Environments
- Thomas Lingefjärd, Jonaki Ghosh, Aaloka Kanhere (Technology Working Group of the Indo Swedish Initiative in Mathematics Education): Students Solving Investigatory Problems with GeoGebra—A Study of Students' Work in India and Sweden,

Powell and Dicker presented a model of collaborative, online learning with a dynamic geometry environment that supports collaboration around mathematical problem solving and development of significant mathematical discourse. The

authors especially intend to motivate in-service secondary teachers in designing curricular units that develop students' significant mathematical discourse as they develop geometric ideas.

Lingefjärd, Ghosh and Kanhere started with the hypothesis that the use of technology in mathematics instruction might lead from an experimental mathematics, that is, verification and conjecturing, to theoretical mathematics, that is, formal abstract concepts and proofs. The authors had done a parallel experimental study in Sweden and India using a dynamical geometry environment and getting quite similar results concerning the working styles of students in these two countries.

Group D: Topics in Mathematics

- Christian Bokhove (St. Michaël College, Zaandam, the Netherlands/Freudenthal Institute, Utrecht University, Utrecht, the Netherlands), Paul Drijvers (Freudenthal Institute, Utrecht University, Utrecht, the Netherlands): Effects Of A Digital Intervention On The Development Of Algebraic Expertise
- Jens Jesberg, Matthias Ludwig (Goethe University Frankfurt, Germany): MathCityMap—Make mathematical experiences in out-of-school activities using mobile technology

Bokhove and Drijvers especially wanted to answer the question about the effect of an intervention, consisting especially of diagnostic digital modules, on the development of algebraic expertise, including both procedural skills and symbol sense. They observed "a large effect on improving algebraic expertise" after an intervention of just 5 h.

Jesberg and Ludwig presented a "MathCityMap-project", which is based on a GPS technology. High school students experienced mathematics at real locations and in real situations within out-of-school activities, with the help of GPS-enabled smartphones and special math problems.

Conclusions

More than thirty years have passed since the first ICMI study group on technology. Papers presented in this TSG show that the work with technologies can present new trends even though one can no longer refer to digital technologies as "new technologies". Digital tablets and devices that increasingly enhance the possible interactions between humans and technology were presented as means for transforming the way students can know. Many of these devices imply changes in curriculum and challenge the structure of time in school. In other words, if they are to be used in school, students will either have to be outside class using mobile

technology, or in class using them for longer periods of time. TSG 19 was diverse enough that many papers also proposed how technology can be used now, without many changes in the way school is organized. “Geogebra” is one of those key applications used at this conference. The free software seems to have found many different followers in different countries and it has been used in different manners. Some have incorporated it into online learning environments, while others are developing ways of annotating the screen of Geogebra.

Last but not least, findings of new technological developments and of research results were discussed in small groups, overcoming language barriers. The situation is the same in mathematics classrooms all over the world. Apart from special and valuable cultural divergence and distinctions new technologies reveal the same or at least similar problems in mathematics learning all over the world and they may be a catalyst to forward important developments in mathematics classroom activity.

The Role of History of Mathematics in Mathematics Education

Renaud Chorlay and Wann-Sheng Horng

Report

At the ICME 2012 Conference, *history of mathematics* (HM) *in maths education* was specifically discussed in several contexts: one of the 37 *Topic Study Groups* (chaired by W.S. Horng and R. Chorlay); one discussion group on "Uses of History of Mathematics in School (Pupils aged 6–13)" (organised by B. Smestad); one regular lecture on "History, Application, and Philosophy of Mathematics in Mathematics Education: Accessing and Assessing Student's Overview & Judgment" (by U. Jankvist); and one general presentation of The HPM international study group, among the organizations affiliated with the ICMI. A parallel TSG dealt with *the history of mathematics teaching and learning* (chaired by K. Bjarnadottir and F. Furinghetti).

Eleven talks and fourteen posters were presented in the context of TSG 20, with participants from (nearly) all continents; unfortunately, the African continent was not represented. The TSG was a great success if success is to be measured by attendance.

Being of a multi-faceted nature, the topic was addressed from a great variety of viewpoints, which testifies to the richness of our field. Our goal here is not to

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summarize the talks (which are still available on-line at http://www.icme12.org/) but to stress this variety of viewpoints and research perspectives.

Research in the history of mathematics was represented by A. Cauty's talk on Aztec calendars, providing the rest of the community with fresh material for future, more teaching-oriented work. At the other end of the spectrum, several innovative teaching or training experiments were presented and discussed: a course for undergraduate students, with a focus on the role of mathematics in European culture (J. Wanko); an undergraduate course on propositional logic and the meaning of "if-then" statements, emphasizing student work on original sources (J. Lodder); a course designed for newly qualified teachers, with an emphasis on the role of HM as a means to foster mathematical content knowledge (S. Lawrence); a course on the history of mathematics for pre-service teachers in Norway, with a focus on the interactions between historical content knowledge, image of mathematics, and attitude toward the inclusion of HM in teaching (B. Smestad).

Finding the right tools (be they conceptual, or quantitative) to describe, analyze and assess teaching practices is another endeavour that calls for further research. These questions are by no means specific to the HPM community, and it is well-worth investigating the extent to which shared tools are relevant in an HPM context. Along this line of research, M. Alpaslan presented his on-going doctoral work on the assessment of a pre-service teacher-training course in HM in Turkey, with a view to improving its design in a context of institutional reform. U. Jankvist presented a joint work (with R. Mosvold, J. Fauskanger, and A. Jakobsen) on the MKT framework (Mathematical Knowledge for Teaching), and argued for its usefulness both as an analytical tool and as a means of communication with the math-education community at large.

Four case-studies were presented, which used specific historical texts to address didactical/epistemological research questions. The role of visualization in proofs was studied on the base of Archimedes' "mechanical proof" of the theorem on the volume of the sphere (M. del Carmen Bonilla); CABRI 3D was used as a visualization tool. S. Xuhua argued that several justifications for algorithms in the multiplicative theory of fractions that can be found in the Chinese classic *The Nine Chapters* could improve students' understanding of the standard rules, and help fight well-known systematic errors. T. Kjeldsen reported on an experiment conducted at high-school level, in which students were asked to make sense and compare two historical texts bearing on the notion of function (Euler, Dirichlet); among other effects, this unusual task was shown to help make "meta discursive rules" more explicit. Finally, A. Michel-Pajus presented a collection of algorithmic texts—some well-known, some excitingly new—and studied them from an epistemological and comparative perspective; the algorithms were studied both in terms of expression (algorithmic *texts*, in a semiotic and instrumental context), and justification.

It should be stressed that in the ICME context, the TSG on HM in maths-education attracts many newcomers to the field of HPM, thus challenging members of the HPM community to make their "common culture" and their quality requirements more explicit. For instance, the fact that most of us stress the

importance of the use of *original* sources may have come as a surprise to some; even without considering use in the classroom, the fact that original sources *are* available (availability being highly dependent on language) is not always so well known. When original sources are considered, working with them does require some know-how. We hope this TSG was instrumental in raising awareness on these aspects; we were pleased to see that many participants, including newcomers, could attend the HPM meeting in Daejeon (16–20 July 2012).

The chairpersons would very much like to thank all those who helped organize this TSG, in particular the members of the "team": Hyewon Chang, Kathy Clark, Abdellah El Idrissi, and Manfred Kronfellner; and, Evelyne Barbin, who acted as liaison with the IPC.

Research on Classroom Practice

Yeping Li and Hélia Oliveira

Introduction

Classroom practice, as a process, involves multiple agents and their interactions within the classroom as a system. The process can be manifested in diverse formats and structures, and its effectiveness can be influenced by numerous factors both internal and external to the classroom. Research on (mathematics) classroom practice can thus take different perspectives, and much remains to be examined and understood as we all try to improve mathematics teaching and learning through classroom practice.

Although it has long been recognized that research on classroom practice is important, large-scale systematic research on classroom practice in school mathematics is a relatively new endeavour. In fact, this Topic Study Group is only the second time in the ICME history to take a primary focus on classroom practice. As the quality of classroom instruction is a key to students' mathematics learning, this Topic Study Group focuses on finding ways for understanding, assessing, and improving the quality of classroom practice.

The entire organizing team worked together before the congress in planning and organizing TSG 21. The TSG 21 was well attended in all four 90-min sessions,

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which indicates strong interest in this topic by congress delegates. This report provides an overview of the aim and focus of TSG21 and a summary of the discussion that occurred throughout the sessions.

Aims, Focuses, and Themes

As set by the organization team, the general aim of TSG 21 was, in the international mathematics education community, to elevate people's understanding of the importance, specific nature, and challenges in research on classroom practice, to promote exchanges and collaborations in identifying and examining high-quality practices in classroom instruction across different education systems, and to enhance the quality of research and classroom practice. More specifically, through its official program during the congress and other activities (including those before and after the congress), TSG 21 was intended to provide an international platform for all interested parties (e.g., mathematics educators, mathematics teachers, educational researchers, etc.) to disseminate findings from their research on classroom practice with the use of various theoretical perspectives and methodologies, and to exchange ideas about mathematics classroom research, development, and evaluation.

The main focus of TSG 21 was a discussion of research related to mathematics classroom practice, which includes activities of learning and teaching processes located within the classroom as a system. This requires a study of the interactions among the mathematical content to be taught and learned, the instructional practices of the teacher, and the work and experiences of the students. In the interaction processes, mathematical content is contextualized through situations, the teacher plays an important instructional role drawing on his/her knowledge, and the students involve themselves in the learning processes. It is important to understand through research the nature and extent of these interactions, the complexity of the didactic system, the roles of the teacher and students in the interaction processes when the mathematical content is taught and learned, and the complexity of the activities in the classroom.

The 39 accepted papers were assembled into the following eight themes for presentation and discussion during the congress:

- Theme 1: Theoretical and methodological considerations
- Theme 2: Instructional context, reflection, and improvement
- Theme 3: High-quality instructional practices
- Theme 4: Students' perception, class work, and learning
- Theme 5: Teaching and learning elementary mathematics
- Theme 6: Teachers' questioning and response in classroom instruction
- Theme 7: Instructional design and practice
- Theme 8: Curriculum/task implementation

In addition, there were nine proposals accepted for poster presentations in a separate session organized by the Congress.

Each of the four 90-min sessions (July 10, 11, 13, and 14 in 2012) was devoted to two of these eight themes (4–5 papers for each theme), which were carried out simultaneously in two separate rooms. In the following sections, we briefly summarize the paper presentations and discussions during these sessions.

Session 1 (Theme 1: Theoretical and Methodological Considerations)

Gade adopted a theory/practice approach based on Vygostky for researching classroom practice, with the potential of informing practitioner's inquiry in ongoing classrooms. Morera and Fortuny illustrated the use of an analytical method of classroom episodes as a proposal to develop systematic research on whole-group discussions. Mesa, Lande and Whittemore argued for the need to attend to two dimensions of classroom interaction when describing it, by one study where they, simultaneously, analyzed the complexity of mathematical questions and the interactional moves that the teachers use to encourage student involvement in the lesson. Canavarro, Oliveira and Menezes illustrated the use of an analytical tool for lessons driven by an inquiry-based perspective in the case of one teacher who adopted a four phase model for the lesson structure. Xolo reported one expanded coding scheme that focuses on learning outcomes and teachers' didactic strategies from video recordings of sequences of lessons, intended to capture a greater degree of nuance in classroom practice.

In synthesis, these papers propose new analytical tools to investigate the classroom practice that contribute to having a better picture of what is happening in the classroom, showing a deep concern for acknowledging the teachers' work.

Session 1 (Theme 2: Instructional Context, Reflection, and Improvement)

Andersson presented a study of disengaged students' identity narratives in the senior secondary years. The instructional context was defined by tasks, situations (tools, activities, participants), school structures, the socio-political context, and the societal context. Olfos and Estrella described the use of a short video rich in potential problem situations to help primary school teachers initiate a lesson on fractions via problem posing. The lesson study approach resulted in the lesson being successively improved as each teacher implemented it. Oliveira, Menezes, and Canavarro reported on a project that created multimedia cases to stimulate reflective analysis of lessons in teacher education. Lee and Kim analysed one teacher's discourse during lessons involving small group work and used the lesson video to stimulate the teacher's reflective thinking towards improvement. Vanegas,

Giménez, and Font i Moll illustrated the use of a two dimensional grid for identifying nine types of democratic mathematical practices in the classroom.

In synthesis, each of the papers presented in this theme reported on attempts to support democratic, equitable and critical classroom practices. The authors also investigated processes of teacher change by supporting teachers' systematic reflection and iterative improvement of their lessons.

Session 2 (Theme 3: High-Quality Instructional Practices)

Zhao and Ma found that lessons taught decades apart had similar content and teacher-student interaction but different types of tasks. Lee reported on the classroom practice of a teacher with high levels of mathematical knowledge for teaching. This study highlights a need for sensitivity in building respectful relationships between the researcher and teacher when classroom observation is also used for teacher evaluation. Zhao examined secondary school teaching practices in China where few teachers have a background in statistics. Focusing on teachers' interpretation of statistical graphs, the study found that teachers had limited understanding of key statistical concepts and gave more emphasis to procedures than conceptual understanding. Lewis, Corey, and Leong compared research from Japan, Singapore, and the US and found similarities in the categories used to define high quality practice. Li asked what could be learned from culturally valued classroom practices in China, and proposed a framework comprising macro pedagogy and micro pedagogy perspectives for understanding classroom instruction.

In synthesis, the papers in this theme proposed a variety of analytical frameworks for observing lessons and evaluating the quality of instruction. But each was concerned with the question of what counts as "high quality" instruction, and whether there are common or different criteria across countries and cultures.

Session 2 (Theme 4: Students' Perception, Class Work, and Learning)

Olteanu presented some results from a longitudinal study whose aim was to provide and develop a repertoire of reliable practices and tools to solve immediate problems in teachers' daily professional lives; namely, to improve students' learning in mathematics. Gao and Tian concluded that the students in the class where an open inquiry to problem solving was adopted were more accurate and succinct, quicker, and more fluent in language than the students in the class that followed a guided inquiry. Yang and Leung found that secondary students generally do not perceive their mathematics classroom environment very favorably. Gender differences were also found. Yau and Mok reported five consecutive lessons that showed that most

students imitated the teacher's examples completely or partly. The authors argued that the strong direct role of teacher might help the students master their mathematical content. Araya, Varas, Giaconi and Foltz analyzed pupil's perceptions about mathematics, math learning and teaching in Chile and Finland. Considering the significant difference between these two countries, results showed surprising similarities connected to prototypical ideas.

Session 3 (Theme 5: Teaching and Learning Elementary Mathematics)

Silvestre and Ponte showed that the teaching/learning experience supports the conjecture that proportional reasoning develops when students explore, solve problems, and work with different representations. Yong, Zanzali, and Jiar showed that by developing a favorable learning environment and through scaffolding the students (low achievers) could progressively adapt themselves to a child-centered approach and begin to think more autonomously. Goos, Geiger, and Dole presented a model of numeracy whose elements comprise mathematical knowledge, dispositions, tools, contexts, and a critical orientation to the use of mathematics, and applied it to analyze changes in one teacher's planning, classroom practice, and personal conceptions of numeracy. Kwon and Thames showed that despite variations in the use of the task and the collective work with students, the work of teaching involves several core features: hearing mathematical reasoning, mathematical needs, and key mathematical concepts; and comparing different solutions and making alternative solutions reasonable. Pinto studied the development of the meaning of multiplication and division of non-negative rational numbers, arguing that problem solving helps students to overcome some difficulties and to understand and to formalize mathematical concepts.

In synthesis, these papers illustrate good practices that draw on the use of powerful mathematical tasks alongside with approaches that promote students' autonomy and critical orientation in solving problems.

Session 3 (Theme 6: Teachers' Questioning and Response in Classroom Instruction)

Lee analyzed the changes in one pre-service teacher's questioning practices, as she starts to give her students the opportunity for explaining and justifying their mathematical ideas. Subramanian illustrated several forms of questioning by one Indian teacher, which she argues is a culture-influenced pedagogy in that country and thereby widely practiced in the classrooms. Fox reported how two teachers who were observed throughout one unit of instruction were able to handle unanticipated questions by posing counter examples or simpler related questions. Sun compared

the questioning practices of two teachers (Chinese and Czech) by observing the video of one lesson from each teacher. The questions posed by the Chinese teacher tended to require only a short answer, in a short period of time, and without the teacher's help. On the contrary, the questions by the Czech teacher were more cognitively demanding, but he provided no scaffolding. Aizikovitsh-Udi, Star, and Clarke presented two case studies demonstrating that good teacher questioning involves more than just good questions.

In synthesis, these papers show the growing interest in the teachers' questioning practices as a consequence of the recognition of its pedagogical value for the students' learning. Some professional cultures seem to value the power of questioning for a long time, but the nature and objectives of the questions the teachers pose differ substantially from setting to setting.

Session 4 (Theme 7: Instructional Design and Practice)

Mogensen shared recent efforts in Denmark to focus on mathematical pedagogical goals and mathematical points in mathematics teaching. Choquet analyzed the practice changes of a primary school teacher resulted from using '*problème souverts*' (open problems). Sekiguchi examined how Japanese mathematics teachers handle multi-dimensions of coherence and coordinate coherence and variation. Japanese mathematics teachers seemed to achieve multidimensional coherence by utilizing a double-anchored process schema, and their deliberate use of variation seemed to facilitate students' reflection. Lin described a general procedure of conceptual variation via either diagram form (more on perceptual knowledge), or verbal/symbolic form (more on rational knowledge). A lesson plan of conceptual variation on the topic of elliptical definition was also given to illustrate how to use the general procedure to design conceptual variation. Varas, Martínez, Fuentealba, Näveri, Ahtee, and Pehkonen presented results from a three-year follow-up Finland–Chile research project that introduced open-ended problem solving activities in third grade classes.

In synthesis, these papers present different perspectives and approaches used in developing and designing classroom instruction, with particular focuses on the use and organization of mathematical ideas/points, open-ended problems, instructional coherence and variation.

Session 4 (Theme 8: Curriculum/Task Implementation)

Huang, Li and Yang reported one study with three primary teachers in the context of the implementation of a new mathematics curriculum, in which the notion of variable was taught. All teachers promoted students' use of numbers and letters to describe realistic problems and explain conclusions, but they provided few opportunities for students to experience the problem-solving process. Moreira and

Campelos discussed the implications of the implementation of a new mathematics curriculum on teachers' practices, focusing on the balance between the collective and individual component of the practice. Grow-Maienza presented the results of one program that promoted the teachers' integration of principles abstracted from a Korean curriculum into the curriculum in use in one elementary school in the USA. Bingolbali and Bingolbali analyzed one teachers' practice concerning the implementation of one task in the classroom, arguing that a low fidelity to the task plan may be an expression of the teacher's flexibility to attend to students' needs.

In synthesis, these papers show that curriculum reforms are fruitful contexts to research the classroom practice, that may provide good opportunities to rethink the professional development of teachers, but that it is also necessary to understand how the intended innovations relate with the collectively and individually established teachers' practices.

Closing Remarks

Among the main points discussed across the four sessions we highlight the following ones:

- The search for what characterizes "high-quality" practices and the frameworks used to evaluate these practices taking into account the cultural and national diversity;
- The evolving classroom practices in many countries that reflect a move from the traditional instruction to innovative ways of teaching, and the demanding teacher's role associated with that transformation;
- The practices of questioning and inquiry-based approaches in different countries and their commonalities and differences;
- The teachers´ practices concerning the work with mathematical tasks, namely their concern about the role played by the contexts, and the students´ dispositions and perspectives concerning mathematics;
- The "Chinese paradox" and other countries' paradoxes concerning the relationship between students' achievement and classroom practice;
- The development of new analytical tools to do research on classroom practice.

Naturally, in such a broad topic as classroom practice, many questions remain to be addressed. The diversity of themes and focuses presented suggests many different perspectives that contributors took on what constitutes "classroom practice", which aspects of classroom practice are to be focused, and how "practice" is conceived using different analytical frameworks. The participants shared a strong interest in continuing the TSG's dynamics, and proposed the possibility of exploring joint projects in different countries and new publications focusing on some of the main themes discussed, and of gathering at other international conferences to do informal meetings to continue to do some work together.

Learning and Cognition in Mathematics

Gaye Williams and Hsin Mei Huang

Aims

Learning and cognition is a classical and very vital area in research on mathematics education. Researchers have published many valuable research findings that have contributed to significant development in this area. The continued efforts of researchers now and in the future will, we hope, lead to extensive 'pay-offs'. Different to many other special and related TSGs, such as teaching and learning of algebra, geometry, measurement, statistics, calculus, reasoning, proving and problem solving, to mention a few, TSG22's participants will contribute a more general focus on learning and cognitive activity, and insights into students' characteristics; their strengths and weaknesses in the process of mathematics learning. The TSG focus can include any teaching and learning contexts: from kindergarten to tertiary level, adult education, and teacher professional development. TSG22 discussions should be balanced between theories and their practical applications in mathematics teaching and learning.

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Focus and Themes

Psychological characteristics of students that influence their inclination to think creatively in mathematics

- Effects of psychological characteristics on students' test performances
- The role of optimism (resilience) in mathematical problem solving

Cognitive processing associated with the creative constructing of knowledge

- What aspects of curriculum development/materials contribute to developing learners' mathematical thinking, mathematical inquiry or mathematical creativity?
- What cognitive processes are associated with autonomous student development of new knowledge and what 'teacher moves' can promote such activity?

Mathematical thinking accompanied by affective elements

- In what ways are cognitive, social, and affective elements connected during the development of new knowledge?
- The nature of affective elements that can accompany creative mathematical thinking.

Social interactions associated with creative mathematical thinking

- What aspects of teaching mathematics (teaching behaviors) contribute to developing mathematical thinking, mathematical inquiry or creativity in mathematics?
- What characteristics of classroom interaction or discourse (students-students; teacher-students) facilitate or contribute to knowing mathematics or developing thinking or inquiry abilities?
- What mathematical problems are there that have good use in the classroom by teachers that contribute towards developing cognition in mathematics?

The nature of mathematical understanding

- Children's interpretation of and performance on national and international math tests
- The rationale behind selecting a wrong answer in multiple-choice items in mathematics assessments.
- Contexts for developing mathematical understanding

Number of Submissions and Attendants

31 papers were reviewed and the following decisions were made: 5 long presentation (16 %), 8 short presentation (26 %), 11 posters (35 %), Overall acceptance rate was 77 %.

The number of attendants at each session was between 30 and 45. Each of the four TSG22 sessions attracted a large audience and this added to the stimulating nature of the discussions. With so many thought provoking contributions, and the differences in perspectives communicated, there was insufficient time to pursue all of the interesting questions and issues that arose. The panel's post-ICME communications with participants and others visiting the ICME TSG22 site illustrate ongoing interest and reflections arising from the work of TSG22.

Schedule of TSG22

Session 1, Tues 10, 10:30-12

Welcome, Overview

Luis Radford (Invited Plenary), Sensuous Cognition: Mathematical thinking as a Body- and Artifact-based Social Practice (30 min)

Round Table 1: In school and out of school mathematics learning (12 min)

- Paper 1, Michaela Regecova & Maria Slavickova, How Students' Everyday Experiences Influence Their Mathematical Thinking
- Paper 2, Rankin Graham, Homework: Pre-calculus Algebra Class
- Paper 3, Kadian M. Callahan, Prospective middle School teachers' generalizing actions (reasoning about algebraic and geometric representations)
- Question/Discussion

Poster Session (Parallel to Round Table 1)

Jorge Soto-Andrade & Pamela Reyes-Santander, Mathematical cognition in young offenders

Shin-Yi Lee (Invited Early Career Researcher), Analysis of "look back" strategies in mathematical problem solving

Hsin-Mei Huang, Children's thinking about measuring areas

Plenary Discussion (4 min)

Session 2: Wednesday, July 11, 10:30-12

Introducing Session

Lianghuo Fan (Invited Plenary), Learning of Algorithms: A Theoretical model with focus on cognitive development (30 min)

Rosa Ma. Garcia & Mariana Saiz (Electronic), Listening to children explain wrong answers

Terezinha Nunes & Peter Bryant, Children's' Understanding of Probabilities

Yasufumi Kuroda & Naoko Okamoto, How can brain activity contribute to understanding of mathematical learning process

Plenasry Discussion (10 min)

Session 3: Friday, July 13, 11-12:30

Introducing Session

Rina Hershkowitz, Tommy Dreyfus, Michal Tabach, Chris Rasmussen, Megan Wawro (Invited Plenary Team) (55 min)

- Hershkowitz, Dreyfus, & Tabach, Exponential growth: Constructing knowledge in the classroom
- Chris Rasmussen, Megan Wawro Documenting collective activity in the classroom
- Michal Tabach, Rina Hershkowitz, Chris Rasmussen, & Tommy Dreyfus, Exponential Growth: Co-ordinating Construction of Knowledge and Documenting Collective Activity in the Classroom
- Question/Discussion

Hong Seek Eng, Lee Ngan Hoe, & Darren Yeo Jian Sheng, Metacognitive approach: Kick- starting problem solving activity

Gaye Williams (Co-chair) Linking confidence, persistence, and optimistic problem solving activity

Plenary Discussion (13 Mins)

Session 4: Saturday, July 14, 10:30-12

Introducing Session

Alan Schoenfeld (Invited) (30 Mins) Social dynamics for supporting creative mathematical thinking and problem solving

RT2a: Promoting creative thinking: international perspectives (12 min)

- Paper 1, Xianwei Yuan Van Harpen: Creativity and problem posing in US and China
- Paper 2, Yeojoo Jin: Problem solving in Korea
- Paper 3, Cristina Frade, Steve Lerman, Luciano Meira, Peter Winbourne: Working with the ZPD to Identify Learning as Participation in Mathematical Practices
- Question/Discussion

RT2b: Developing understandings of complex mathematical ideas (Parallel to RT 2a) (12 min)

- Paper 1, Revathy Parameswaran: Expert mathematicians approach to understanding definitions
- Paper 2, Megan Wawro: Student reasoning about invertible matrix theorem in linear algebra
- Paper 3, Jun Mun Kyeong Semantic and syntactic reasoning on the learning of algebra
- Question/Discussion

Yuka Koizumi & Keiko Hino Social interactions of competent teacher: Stimulating creative thinking

Plenary Discussion (11 Mins)

Where to Now? (20 Mins)

Brief Summary of Outcomes

The ICME TSG22: Learning and Cognition co-chairs Hsin-Mei Huang and Gaye Williams provide a brief overview of what occurred in preparing for and participating in the ICME-12 TSG22 Learning and Cognition. The TSG22 Panel invited five researchers/research team presentations (Luis Radford; Alan Schoenfeld; The Tommy Dreyfus, Rina Hershkowitz, Michal Tabach, Chris Rasmussen, Megan Wawro Team; The Terezinha Nunes, Peter Bryant Team; and Lianghuo Fan), and two Early Career Researchers (Michal Tabach and Shin-Yi Lee) to highlight cutting edge research in this TSG. The announcement of the 2011 ICMI Awards (Hans Freudenthal Award: Luis Radford; Felix Klein Award: Alan Schoenfeld) contributed further to the interest already shown in this TSG. The quality and number of papers submitted through the reviewing process created dilemmas: how could we enable the sharing of the rich contributions proposed? We decided upon round tables presented simultaneously with many short presentations. Researchers rose to the challenge of showcasing their studies succinctly but with sufficient depth to allow others to follow up on their work. The TSG22 Poster Sessions were well attended and contributed further to TSG22 research.

Some of the connections identified between various presentations are now identified. For example, Radford, and Koizumi and Hino, and Schoenfeld focused differently on 'culture'. Radford on cognition as a 'culturally and historically constituted form of creative responding' with 'sensation considered as a substrate of the mind …', Koizumi and Hino on the learning culture set up by the teacher to 'stimulate[s] children's creative mathematical thinking', and Schoenfeld on the development of classroom cultures in which 'the students had internalized the relevant mathematical standards' to become 'accountable to the discipline (as opposed to, or in addition to, accountable to the teacher)' and 'able to speak with mathematical authority'. The Hershkowitz Team adapted existing methodological tools to network theories in studying 'the role played by individuals and groups in the class as well as by the class as a whole, in the knowledge constructing process'. Williams examined psychological influences on processes associated with creative construction of new knowledge during problem solving, and Ngan Hoe Lee's Team, and Shin-Yi Lee examined metacognitive processes associated with problem solving. Mathematical understanding and how it develops was explored in probability (by Nunes and Bryant), and in children's developing understandings of area formulae (by Huang). A 'theoretical model for the learning of algorithm with focus on students' cognitive development' was presented by Fan. Soto-Andrade and Reyes-Santander illustrated creative mathematical activity amongst young offenders thus identifying a fruitful area for further research, and Yasufumi Kuroda and Naoko Okamoto's research on brain activity provided a reminder of an expanding area of research in learning and cognition.

The 2014 ZDM Special Edition 'New Perspective on Learning and Cognition in Mathematics Education' (presently under construction) extends many invited presentations and long presentations within TSG22 along four broad themes:

- Contributions of ‘Culture’ to Cognition;
- Cognitive, Social, and Psychological Elements of Knowledge Construction;
- Influences of the Mathematics as ‘Taught’ on Mathematical Thinking and Mathematical Understandings; and
- Focusing Students on Learning Processes Including Problem Solving Processes.

Mathematical Knowledge for Teaching at Primary Level

Len Sparrow

Overview

The group generated considerable interest with 30 papers and abstracts being submitted. A review system was established by the TSG 23 Chair Christoph Selter whereby each paper was read and reviewed by one of the Co-Chairs and a Team Member. From this process 19 papers were accepted for presentation in Seoul.

The presentations were given over four days with each day being allocated 90 min in the main program. These sessions were chaired by Len Sparrow with help from Pi-Jen Lin on Day 2. Due to the high number of papers, and a wish of the organising team for as many colleagues as possible to experience presenting at the Congress, paper presentations were short (15 min). Each presentation had an allowance for questions and comments by the TSG participants. Papers were grouped under similar themes so that there was an element of coherence each day. The Chair summarised the issues and questions for each day and presented these to the TSG members for comment at the next session. They are copied below. Attendance at the presentations was typical of such groups with a group of stalwarts attending every presentation and every day while others attended only for their presentation. The group attracted a range of participants from early researchers to highly experienced professors and was enriched by this diversity.

Organizers Co-chairs: Christoph Selter (Germany), Suck Yoon Paik (Korea); Team Members : Catherine Taveau (France), Pi-Jen Lin (Taiwan), Len Sparrow (Australia); Liaison IPC member: Mercy Kazima (Malawi).

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Schedule

Session 1: Tuesday, 10th July, Teachers' mathematical knowledge

10:35 Christine Browning, Understanding Prospective Elementary Teacher Content Knowledge: Common Themes from the Past Decade.

10:50 Siew Yin Ho, Pre-service teachers' specialised content knowledge on multiplication of decimals.

11:05 Pi-Jen Lin, Future teachers' proof of universal and existential elements.

11:20 Di Liu, A comparative study of Chinese and US pre-service teachers' mathematical knowledge of teaching in planning and evaluating instruction.

11:35 Cheng-Yao Lin, Enhancing pre-service teachers' computational skills through open approach instruction.

11:50 Eva Thanheiser, Preservice elementary teachers' understanding of multi-digit whole numbers: Conceptions and development of conceptions.

Session 2: Wednesday, 11th July, Teachers' knowledge about children's mathematical thinking and reasoning.

10:40 Jeong Suk Pang, Novice Elementary Teachers' Knowledge of Student Errors.

10:55 Yusuke Shinno, Issues on prospective teachers' argumentation for teaching and evaluating at primary level: Focussing on a problem related to discrete mathematics.

11:10 Mi Sun Pak, Teachers' knowledge and math teaching in a reform curriculum.

11:25 Mustafa Alpaslan, Preservice mathematics teachers' conceptions regarding elementary students' difficulties in fractions.

Day 3 Friday 13th July—Teachers' beliefs, attitudes and orientations

15:10 Audrey Cooke, Anxiety, awareness and action: Mathematical knowledge for teaching.

15:25 Ronald Keijzer, Mathematical knowledge for teaching in the Netherlands.

15:40 Sharyn Livy, Foundation and connected mathematical content knowledge for second year primary pre-service teachers developed in practice.

15:55 Hyun Mi Hwang, Korean elementary teachers' orientations and use of manipulative materials in mathematics textbooks.

Session 4 Saturday, 14th July, Theoretical conceptualisation of teachers' knowledge

10:40 Minsung Kwon, Mathematical knowledge for teaching in the different phases of the teaching profession.

10:55 Tibor Marcinek, Learning to interpret the mathematical thinking of others in preservice mathematics courses: Potential and limitations.

11:10 Miguel Ribeiro, Teachers' mathematical knowledge for teaching and its role on practice.

11:25 Arne Jakobsen, Using practice to define and distinguish horizon content knowledge.

Summary of Issues Raised in Topic Study Group

Session 1:

- We already know a lot about the content knowledge of preservice primary/elementary teachers in USA.
- Similar information is available from Non-USA countries.
- Generally, they lack deeper forms of conceptual knowledge especially in number related areas.
- What causes these limitations? Procedural teaching? Other?
- What are the consequences of this? Why is it a problem?
- Results in procedural teaching and a continuation of the cycle of procedural teaching?
- What are strategies to overcome this limited knowledge?
- Is it important to overcome these limitations?
- What has already been done? National testing of pre-service teachers in UK —Evidence that it is effective? Teaching primary mathematics content in University programs/courses/units.
- Is this phenomenon in all countries? If not, how are they different? Singapore? China? Finland? Korea?
- What mathematics should pre-service teachers know?
- Should there be an entry standard in mathematics for pre-service primary teachers? If so, what should it be? Higher level mathematics?

Session 2:

- What mathematics should primary teachers know? Pre-service/In-service?
- How will they come to know this?
- How will others know they know?
- Should we employ mathematics specialists?
- How does better teacher mathematics knowledge impact the classroom/children's mathematics learning?
- How will they come to gain knowledge of children's errors, thinking, misconceptions?
- Is it important that primary teachers know about and undertake investigations, proof, explanations in mathematics?

Session 3 and 4:

- Is it possible to teach sufficient mathematics content while teaching about mathematics pedagogy?
- How can you motivate pre-service/in-service teachers to learn the mathematics needed for primary teaching?
- Do teachers need knowledge of how to use materials for teaching mathematics?
- Should we develop teachers' numeracy or mathematical knowledge?

- How do you find out what mathematics pre-service/in-service teachers know/understand?
- Is developing teacher confidence in mathematics the key?
- How can you tell which teachers are in denial or are just unaware of their limited mathematical knowledge?
- What are situations that help pre-service/in-service teachers identify gaps in their knowledge?
- How do you help when you/they spot gaps in knowledge?
- What knowledge do teachers need to make practice 'mathematically demanding' and 'pedagogically exciting'?
- How can one help develop horizon content knowledge?

Mathematical Knowledge for Teaching at the Secondary Level

Aihui Peng and Hikma Smida

Overview

TSG 24 at ICME-12 aimed to especially examine current scholarship and research on mathematical knowledge for teaching at the secondary level by collecting, comparing and discussing research experiences in this area, through the following three questions: What mathematical knowledge is needed for teaching at secondary level? What are the status quo of knowing and using mathematical knowledge for teaching at secondary level? How should we move forward (or what we have done) towards better equipped with mathematical knowledge for teaching at secondary level? In ICME 12, TSG 24 gathered 23 oral presentations from Canada, China, Finland, France, India, Ireland, Korea, Norway, South Africa, Spain, Sweden, and Turkey. They were presented in terms of four subtopics.

Organizers Co-chairs: Aihui Peng (China), Hikma Smida (Tunisia); Team Members: Hakan Sollervall (Sweden), Dongwon Kim (Korea), Karin Brodie (South Africa); Liaison IPC Member: Mercy Kazima (Malawi).

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Theoretical Perspective and Conceptual Framework for Mathematical Knowledge for Teaching at Secondary Level

The first presentation entitled "secondary school teachers' mathematical problem-solving knowledge for teaching" was presented by Olive Chapman (Canada). The study identified the nature of mathematical problem-solving knowledge for teaching and how this knowledge could support students' development of proficiency in problem-solving, which has significant implications for teacher education. In particular, the author discussed what should teachers know to teach for problem-solving proficiency and what knowledge should teachers hold to help students to become proficient in problem solving. These questions were addressed from a theoretical perspective and from a study that investigated secondary school teachers' knowledge in terms of their conceptions and teaching of problem solving in relation to contextual problems.

The second presentation "the ladder of knowledge: A model of knowledge for second level mathematics teachers" was presented by Niamh O'Meara (Ireland). In this study, the authors developed a new model of knowledge to meet the needs of curricula with a strong focus on mathematical applications.

The third presentation "competence in didactic analysis in the pre-service training of secondary school mathematics teachers in Spain" was presented by Vincent Font (Spain). The study illustrated how one of the components of the broad competence in didactic analysis (identifying potential improvements to be implemented in future classes) was developed within the context of the University of Barcelona.

The fourth presentation "coordinating theories to analyze the relationship between teachers' actions and teachers' knowledge—a presentation of a methodological approach" was presented by Erika Stadler (Sweden). The study presented a tentative methodological framework to analyze what kind of mathematical knowledge for teaching, MKT, novice mathematics teachers use when teaching. The main idea of the framework is to coordinate three different theoretical frameworks, which provide a methodological tool for analyzing the relationship between teachers' teaching actions and mathematical knowledge.

The fifth presentation "the structure of knowledge of teaching of student teachers on the topic of distance formula" was presented by Lin Ding (China). The presentation provided a new approach of interpreting knowledge of teaching (KOT) of secondary mathematics student teachers by examining its structure (i.e. mathematics, student and pedagogy). A brief analysis on two examples regarding the structure of KOT was provided in order to illustrate how this approach works.

The sixth presentation "A pre-analysis of the creation of teacher's resources for developing instruction in basic logic in French high schools" was presented by Zoe Mesnil (France). The author presented studies on the role of logic in mathematics education in order to show how it can help students to improve their skills in

language and expression. Through the analysis of curricula and textbooks, the study presented an overview of the process of didactic transposition for teaching the concepts of logic.

Pre-service Mathematics Teachers' Knowledge

This subtopic consists of five presentations from USA, Ireland and Turkey. The first presentation "secondary teacher candidates' mathematical knowledge for teaching as demonstrated in their portfolios" was presented by Hari Koirala (USA). Their study focused on prospective secondary school teachers' mathematical knowledge and their ability to demonstrate how their learning of mathematics from their university courses applies to the teaching of secondary school mathematics.

The second presentation "Chinese and US pre-service mathematics teachers' knowledge for teaching algebra with a focus on representational flexibility" was presented by Rongjin Huang (USA). Their study examined Chinese and U.S prospective middle grade teachers' knowledge of algebra for teaching with a focus on representational flexibility. It was found that the Chinese participants not only demonstrated sound knowledge needed for teaching the concept of function, but also had the flexibility in using representations appropriately. In contrast, the U.S. counterparts showed their weakness of using these concepts to solve problems and using appropriate representations.

The third presentation "whose fault is it anyway? The truth about the mathematical knowledge of prospective secondary school teachers and the role of mathematics teacher educators" was presented by Miriam Liston (Ireland). The author presented an empirical research study which aims to contribute to the understanding of prospective secondary level mathematics teachers' mathematical knowledge for teaching. The findings suggest that prospective mathematics teachers may not have sufficient subject matter knowledge to alter their teaching strategies and ultimately teach for understanding.

The fourth presentation is "pre-service secondary school mathematics teachers' specialized content knowledge of complex numbers" presented by Fatma Aslan (Turkey). The author reported the findings of a study of pre-service secondary school mathematics teachers' learning of complex numbers during a content course. According to the author's findings, participants were able to build connections between their mathematical understanding as teachers with their teaching practice and students' mathematical ideas.

The fifth presentation "a comparative analysis of the content knowledge for secondary pre-service mathematics teachers" was presented by Wei Sun (USA). In his study, it focused on the knowledge that the secondary pre-service teachers gain during their study in the teacher education program. Two mathematics teacher preparation programs were examined, one from China and the other from the US,

with the intent to shed light on this important issue and help mathematics educations understand mathematics teacher education from a broader (international) perspective.

In-service Mathematics Teachers' Knowledge

The first presentation "seeing mathematics through processes and actions: investigating teachers' mathematical knowledge and secondary school classroom opportunities for students" was presented by Rose Mary Zbiek (USA). The study described the processes and actions approach. The authors proposed a more general way to characterize MKT than is typically used.

The second presentation "what is pre-service and in-service Teachers' MKT in concept of vector" presented by Hyunkyoung Yoon (Korea) was to investigate the mathematical knowledge for teaching (MKT) of pre-service and in-service mathematics teachers on the concept of vector. 80 pre-service and 124 in-service mathematics teachers were asked to perform three questions based on MKT's subdomain. The results show that pre-service teachers have stronger common content knowledge. On the other hand, in-service teachers have stronger specialized content knowledge, knowledge of content and teaching.

The third presentation "pedagogical knowledge, pedagogical content knowledge, and content knowledge for teaching mathematics: how do they shape teaching practices?" was presented by Hee-Jeong Kim (USA). This empirical study offered a case of a proficient middle school mathematics teacher, well known as a highly skilled teacher in her district, and explored the teacher's decision-making in different teaching contexts. The author discussed what the contributions of different kinds of knowledge were and implied how we can support teachers with regard to knowledge for better mathematics teaching.

The fourth presentation "hypothetical teaching trajectories (HTT): analysing contingency events in secondary mathematics teachers' practice" was presented by Jordi Deulofeu (Spain). This paper showed through the work done by a future secondary mathematics teacher called Gabriel in his initial training at the university, how analyzing HTT can serve a double role: giving information about the prospective teacher's mathematical knowledge and helping to validate an instrument that serves teachers to reflect on their own mathematical knowledge in practice.

The fifth presentation "developing craft knowledge in mathematics teaching" was presented by Inger Nergaard (Norway). Her study focused on teachers' opportunities to develop craft knowledge through their engagement with students. Using video recordings of mathematics lessons and following up conversation with the teachers, two episodes of teaching were considered. In the first episode the teacher appears to close down opportunities for discussion of the unanticipated situations that arose and thus she denied herself opportunities to learn from the situation, while the second episode concerned a teacher who invites students into her teaching and thus enable further development of existing knowledge.

The sixth presentation "understanding teachers' knowledge of and responses to students' mathematical thinking" was presented by Shikha Takker (India). She reported a case study which aimed at understanding teachers' knowledge about students' mathematical thinking in situ. Teacher's response to students' mathematical thinking was characterized based on classroom observations, task-based interviews, complemented with the anticipation and reflection of students' responses to 'proportion' problems. It was found that such a framework helps in creation of conflict in the teacher and is a potential source of teacher reflection.

Methodology Issues on Mathematics Teachers' Knowledge

The first presentation "using scenarios validated as measures to explore subject matter knowledge (SMK) in an interview setting" was presented by Sitti Patahuddin (South Africa). The presentation focused on one scenario adapted from LMT (e.g. from the Learning Mathematics for Teaching—LMT—project) in order to explore how teachers interviewed engage with each of the responses offered.

The second presentation "instruments for improving teachers' use of artifacts for the learning of mathematics" was presented by Håkan Sollervall (Sweden). The author argued that teachers' mathematical knowledge has to include instruments for controlling how the artifact used become involved when students engage in solving mathematical tasks. The authors proposed to meet this demand by coordinating the matching notions of affordances (planning) and objects of activity (evaluation). They briefly illustrated how these notions can be used as analytical instruments in a fashion that connects to what teachers already do in their daily work.

The third presentation "exploring the influence of teachers' use of representation on students' learning of mathematics" was presented by Emmanuel Bofah (Finland). The aim of the study was to examine how teachers' use of different mathematics representations, in the domain of functions, affects students' behavior in the process of doing and learning mathematics.

The fourth presentation "consensuating the best profile of a mathematics teacher in the transition to secondary school; a discussion of experts using the Delphi method" was presented by Sainza Fernandez (Spain). This on-going investigation, embedded in a larger project that targets primary-secondary transition in mathematics, explored the knowledge of a group of expert mathematics teachers and experts involved in teachers' education using the Delphi method. The results arisen point at secondary teachers as more responsible for the success or failure of the process and their sensitivity as professionals of mathematics education as particularly determinant.

The fifth presentation is "a case study on the status quo of the development of Tibetan mathematics teacher's pedagogical content knowledge (PCK) in Lasha" (China). A case study was used to analyze the development of Tibetan mathematics teacher's PCK in secondary school in Lasha.

The sixth presentation "The project: collaborating to advance secondary teachers' mathematics proficiency for teaching" was presented by Pier Junior Clark (USA). Using the *Provisional Framework for Proficiency in Teaching Mathematics* as a guideline, the author examined the changes in the secondary teachers' mathematics proficiency and efficacy for teaching data analysis and statistics over a year-long professional development project.

Summary

TSG24 included presentations from many points of view:

Conceptual frameworks for mathematical knowledge for teaching at secondary level, e.g., what is the nature of mathematical knowledge for teaching at secondary level? What mathematical knowledge needs to know and how to use it from an advanced perspective for a secondary school teacher? What are the approaches, from the practice point of view, that could support teachers developing their mathematical knowledge that they need to know and know how to use it?

Empirical researches that aim to contribute our understanding of what mathematical knowledge is needed or how it is assessed in different scenarios, e.g., teachers' mathematical knowledge for teaching in specific activities, teachers' mathematical knowledge for teaching in specific domain, teachers' mathematical knowledge for teaching in special situations, such as information and communication technology environment, innovative and creative approaches of developing mathematical knowledge and the instruments for assessing these approaches specifically.

Empirical researches to explore relationships between teachers' learning of teaching (both pre-service and in-service) and students' learning of mathematics, e.g., the effect of mathematics knowledge for teaching on student achievement, the innovative and creative approaches of developing the effect of mathematics knowledge for teaching on students' learning and achievement.

In-Service Education, Professional Development of Mathematics Teachers

Shuhua An and Andrea Peter-Koop

The aim of TSG 25 at ICME-12 was to discuss the experiences and approaches developed in different countries to support the professional development of teachers for practice, in practice and from practice. The study group 25 received 74 paper submissions from scholars, graduates, and practitioners in various countries and regions, and accepted 69 papers. A total of 63 papers were presented at 10 sessions at ICME 12 conference. Participants discussed research based practices and state-of-the-art approaches to the in-service education and professional development of teachers from a multi-national and globe perspectives. This report will address some key ideas in the following topics from TSG 25:

- Research studies and projects in professional development of primary and secondary school teachers
- Research studies and projects in in-service education and teacher education programs
- Classroom teaching research and lesson study in professional development of primary and secondary school teachers
- In-service education in STEM field in secondary school settings—Research studies and projects
- Mentor and coaching programs in professional development of primary and secondary school teachers

Organizers Co-chairs: Shuhua An (USA), Andrea Peter-Koop (Germany); Team Members: Barbara Clarke (Australia), Yimin Cao (China), Gooyeon Kim (Korea); Liaison IPC Member: Gabriele Kaiser (Germany).

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Professional Development of Primary and Secondary School Teachers

One of the challenges in teacher professional development is the nature of the research and the differing agendas of stakeholders. Much of the research takes the form of evaluation of teacher development projects and while they build on a growing body of research, the contexts in which they occur are complex. As a result it is difficult to synthesize the findings in ways that can inform future planning. How can our small pieces of research contribute to our understanding of the whole picture?

The role of teacher attitudes within the context of professional development is important but can be overemphasized at the expense of actions. A number of the papers helped focus on the role of practice in teacher development. The value of ensuring that participants have a voice was a common theme.

The important discussion focused on content of professional development and measurement of effects of professional development. A number of papers indicated the needs of paying attention to specific knowledge, such as error analysis, and measuring teachers' knowledge and teacher learning from error analysis and engaging learners in avoiding the errors.

Participants discussed the forms of professional development. Presentations shared different forms of lesson studies, such as Teacher Research Group in China, an important form of school based professional development.

In-Service Education and Teacher Education Programs

The presentations shared their effective approaches in in-service education and teacher education programs. However, the discussions indicated the challenges in in-service education and teacher education programs. The examples of the challenges: (1) How can we best prepare math teachers? (2) How to measure teachers' pedagogical content knowledge, (3) How to support new teachers and teacher retention issue, (4) Design different models of professional development that support teachers in new initiatives, (5) Relationship between professional development and classroom teaching, (6) Teaching work load and time to plan lesson in US, (7) Tools for reflection, and (8) Leadership roles.

Classroom Teaching Research and Lesson Study in Professional Development and Teacher Education Programs

Classroom teaching research and lesson studies have various forms in different countries. The following focused questions regarding classroom teaching research

and lesson study in professional development of primary and secondary school teachers were asked during the discussions:

- What is effective classroom teaching?
- What math teacher educators should know about effective classroom teaching?
- How can we best prepare math teachers to teach math effectively?
- How do we enhance the effectiveness of professional learning communities for math teachers?

In-Service Education in STEM Field in Secondary School Settings

In-service education in secondary schools with a focus on integrating science and technology is an interesting topic of TSG 25 sessions. A range of contexts and countries were represented both in the papers and the discussions and there was considerable overlap in the issues of concern. The role of technology provides an added challenge as both software and hardware is constantly being updated. The comfort zone of teachers was a common issue and the acknowledgement that in-service education and professional development often requires teacher to move out of their comfort zone. This is particularly relevant in technology rich or cross discipline environments.

Mentor and Coaching Programs in Professional Development of Primary and Secondary School Teachers

There were a range of papers focusing on leading teacher change through a variety of models. Mentoring and coaching models are increasingly being used in many countries. One model that was particularly promising was "teacher researchers" in China. They are a form of master teacher with considerable expertise who supports teacher development. This systemic approach also provides for teacher progression within the profession that is not available in many countries.

Whole Group Discussion

The whole group discussion focused on key issues, major findings, insights, international trends in research and development in professional development and in-service education, and indicated open questions to be addressed in the future.

Questions to be addressed in the future

- How do we support new teachers in the new initiatives?
- What are the different models of professional development? Especially, what are good models for new initiatives?
- What is the relationship between professional development and effective classroom teaching?
- What are the common strategies in professional development and classroom teaching in different countries? Diverse issue is needed to address also.

Discussion on future planning: Publications arising from TSG 25

- Publication of selected papers in an edited volume to be published by Springer Mathematics Teacher Education and Development series (Research based papers)
- Publication of selected papers in a special issue in Journal of Mathematics Education (USA) (Research based papers)

More opportunities:

- Routledge Education, Taylor & Francis expressed their interest in publishing TSG 25 papers
- A journal editor from Singapore also expressed her interest in publishing TSG 25 papers in a special issue

Joint project

- Participants supported the idea to work together for a joint project that compares in-service education and professional development of mathematics teachers in different countries.

Pre-service Mathematical Education of Teachers

Sylvie Coppé and Ngai-Ying Wong

Overview

The topic study group on pre-service mathematical education of teachers is dedicated to sharing and discussing of significant new trends and development in research and practice about the various kinds of education of pre-service mathematics teachers and of pre-service primary teachers who teach mathematics and are trained as generalists. It aimed to provide both an overview of the current state-of-the-art as well as outstanding recent research reports from an international perspective. The group discussed research experiences with different practices of pre-service mathematical education of (mathematics) teachers throughout the world, i.e. similarities and differences concerning the formal mathematical education of teachers, types and routes of teacher education, curricula of (mathematics) teacher education, facets of knowledge and differences in their achievements and beliefs about the nature of their training, and a variety of factors that influence these differences.

Organizers Co-chairs: Sylvie Coppé (France), Ngai-Ying Wong (Hong Kong); Team Members: Lucie De Blois (Canada), Björn Schwarz (Germany), Insun Shin (Korea), Khoon Yoong Wong (Singapore); Liaison IPCMember: Gabriele Kaiser (Germany).

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Session Schedule

We received 51 proposals from different countries, 6 were rejected and at last we had 40 papers and only 37 presentations. As we had four 90-min sessions (July 10, 11, 13, and 14), two groups ran parallel in order to let 10 min to each presentation.

Each session was devoted to different issues in affect research in mathematics education.

Session 1: Tuesday, July 10, 10:30–12:00

Group A:

Buchholtz Nils, Studies on the effectiveness of university mathematics teacher training in Germany

Francis-Poscente Krista, Preparing elementary pre-service teachers to teach mathematics with math fair

Jennifer Suh, 'Situated learning' for teaching mathematics with pre-service teachers in a math lesson study course

GwiSoo Nah, A constructivist teaching experiment for elementary pre-service teachers

Qiaoping Zhang, Pre-service teachers' reflections on their teaching practice

Group B

Liora Hoch, Miriam Amit, When math meets pedagogy: the case of student evaluation

Hugo Diniz, Math Clubs: space of mathematical experimentation and teacher formation

Huk Yuen Law, Becoming professional mathematics teachers through action research

Levi Elipane, Integrating the elements of lesson study in pre-service mathematics teacher education

Müjgan Baki, Investigating prospective primary teachers' knowledge in teaching through lesson study

Session 2: Wednesday, July 11, 10:30–12:00

Group A

Zhiqiang Yuan, Developing prospective mathematics teachers' technological pedagogical content knowledge (TPACK): a case of normal distribution

Roslinda Rosli, Elementary pre-service teachers' pedagogical content knowledge of place value: A mixed analysis

Steve Thornton, Saileigh Page, Julie Clark, Linking the mathematics pedagogical content knowledge of pre-service primary teachers with teacher education courses

Rachael Kenney, Writing and Reflection: Tools for developing pedagogical content knowledge with mathematics pre-service teachers

Group B

Jan Sunderlik, Soetkova, Identification of learning situations during prospective teachers' student teaching in two countries

Yali Pang, Using a Video-based Approach to Develop Prospective Teachers' Mathematical Knowledge for Teaching and Ability to Analyze Mathematics Teaching

Xiong Wang, The Video Analysis of the Authentic Classroom as an Approach to Support Pre-service Teachers' Professional Learning: A Case from Shanghai Normal University, China

Namukasa Immaculate, Measuring teacher candidate's conceptual, procedural and pedagogical content knowledge

Session 3: Friday, July 13, 15:00–16:30

Group A

Hyun Young Kang, Korean Secondary Mathematics Teachers' Perspectives on Competencies for Good Teaching

Rongjin Huang, Pre-service secondary mathematics teachers' knowledge of algebra for teaching in China

Björn Schwarz, Relations between future mathematics teachers´ beliefs and knowledge with regard to modelling in mathematics teaching

Yeon Kim, Challenges to teach mathematical knowledge for teaching in mathematics teacher Education

Group B

Yuki Seo, Enhancing mathematics thinking for training mathematics teachers: a case at the department of engineering

Kiril Bankov, Curriculum for preparation of mathematics teachers: a perspective from TEDS-M

Lin Ding, A comparison of pre-service secondary mathematics teacher education in Hanover (a city in Germany) and Hangzhou (a city in China)

Khaled Ben-Motreb, Pre-service teachers' teaching practices and mathematics conceptions

Ildar Safuanov, Master programs for future mathematics teachers in Russian federation

Session 4: Saturday, July 14, 10:30–12:00

Group A

Claire Berg, Barbro Grevholm, Use of an inquiry-based model in pre-service teacher education: Investigating the gap between theory and practice in mathematics education

Loretta Diane Miller, Brandon Banes, Teaching pre-service elementary teachers mathematics through problem-based learning and problem solving

Ji-Eun Lee, Towards a holistic view: analysis of pre-service teachers' professional vision in field experiences

Diana Cheng, Discourse- based instruction in small groups of pre-service elementary teachers

Kwang Ho Lee, Eun-Ha Jang, The research on PBL Application in mathematics method course

Group B

Ceneida Fernandez, Julia Valis, Salvador Linares, An approach for the development of pre-service mathematics teachers' professional noticing of students' mathematical thinking

Erika Löfström, Tuomas Pursianen, "I knew that sine and cosine are periodic… but I was thinking how I could validate this": A case study on mathematics student teachers' ersonal epistemologies

Ju Hong Woo, The change of mathematics teaching efficacy beliefs by student teaching

Mi Yeon Lee, Preservnrique Galindo, Pre-service Teachers' Ability to Understand Children's Thinking

Ravi Somayajulu, Manjula Joseph, Candace Joswick, Characterizing secondary pre-Service mathematics teachers' growth in understanding of student mathematical thinking over a three-course methods series

Main Questions Discussed

Main questions were discussed such as:

- What are fundamental concepts to study the field of pre-service teacher in comparison of in-service teacher? What are special challenges for respective studies arising from the particular characteristics of pre-service teacher education and how to face them?
- What knowledge contribute to the development of the pre-service teacher? Which actions push the pre-service teacher to lost their initial experience of pupil to integrate new epistemological posture?
- What are the contribution of the different tolls (technology, writing, reflection, video) during the teacher training? How can a common core of the concept of "pedagogical content knowledge" be described against the background of its different conceptualizations?
- Are the challenge different in function of countries? What is the influence of the curriculum on practice of pre-service teacher?
- What kind of mathematic could contribute to the development of pre-service teacher? And how can it be taught adequately?
- Why do we teach mathematics and why this answer influence the teacher training?

Issues and Findings

Quite a number of issues on pre-service teacher education were identified, which includes considerable drop out rate, lack of knowledge and even lack of interest in mathematics among potential teachers in some countries. There also exists

disagreement between goal and reality. For instance, while constructivism is advocated in the school curriculum, teacher education programmes did not provide such experience to student-teachers.

A number of means were introduced to address the above, arriving at promising results. The use of math fair, lesson studies, situated learning, ICT, writing, enquiry/problem based learning and reflections are some of them. We observed the influence of the cultural context concerning education or mathematics teaching/learning from different countries or different parts of the world.

A salient focus among the presentations is teacher's knowledge, ranging from subject content knowledge, pedagogical content knowledge to belief. There were discussions on how teacher education programme can strike a balance between the mathematics component and the pedagogical component and how these two can be linked together.

Probably, the use of video in teacher training sessions is revealed as an important tool which could create or contribute to create these links. But we concluded that using video in pre service teacher training is not easy. We need to elaborate research programs to study how it could be possible to develop video based training. There were discussions on the different kinds of video (for example, showing expert or novice teachers, ordinary lessons or experimental), on the different goals (to show, to analyze, to observe the teacher or the students) on the different points of view (the teacher or the students) on the different conditions and on the limits. These remarks led to another issue: how could the teacher trainer introduce and use video to help the pre-service teacher to develop different kinds of knowledge or skill for mathematics teaching? How could the video give some informations on the student learning…

As for the recurrent issue of PCK, it was realised that it is cultural/context and student dependent. In other words, for a single subject matter, it depends on the 'target audience' for searching for the best way to have it presented. Rather than instoring potential teachers with a bundle of PCK (corresponding to a single SK), it might be more realistic and effective to equip them with the ability to adjust the presentation (of SK) spontaneously according to the subtle variations of their students. Again reflection comes into play.

How to build a path from fun to formal mathematics, from elementary mathematics to advanced mathematics is another issue of concern. All these involve all the parties: the student-teacher, the teacher trainer, the mentor and the pupils (during field experience). All these would not only result in reflections among student-teachers, professors and even teacher education curriculum developers should have their reflections too.

Summary

There were a lot fruitful discussions in this topic study group. We appreciated the different topics of the papers. We observed that there were a lot of very interesting issues which are very similar from a country to another and we hope our discussion

will continue to bear fruits and impacts on our future programme for pre-service mathematics education. We learned from the different points of view and the cultural contexts.

Motivation, Beliefs, and Attitudes Towards Mathematics and Its Teaching

Birgit Pepin and Ji-Won Son

Report

Affect has been a topic of interest in mathematics education research for more than 30 years. More recently, and as emphasized in the last ICME 11 report, beliefs has turned from a 'hidden' to a more 'visible' variable. Today we know that affective variables can be regarded as explicit factors which influence mathematics learning outcomes as well as instructional practice. The different research perspectives used in studies of affect include psychological, social, philosophical, and linguistic. Those various views were represented in the ICME 12 research presentations. It also became clear during the conference, and this was expected, that the construct of 'affect' encompasses related constructs such as 'motivation', 'beliefs', 'values' and 'attitudes', to name but a few. We invited, and received, presentation proposals on all areas of affect in mathematics learning and teaching.

The organizing committee organized the accepted papers and posters for TSG 27 in the following ways:

Organizers Co-chairs: Birgit Pepin (Norway), Ji-Won Son (USA); Team Members: Bettina Roesken (Germany), Inés Mª Gómez-Chacón (Spain), Nayoung Kwon (Korea); Liaison IPC Member: Bill Barton.

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- One 'elicited' Roundtable on 'Methodological issues in Affect Research';
- Six groups of short paper presentations and discussions (15 min);
- Three long paper presentations (30 min);
- Posters in the general poster session.

We had a large number of proposals and rigorously reviewed them, each proposal being evaluated by three reviewers (members of the TG27 team) according to a common set of criteria (agreed review scheme). At the advice of the ICME organizing committee we accepted most, only rejecting about six proposals, and arranged the accepted proposals in sessions. As we had four 90-min sessions (July 10, 11, 13, and 14) available, we decided to run parallel sessions, allocating 20–30 min for long and 10–15 min for short presentations. Each session was chaired by one of the co-chairing team members (unfortunately Inés Mª Gómez-Chacón could not attend ICME 2012). Posters were allocated to the poster session, which was common for all TSGs. One of the highlights of the TSG 27's sessions was the 'elicited' Roundtable on methodological issues, which had a 60-min time allocation.

The following will provide a 'taste' of the presentations and issues discussed.

On the 10th July the co-chairs opened up the first of four one and half hour sessions. Subsequently, Jill Cochran presented her research asking questions concerning values and ideals in mathematics education. She argued that teachers, policy makers, curriculum developers, and other professionals often held ideals that were in opposition to each other, and that this created conflicts of interest, in particular for classroom teachers. The following two sessions ran parallel, and each parallel session included three short presentations on the following topic areas: 'Students' views of mathematics'; and 'Mathematics teacher knowledge and efficacy'. Each series of presentations was followed by a discussion of the presentations.

On the 11th July the (elicited) Methodology Roundtable and one short presentation were scheduled. The panel members of the Roundtable were all well-known researchers in the field of affect in mathematics education: Markku Hannula; Gilah Leder; Ilana Horn; and Guenter Toerner. Each outlined their insights concerning methodological issues, and Markku Hannula presented a theoretical framework for the inclusion of the different 'lenses'. Then questions about the framework and relevant issues were discussed.

The 12th July session started with a (long) presentation by Mac an Bhairs Ciaran and colleagues on the 'effect of fear on engagement with mathematics'. They reported on a comparative study of first year undergraduate mathematics students: one group had failed their first year examinations; the second had successfully completed the first year. It was argued that whilst both groups named 'fear' as a factor for engagement (or not) with mathematics, for one group it emerged as a positive motivation, in the sense that it formed part of their coping mechanisms when dealing with the various obstacles that they encountered. The subsequent parallel sessions included six (short) presentations, under the headings of 'Motivation and conditions

for pupil learning' and 'Teacher beliefs concerning curriculum and tracking'. Again, each series of presentations was followed by a discussion of the presentations.

The last TGS 27 session had a similar structure, albeit more time was allocated for discussion of the whole TGS, insights gained and implications for future research (as this was the last session of the TSG). In an opening (long) presentation Birgit Pepin reported on a study of 'Affective systems of Norwegian mathematics students/teachers in relation to 'unusual' problem solving'. She argued that results from the three different groups (each at different stages of their educational and professional development) showed that positive engagement structures were linked to working together in a group and previous (positive) experiences, whereas 'giving up' was connected to 'working alone' and the 'unusual' problem-solving situation. The subsequent two parallel sessions (including altogether four (short) presentations) were in the two themes of 'Teacher beliefs and practices' and 'Teachers' views on mathematical tasks'.

In a final discussion the following issues were raised:

1. Five minutes for (short) presentations is not sufficient. Hence, either a different mode of running the TGS should be found, or (fewer) presenters should be given more time, also for discussion. This has implications for acceptance of future proposals: this ICME the TGS 27 had a very large number of proposals, and approximately half were accepted as short or long presentations (19), and approximately half accepted as posters (with a small number of rejections). Hence, questions arise: should the reviewing process (TGS 27 had three reviewers and developed its own evaluation schedule) be more rigorous, and more papers be rejected? Or should the TSG be 'inclusive' and find another mode of running the group?
2. The question of 'publication' was raised: presentations were 'published' in the ICME 12 pre-proceedings, but how does this count/is acknowledged in terms of publications?
3. It was suggested to be more selective about the accepted papers and support, and perhaps elicit, more 'novelty' topic areas: e.g. affect and mathematical thinking (including suitable theoretical frameworks and measurement instruments/methodological tools for this field of research); affect as a dynamic system (including affective systems and 'collectives' in social contexts); intervention studies/design-based research on 'affect and cognition'.
4. TGS 27 was provided with two rooms close to each other (and this was beneficial for participants to be able to attend sessions). However, it was difficult for the group to 'merge' as a whole, as many discussions took place in separate sessions, and some participants wanted to share their ideas in a whole group discussion.
5. Overall, it was emphasized that this ICME's TGS on affect went well (as did previous groups) and that this group is now an established and well-recognized part of ICME.

List of Groups, Presentations and Presenters of Long and Short Presentations

Tuesday, 10th July

Jill Cochran, Does a balanced philosophy in mathematics education exist? Student views of mathematics:

- **Mario Sanchez Aguilar**, Alejandro Rosas and Juan Gabriel Molina Zavaleta, Mexican students' images of mathematicians
- **Sally Hobden**, After graduation? The beliefs of alumni bachelor of education students reading mathematics and the formation of mathematical knowledge
- **Veronica Vargas Alejo**, Cesar Cristobal Escalante and Jamal Hussain, Beliefs and attitudes toward mathematics at university Level, development of mathematical knowledge

Teacher knowledge and efficacy

- **Janne Fauskanger**, Teachers' epistemic beliefs about HCK
- **Giang-Nguyen Nguyen,** Diagnosing student motivation to learn mathematics: A form of teacher knowledge
- **Ayse Sarac** and Fatma Aslan-Tutak, The relation of teacher efficacy to students' trigonometry achievement

Wednesday, 11th July

Methodological issues

Dohyoung Ryang, The viability of the mathematics teaching efficacy beliefs instrument for Korean secondary pre-service teachers

Roundtable (Co-chairs: Bettina Roesken and Birgit Pepin; Panel members: Markku Hannula, Gilah Leder, Ilana Horn and Guenter Toerner)

Methodological issues in Affect Research: distinguishing between 'state' and 'trait' in mathematics education research.

Friday, 13th July

Mac an Bhaird Ciaran, The effect of fear on engagement with mathematics

Motivation and conditions for pupil learning

- **Chonghee Lee**, Sun Hee Kim, Bumi Kim, Soojin Kim and Kiyeon Kim,
- **Denival Biotto Filho** and Ole Skovsmose, Researching foregrounds: About motives and conditions for learning
- **Nelia Amado**[1] and Silvia Reis, A young student's emotions when solving a mathematical challenge
- **Suela Kacerja**, "Cultural products are girls' things!" Interests Albanian students retain for real-life situations that can be used in mathematics

Teacher beliefs concerning curriculum and tracking

- **Qian Chen**, Teachers' beliefs and mathematics curriculum reform: A comparative study of Hong Kong and Chongging

- **Benjamin Hedrick** and Erin Baldinger, Beliefs about tracking: Comparing American and Finnish prospective teachers

Saturday, 14th July

Birgit Pepin: "Exploring affective systems of Norwegian mathematics student/ teachers in relation to 'unusual' problem solving"

Teacher beliefs and practices

- **Dionne Cross** and Ji Hong, "I'm not sitting here doing worksheets all day!": A longitudinal case study exploring perceived discrepancies between teachers' beliefs and practices
- **Ralf Erens**[1] and Andreas Eichler[1], Teachers' curricula beliefs referring to calculus

Teacher views on mathematics tasks

- **Esther Levenson**, Affective issues associated with multiple-solution tasks: Elementary school teachers speak out
- **Anika Dreher** and Sebastian Kuntze, Pre-service teachers'views on pictorial representations in tasks

Language and Communication in Mathematics Education

Tracy Craig and Candia Morgan

Introduction

The topic of "Language and Communication in Mathematics Education" covers a wide range of areas of interest, ranging from the question of what constitutes "language" in mathematics, through investigations of communicative interactions in mathematics classrooms and study of issues involved in teaching and learning mathematics in multilingual settings. This breadth was well represented in the papers accepted for presentation in the Topic Study Group at ICME12. In order to facilitate discussion, the paper presentations in each session were divided into two sets, with participants choosing which set to attend. This allowed the discussion to focus in greater depth on common themes. In addition, one session of the TSG was devoted to a panel discussion on the topic of "Theoretical and methodological issues in studying language in mathematics education" and a final plenary meeting enabled participants to reflect on the TSG as a whole, the common issues addressed, the lessons learnt and aspirations for future work on the topic. In this report, we present an overview of the major themes arising in the papers presented and in the discussions during the congress.

Organizers Co-chairs: Tracy Craig (South Africa), Candia Morgan (UK); Team Members: Marcus Schuette (Germany), Rae Young Kim (Korea), David Wagner (Canada); Liaison IPC Member: Oh Nam Kwon (Korea).

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Classroom Interactions

The nature of classroom interactions and their relationship to the doing and learning of mathematics is a major area of research, forming the focus of many of the papers presented in the TSG. The majority of these papers were concerned with the construction of mathematics and mathematical thinking and, in particular, the ways that teachers and teaching methods shape the possibilities for students' mathematical thinking and the ways in which mathematical knowledge is developed in interactions between teacher and students and among groups of students.

Drageset characterised different ways in which teachers respond to student contributions, offering a framework for analysing how different practices may have potential to help student thinking to progress. Milani also discussed how different forms of interaction may relate to learning, identifying dialogic questioning as a form that involves students as active participants in the learning process. Focusing on the development of spatial perception in young children, Schuette's study investigated the different ways in which this domain is talked about in the three contexts of primary school, infant school and in the home. Park used a semiotic approach to analyse and describe students' proportional reasoning, finding that multiplicative strategies were more successful than either additive or formal strategies.

Lee et al. looked at the effects of using "story-telling" instead of formal proof when teaching about transformation of functions, suggesting that students have similar success with both methods but that the story-telling approach has affective benefits. Investigating students' ability to present their solution methods and explanations in writing, Misono and Takeda identified a need for teachers to work with students to develop their use of mathematical language and their communication skills. Another approach to thinking about teaching methods was provided by O'Keefe and O'Donoghue, who offered a linguistic analysis of textbooks, using this to characterise how the nature of mathematics is portrayed.

Looking in detail at a teacher working with a small group of children, Gellert analysed an episode in which a disagreement arises, identifying the epistemological development and how the teacher and students negotiate mathematically. In Barcelona, a group of researchers is investigating classroom interaction from the point of view of studying the social construction of mathematical knowledge. This group presented two papers looking deeply at the mathematical activity of students when working in pairs (Badillo, Planas, Goizueta and Manrique) and in whole group discussion (Chico, Planas and Goizueta).

Language is not only used for communicating knowledge but is also a means for establishing our identities and relationships. This function of language was addressed by Heyd-Metzuyanim, whose paper presented an analysis of the "identifying" and "mathematizing" interactions in two small groups of students while they were engaged in problem solving. She suggested that, for the lower attaining group, the struggles over identification may have hindered their progress in learning.

Multilingualism in Mathematics Education

There has been a longstanding interest in the issues involved in teaching and learning mathematics in different languages. This originated to a large extent in the context of post-colonialism at a time when many countries with a legacy of education in the language of the ex-colonial power were struggling to value their own national and local languages and to develop the use of these languages in education. Political struggles over choice of language of instruction continue, while research is adding to our understanding of how characteristics of specific languages may affect the nature of the mathematics that is done using the language as well as how they may affect student learning. Two papers by Edmonds-Wathen and by Russell and Chernoff both addressed the differences between Aboriginal Englishes, spoken in indigenous communities in Australia and Canada respectively, and the standard forms of English spoken by the majority of their teachers and used in the classroom. While appearing similar in some respects, these languages carry different cultural and conceptual underpinnings with consequent possibilities for meaning making that teachers need to be aware of.

With increased mobility of populations as well as national decisions to offer mathematics education in a range of languages, mathematics educators across the world are increasingly needing to deal with classrooms in which students speak more than one language and have varying levels of competence in the main language of instruction. While this is often portrayed as being a 'problem', the papers presented in the TSG demonstrate that mathematics educators are dealing in subtle and important ways with the complex issues involved. Indeed, the research reported by Ní Ríordáin and McClusky from Ireland indicates that bilingual students with good competence in both languages (Irish and English) outperformed those for whom one language was dominant. Investigation of the students' language use while problem solving suggested that bilingualism was associated with enhanced metacognitive ability. The benefits of bilingualism are one of the motivations behind the introduction of Content and Language Integrated Learning (CLIL), a policy supported by the European Commission, involving teaching curriculum content through the medium of a foreign language. Maffei, Favilli and Peroni reported on the introduction of CLIL in Italy, teaching mathematics through the medium of English in secondary schools.

Whereas the students investigated by Ní Ríordáin and McClusky and by Maffei et al. experienced teaching and learning in both languages, Craig's study looked at the experience of university students in South Africa, studying mathematics through the medium of English only, in spite of the fact that for some of them this was not their main language. She introduced writing activities into the classroom as a means of developing students' understanding of mathematical concepts and found that both English and non-English main language students grappled similarly with the mathematical content but that language was a source of difficulty and a potential obstacle for less well-prepared students. The question of how pedagogic methods may have differential effects for students from different linguistic and cultural

backgrounds was also addressed by the study proposed by Björklund Boistrop and Norén. Their concern was to investigate teachers' assessment practices in interactions with students in multilingual classrooms in Sweden.

Theory and Methodology

A wide range of theoretical perspectives and methodologies was apparent in the papers presented and this was a focus of much discussion during the TSG sessions as participants sought to understand the basis for analyses and conclusions and to interrogate and develop the rigour of the methods used to study language and communication. Two presentations took as their main topic the use and development of theory and methodology. Nachlielli and Tabach addressed the combination of two theories: the social semiotics and Systemic Functional Linguistics of Halliday (1974), a general semiotic and linguistic theory, and Sfard's theory of commognition (2008), which addresses the nature of mathematical discourse specifically. They used these theories to develop a framework for analysing classroom interaction. Similarly, Tang, Morgan and Sfard drew on the same two theories to present the development of an analytical framework for studying examination papers and the nature of the mathematical activity that students taking these examinations are expected to engage in.

Given the widespread interest in theory and methodology among those attending the TSG, a plenary panel discussion on this topic was organised. Three presenters, Einat Heyd-Metzuyanim, Candia Morgan and Máire Ní Ríordáin were asked to identify and reflect upon the theoretical and methodological issues that had arisen for them in their research programmes, the choices they had made and the ways these choices may have affected the outcomes of the study. The presenters also questioned each other and responded to these questions and to those raised by other members of the TSG. Issues raised included the definition and operationalization of constructs, use of quantitative and qualitative methods, and the effects of language used by a researcher on the nature of data collected.

Final Reflections

Underpinning many of the presentations were the intertwined themes of politics and culture. It was repeatedly observed that language in education is inherently political, in more than one way. National or cultural politics can influence the choice of language and teaching methods, the roles language plays in the classroom, researcher access to classrooms and the uses to which research findings are put. Language is similarly influenced by culture and is an indicator of cultural identity. Politics, culture, language and teaching and learning are interrelated. Additionally, culture can influence research methodology.

Language, from the point of view of the learner, both gives and limits access to mathematics. Communicative activities in and outside the classroom shape mathematical thinking and thus language mediates access to mathematics. From the point of view of the researcher, language is both a research tool and a focus for research into mathematics teaching and learning. There is a relationship between language and learning, but also one between language and pedagogy. Analysis of communicative activities in the context of mathematics teaching and learning allows us to understand both. For successful learning to occur the teacher needs to effectively communicate mathematics, bringing issues such as open and closed discourses, specialised and everyday registers, multimodality and multilingualism to the attention of the researcher of language.

The practical topics of data collection, processing and analysis were of particular interest. Analysing language issues in the mathematics classroom can be difficult, there are methodological dilemmas and challenges. The logistics of gathering and analysing language data can benefit from further investigation, addressing issues such as how to analyse large corpuses of data when the method of analysis calls for detailed attention to small amounts of text. Large bodies of language data could benefit from being made accessible to large groups of people to work collaboratively, but that in itself brings in complications of ethics and multiple languages. Also, context is key to understanding and, in data sharing, the context of the data collection could be obscured. The role of language in the communication of mathematics is complex; in trying to capture that complexity we tend to reduce it for ease of understanding. This introduces a tension for researchers as something is inevitably lost in that reduction. Analysis of language as communication of mathematics benefits from the insights offered by cross-disciplinary perspectives, such as from linguistics.

The Topic Study Group closed with an appreciation of the small community which had formed at ICME, a hope to collaborate (and data share?) in future and a call to pool our skills and knowledge with one another.

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Gender and Education

Olof Steinthorsdottir and Veronique Lizan

Report

While mathematics are universal, it appears that delicate process in the classroom, but not only there, lead boys and girls to perceive things differently. And from this perception at school depends the future of the jobs. If the teacher, male or female, is conscious of this, what can he/she do to provide to each pupil or student, boy or girl, the opportunity of understanding, participating and finally appreciating mathematics at best?

The subject is not new: it merges explicitly at ICME3 in Karlsruhe (Germany) in 1976. «[…] Moreover, it is recommended that the theme 'Women ans Mathematics' be an explicit theme of ICME 1980.»: this ends the third and last resolution of the Congress. This recommandation became realised at ICME4 in Berkeley in 1980 and goes on since.

From the proposals received for ICME12 from all over the world, the reflection at Topic Study Group «Gender and Education» was organised along four themes: gender issues in research and learning environmental; student's achievement, assessment and classroom activities; self-efficacy and attitudes; gendered views of mathematics.

Organizers Co-chairs: Olof Steinthorsdottir (USA), Veronique Lizan (France); Team Members: Collen Vale (Australia), Laura Martigon (Germany), Sun Hee Kim (Korea); Liaison IPC Member: Cheryl Praeger (Australia).

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The subject deals with the notion of «gender», that has merged in sociology studies during 70s and it took time to work out a definition since gender doesn't reduce to sex. The term appeared in ICMI history first in 1992; it was introduced at ICME7 in Québec by IOWME.

Indeed, gender and mathematics is at the crossing of different subjects (sociology, psychology, biology or anthropology for example) what is not surprising since teaching mathematics to pupils or students generates interactions between the teacher and the classroom but also between classroom members. So it is at the same time a complex but also a completely natural subject, so natural that it can sound unrelevant.

What Do We Learn on «Gender and Mathematics» at ICME12?

The aim of the first session was to establish some basis: precisely define vocabulary, revisit the term « gender » for maths classrooms and develop a methodology to study what happens in a math class when considered from a gendered viewpoint. Indeed, crossing gender with mathematics stakes very delicate process and it is essential to circumscribe the studied objects and the way they'll be studied in any research on the subject.

The second session pointed that different social parameters impact pupils achievement to international tests or national selection process, especially those that concern family background. The type of tests or criteria of selection can also introduce unsuspected bias into selection process. Gender interfers with mathematics achievement not only in the classroom but everywhere from the moment there are human relationships, and more acutely when mathematics are assigned a role of selection, quite a social selection role.

The third session enlighted how important is the way of teaching to catch the interest of pupils—the girls of the study appreciate to be responsibilized and active—and also how important is the involvement of parents for maths studies or topics in pupils' interest for maths and their success, especially concerning girls. In maths teaching process, the content is important of course, but the manner also is of importance as well as the environment knowledge to try to equally imply most if not all pupils or students of a classroom and make them feel concerned by the maths class. Reading ability of course is also a technical factor of success for students in mathematics through their self-assessment—the best they read and the more accurately self-assessment is perceived to perform—Self-efficacy that boys and girls don't live in he same way especially during problem-solving tasks is also a parameter of importance in mathematical activity environment. It is precisely when the maths activity perturbates the pupil, the pupil's security in some way (difficult question or open problem for example) that some aspects of each pupil's personality built since childhood stake. In that sense maths activity actively participates in the personal construction of each pupil.

The fourth session pointed that children at pre-school are already submitted to gendered stereotypes during mathematics activities, and also that gender and mathematics are related to cultural parameters even if statistics show differences between boys and girls achievements in the same sense everywhere: mathematics are abstract and universal but the question is the same everywhere independantly of cultures.

Perspectives for the Future

Different gender activities were disseminated in ICME12 program and one could concoct a quite full time «Gender and mathematics» program during the congress : part of Gilah Leder's talk since gender is one of her interests; an overview «Gender and Mathematics education (revisited)»; 2 IOWME (International Organization of Women in Mathematical Education) meetings; a Girls' day organised by KWMS (Korean Women in Mathematical Sciences) and WISET (Korea Advanced Institute of Women in Science, Engineering and Technology); and of course the topic study group «Gender and Mathematics» and its four sessions. The Girls' day mentoring activity was of special interest because it involved about 110 girls and also mentors, women maths researchers or scientific engineers; it was related to the WISET stand at the Mathematical Carnival and also to activities especially for girls. Analogous days also exist in Australia, France or USA for example, and they constitute a first step to an active treatment of gender and mathematics, or more generally science, topic.

Anyway the public at the topic study group was essentially constituted by people already conscious that mathematical activity at school has not the same social meaning or psychological impact for boys as for girls. But, are all of us that teach mathematics to both female and male conscious (or convinced?) that both publics don't deal with mathematics in the same way? And how to make a math course equally attractive for boys and for girls?

Of course, ICME takes the subject of gender and mathematics into account since its very beginning. Anyway it is not a timeworn leitmotiv since the corpus on gender and mathematical education constitutes along years. On the contrary, it is necessary to wake up that the question is of importance and to become aware that it is closely related to the future of mathematics and science that lack of students for both research, engineering and technology.

Scientists are already active on the subject (Cf. Girls' day and also the work of the devoted associations). When will teachers be systematically trained to consider their pupils also as boys and girls and then when will teachers take into account in their practice gender angle to tackle their classes? And what contents for training teachers on the subject? Perhaps subjects at a plenary talk in a future ICME.

Mathematics Education in a Multilingual and Multicultural Environment

Anjum Halai and Richard Barwell

Introduction

For this topic study group, 35 papers were accepted from a range of different cultural, linguistic and country contexts. The papers were discussed under specific thematic questions. These themes provide an organizing framework for this report that draws its content from the papers and the discussion in the TSG 30 sessions. The submissions illustrated the rich diversity in the kinds of issues that arise in mathematics education in multilingual and multicultural environments. These include challenges for teaching, learning, curriculum, pedagogy, teacher education and use of technology in and for multilingual and multicultural settings. Issues were at the level of policy (e.g. language of instruction) and at the level of classrooms (e.g. teaching methods, curriculum) and teacher education (e.g. models of pre-service and teacher professional development). Diversity was also seen in terms of the geographical spread of the contexts from where papers were presented. The diversity of contexts reflects technologically advanced countries with increasingly large immigrant populations (e.g. Australia, Canada, Germany, Sweden, USA, UK), postcolonial countries with concomitant colonial languages as the medium of instruction (e.g. Ghana, Pakistan, Malaysia, South Africa, Tanzania) and countries with varied indigenous and official languages (e.g. China, India, Indonesia, Mexico,

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New Zealand). The overwhelming prevalence of issues related to quality of mathematics education in multilingual and multilingual contexts illustrates its significance.

Theme One: What Is Distinctive About Learning and Teaching of Mathematics in Multicultural and Multilingual Settings?

Presenters and participants identified several teaching strategies and distinctive elements of multilingual classrooms, highlighting potential for improving learners' mathematical skills. These included the use of group work, judicious questioning, implementation of second language teaching techniques in mathematics classrooms, promoting a positive climate in the classroom, enabling "translanguaging" i.e. to switch between the linguistic resources and cultures that learners have at their disposal (e.g. Farasani's work with British Iranian learners), and "exploratory talk" (e.g. the work of Webb and Webb in South Africa) as a vehicle to promote dialogue to enhance learners' reasoning skills in mathematics. An enduring concern for mathematics learning was students' lack of competence in the language of instruction. It was also noted that the discussion of papers in this theme emphasized issues arising specifically from multilingualism, as compared to multiculturalism.

Theme Two: What Is the Experience of Education Systems that Have Changed the Medium of Instruction in Mathematics?

Experiences were shared of learners and teachers from different country contexts where the medium of instruction was changed or different from the first language of the learners (e.g. Kasmer's and Kajoro's work in Tanzania) and multilingual classrooms with immigrant learners from several different first language backgrounds (e.g. Meyer's work with immigrant learners in Germany). For learners in multilingual postcolonial classrooms, presenters discussed several linguistically and culturally responsive teaching strategies such as the use of pictorial and other representations of mathematical ideas, situating the mathematics tasks in a familiar context, and code switching to facilitate learning. However, it was noted that there were tensions in classroom dynamics where a position of power and prestige was given to the language of instruction while learners' first language was not seen as a language of choice (e.g. Ampah-Mensah's work in Ghana).

In the case of classrooms where learners, often from immigrants communities, came from multiple language backgrounds not shared by the teacher and often not by other learners, it was concluded that an official language of the classroom was

necessary to enable communication in the whole class. However, this necessity need not preclude strategies such as small group work where learners could use their home languages. Empowering the learners to take responsibility for their learning in small groups, and looking at the outcomes of the group work, could be strategies that teachers could employ in such multilingual settings. It was agreed in the discussion that the range of strategies and methods being employed by teachers and learners in the multilingual classrooms needed to be evaluated for their efficiency and effectiveness.

Theme Three: How Can Mathematics Teaching Respond to the Oppression of Cultural and Linguistic Minorities?

Studies in this theme reported different models (e.g. the "bi-cultural curriculum model" in New Zealand presented by Jorgensen), and teaching methods (e.g. Matematika GASING Method in Indonesia by Surya and Moss) for responding to the needs of learners from cultural and linguistic minorities. While there were subtle differences in the orientation and motives of these methods and models, they were mainly premised on the view that all children can learn mathematics provided they have opportunity to do so, and that the opportunity should be to access culturally and linguistically relevant mathematics teaching and learning. It was also recognized by these proponents that language, culture and mathematics pedagogy are integrally bound in a complex relationship. The models and methods proposed certain key elements of teaching that could be employed in mathematics classrooms for learners from culturally and linguistically marginalized or minority groups. For example, exposing learners to multicultural visual representation and conceptual tools before abstract mathematics notation; ensuring "respect" for learners in multiethnic classrooms by creating ample space to listen to them and guide their thinking (e.g. Averill and Clark's work in New Zealand); and taking a "bi-cultural focus" in the curriculum that legitimizes the culture of the school and of the community. However, in the discussion an issue was raised that culture was a broad and potentially nebulous term and needed further clarity in terms of its application to mathematics education.

Theme Four: How Does/Should Teacher Education Take Account of Cultural and Linguistic Diversity?

In this strand, it was pointed out that pre-service teacher education must take account of multilingual classrooms and recognized that a vast majority of learners learn mathematics in a second or third language. Exemplars of teacher education programmes included the presentation by Prediger and team, on the notion of an

inter-disciplinary teacher education course proposing that mathematics teachers need to have didactic and linguistic knowledge and cultural sensitivity to understand the challenges that might be faced by the learners from diverse settings. Likewise interventions in teacher education provided a range of strategies and techniques that could be employed with teachers and students. These included, dialogic strategies and "exploratory talk" to promote mathematical reasoning among students, extended wait time for second language learners of mathematics, need for clarity and avoidance of slang in use of language in multilingual classrooms, utilizing learners' fluency in their main language as well as to garner the aid of a more able peer. The few studies that harnessed the potential of technology to enhance the cultural understanding and experience of learning mathematics in a second or third language included the use of video-conferencing, social media and Skype as a medium to provide experience of teaching in a multilingual setting and enhance cultural understanding (e.g. the work of Moss and Boutwell with pre-service teachers in USA, Singapore and Haiti). A conclusion was that technology provided a relatively easy opportunity for teaching mathematics within a multicultural and multilingual environment. With creativity, connections, and technology, pre-service mathematics teachers could learn about mathematics, teaching, and culture in other countries without leaving their own.

Theme Five: How Do Curricula and Policy Take Account (or not) of Cultural and Linguistic Diversity?

In this theme the focus was more on curricular processes (not necessarily curricular content) embedded in instructional sequence, pedagogy and teaching strategies for improved teaching and learning in diverse contexts. For example a teaching sequence was presented by Xaab Vasquez, based on the philosophy of "Wejën Kajën" in Oaxaca in Mexico, which encourages reflection on the prevailing education processes and the need to make explicit that learners are not isolated but are situated in a wider social and cultural context. Cooperative learning strategies were presented as an approach to create space for marginalized learners to improve achievement in mathematics. Similarly, presentations proposed differentiated instruction sensitive to the needs of minority students and "equitable strategies" that encourage collaborative knowledge production, student authority and ownership of knowledge, and mutual respect (e.g. the work of Manjula and Erchick in USA). Such strategies should be guided by the principle of reducing discontinuities between the lives of students by drawing on their cultural heritage to create an egalitarian context for supporting the learning of all students (e.g. the work of Ryoon Jin Song and team in South Korea). Use of mathematics investigations, films, print literature and internet websites were also seen as ways to accommodate cultural diversity in the classroom. The case was also presented of the International Baccalaureate Diploma Program, IB, which operates in three languages (English,

French and Spanish). It was pointed out that the IB curriculum is integrally concerned with the international dimensions of mathematics and the multiplicity of its cultural and historical perspectives, which in turn helps to discover new perspectives and horizons in international mathematical education.

Theme Six: What Theoretical Perspectives on Cultural and Linguistic Diversity Are Most Helpful in Investigating The Teaching and Learning of Mathematics?

Several theoretical frameworks and conceptual models were presented in this theme to provide tools for understanding and analyses of issues related to teaching and learning of mathematics in contexts of cultural and linguistic diversity. For example these included the presentation by Essien and team on an extension of Wenger's work on "communities of practice" for application to pre-service teacher education for multilingual mathematics classrooms. Likewise an integrated model was presented that integrates three hitherto disparate registers: those of code switching, transitions between informal and academic (mathematical) forms of language within a given language, and transitions between different mathematical representations. However, it was pointed out that further research was required to establish the efficacy of this model. Sevensson's presentation raised issues related to research methodology in ensuring that "students' voices" are heard. Barwell and team presented work that extended Bakhtin's (1981) theory of language and claimed that the theory provides a framework for looking at the tensions in mathematics classrooms in diverse language contexts but go on to state that more research is needed in this area.

Concluding Remarks

Certain key overarching questions or concerns were raised for further deliberation about the quality of mathematics education in diverse linguistic and cultural settings. These include: "Where is the mathematics in talking about the methodological, political and equity issues in multilingual and multicultural classrooms?" It was reiterated that meetings like ICME are primarily about mathematics education and therefore mathematics should be in the foreground. A concern was that meta-concepts like "culture" and "language" were employed in the discussion as if there existed a shared understanding of these concepts. However, there needs to be discussion and debate to problematize these notions and clarify their usage in mathematics education. Also it was noted that even though the title of the TSG 30 and the themes included "multilingualism" and "multiculturalism" the papers and discussion tended to focus on issues related to multilingualism.

Acknowledgments The contribution of the committee members especially Clement Dlamini, authors and participants in TSG 30 are sincerely acknowledged.

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Tasks Design and Analysis

Xuhua Sun and Lalina Coulange

Aims

A critical topic in mathematics education is the design and analysis of open-ended, realistic, and exemplary tasks. Task design and analysis is a relatively new field, appearing for the first time as a topic of study (TSG 34) at ICME-11 in Monterrey, Mexico. It is developing quickly and dynamically as an area of international attention and active research.

Topic Study Group 31 will bring together researchers, developers and teachers who systematically investigate and develop theoretical and practical accounts of task design and analysis. We welcome proposals from both researchers and practitioners and encourage contributions from all countries. Presentations and discussions will target new trends, new understanding, and new developments in research and practice.

We have a particular interest in empirically grounded contributions that underline design principles and theoretical approaches, and give examples of tasks

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designed for promoting mathematical development. We plan to discuss (but are not limited to) the following themes:

- Theoretical and practical development that guides task design and analysis
- Diverse theoretical approaches or principles that guide task design and analysis
- Diverse practical traditions/approaches that guide task design/analysis and their theoretical accounts
- Examples of task analysis for studying the relations between tasks, psychological development, and mathematical development
- Critical literature studies or meta-analysis of task design and analysis

The group will welcome contributions that focus on primary or secondary education. Research and development in task design and analysis presented at ICME-11 is retrievable at (http://tsg.icme11.org/tsg/show/35).

Organizations

On the website of ICME-12 it was possible to follow the planning process and eventually access all relevant documents including the timetable for TSG sessions. Each Session has four 90 min timeslots (on Tuesday, Wednesday, Friday and Saturday mornings). This made TSGs the prime forum for participation. We expected that participants engage in the review process prior to the conference, and we nominated respondents to all presentations in order to enable deeper levels of critical discussion during the conference. The presenters worked in pairs and made short comments or elaborated on each other's work after every presentation. In this way, TSG 31 was an active study group.

Submissions and Theme

The organizing committee received 12 submissions with 100 % acceptance rate (11 short oral presentations and 1 poster). The organizing committee assembled the accepted papers for TSG 31 into four groups for summary, presentation, and discussion:

- *Dynamic Geometry Environments and the Role of Representations*
- *Categorizations of Tasks and Textbooks*
- *Tasks Enacted by the Teacher and Students*
- *Discoveries and Justifications*

Schedule

Session 1 Tuesday, *10th July*, *10:30–12:00*, *Dynamic Geometry Environments and the Role of Representations* (*Number of attendants*: *24*).

Opening remarks: Sun Xuhua susanna and Lalina Coulange (20 min).

Mickael Edwards, Task Design and Analysis using the Measure-Trace-Algebratize Approach (25 min).

Teresa B. Neto, Xuhua Sun, Task design and analysis of on-to semiotic approach (25 min).

Eddie Chi-keung Leung, Hea-Jin Lee, Sun Xuhua (Main discussant speakers): Round-table discussion with the whole group on the 2 contributions (20 min).

Session 2 Wednesday, *11th July*, *10:30–12:00*, *Categorizations of Tasks and Textbooks* (*Number of attendants*: *32*).

Regina Bruder, Eight target structure types of Tasks as background for learning surroundings (25 min).

Hyungmi Cho, Jaehoon Jung, Ami Kim and Oh Nam Kwon, An analysis of the mathematical tasks in the Korean 7th grade mathematics textbooks and workbook (25 min).

Lianzhong Fan, Jiali Yan, Xuhua Sun, The Changes of Task Design for Development "Two-Bases" in China after Ten-year Curriculum Reform (25 min).

Hea-Jin Lee, Nguyen Chi Tanh, Lalina Coulange (Main discussant speakers), Round-table discussion with the whole group on the 3 contributions (15 min.)

Session 3 Friday, *13th July11:00–12:30 Tasks Enacted by the Teacher and Students* (*Number of attendants: 35*).

Rina Namiki and Yoshinori Shimizu, On the Nature of Mathematical Tasks in the Sequence of Lessons (25 min).

Julie Horoks, Analysing tasks to describe teachers' practices and link them to pupils' learning in mathematics (25 min).

Marita Barabash, Raisa Guberman, Multiple informal classifications of geometrical objects as an ongoing process of developing young students' geometric insight (25 min).

Eddie Chi-keung Leung, Nguyen Chi Tanh, Lalina Coulange Main discussant speakers: Round-table discussion with the whole group on the 3 contributions (15 min).

Session 4 Saturday, *14th July10:30–12:00*, *Discoveries and Justifications* (*Number of attendants*: *25*).

Michael Meyer Forming concepts through discoveries and justifications (25 min).

Celine Constantin, Lalina Coulange In search for a specific algebraic task design or how to elaborate a situation highlighting algebraic techniques in second grade (25 min).

Eddie Chi-keung Leung, Nguyen Chi Tanh, Hea-Jin Lee Main discussant speakers Round-table discussions of the session papers (15 min).

Sun Xuhua, Lalina Coulange Closing remarks: Whole group discussion on the work of the group and conclusion (25 min).

On-line Discussion Notes

https://docs.google.com/document/d/1Bll7r2tN7ha8PQrr2J3xJu5gEQ05wPveDFV83EZdpvc/edit.

Mathematics Curriculum Development

Koeno Gravemeijer and Anita Rampal

Introduction

The purpose of TSG 32 was to gather congress participants who are interested in research, policy or design that focuses on mathematics curriculum development. The TSG aimed at including presentations and discussions of the state-of-the-art in this topic area and new trends and developments in research and practice in mathematics education. Curriculum was perceived at two levels. On a national or state level, where the focus is on content and goals for the primary or secondary school mathematics curriculum. And on a more specific level of curriculum design which concerns the developmental trajectories of mathematics content and the best ways to represent them. In relation to this theme, we especially solicited papers that might foster the deliberation on the varied aims of the curriculum and bring concerns and experiences from different contexts.

The papers that were submitted could be arranged in four categories, which were used to structure the sessions:

- Authenticity and Inquiry
- Implementation

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- The Syllabus
- Math Topics

Each session consisted of one long paper and a number of short paper presentations.

Authenticity and Inquiry

The session on authenticity and inquiry started with a presentation by *Anita Rampal* (India) (with *Katie Makar* (Australia)) of a paper on the topic of embedding authenticity and cultural relevance in primary mathematics. She observed that there is an increasing need for a more democratic and universal participation in elementary school, better numeracy among citizens and mathematical competence and expertise in the workforce, but that accountability systems have often worked in opposition to these elements to further suppress authentic problems in favor of those that can be easily tested. In their paper they highlighted approaches to tackle this problem using innovative curriculum materials in two diverse contexts—India and Australia. These materials were designed with the specific intent of increasing students' opportunities for learning mathematics in ways that are relevant to their familiar and local contexts and cultures. Specifically, to increase the use of culturally relevant thematic units in Indian primary school textbooks, and to embed inquiry-based learning using authentic problems in the Australian curriculum. As half of India's children do not complete elementary education owing to the alienation they face in school, a social constructivist approach has been adopted to ensure more inclusive and democratic participation of all children. This has led to the development of new textbooks which, especially at the primary level, attempt to locate mathematics in the diverse socio-cultural contexts of children's lives. A new national curriculum in Australia has sought to align the curricula across the states and territories and to reflect a stronger focus on disciplinary knowledge and proficiencies, general capabilities and cross-curricular priorities. A seven year longitudinal study has been researching teachers' experiences and pedagogical practices as they adopt and adapt inquiry-based teaching in their classrooms, by engaging students in addressing ill-structured problems that required students to continually re-negotiate their understandings of mathematics within a rich context.

This presentation was followed by three short paper presentations:

Shelley Dole (with *Katie Makar*, and *Gillies Robyn*) (Australia) presented a paper on how the inquiry pedagogy of the intended curriculum was enacted in Australian classrooms. To answer this question, they assembled video data, classroom observations, and interviews with teachers involved in a design research-project. This concerned 40 teachers (of Grades Prep to 7) who attended three professional development meetings per year, and taught 3–4 inquiry-mathematics units per year. The teacher meetings provided the teachers with an opportunity to discuss their thoughts about and experiences with inquiry. It showed that during these meetings

teachers identified the benefits of inquiry. The classroom observations showed that the teachers were keen to undertake inquiry in their classrooms, but it showed also that inquiry is difficult for both teachers and students.

Danrong Ying (China) presented a study in which a comparison was made between inquiry tasks in three high school mathematics series in China. Two textbooks were based on the "Obligatory High School Standards", issued by the Chinese Ministry of Education, the other one was based on the "Shanghai Primary and Middle School Mathematics Curriculum Standards". The results reveal that mathematics inquiry tasks in three series mainly focused on "Number and Algebra". And even though the textbooks based on the "Obligatory High School Standards", gave various names to mathematical inquiry tasks, the actual presentation was mainly in the form of pure mathematical problems. In all three selected textbook series, the tasks labeled "experiment" all focused on using information technology to solve mathematical problems, while clear procedures were given.

Yamei Zhu (with *Yun Gan*, and *Yaping Yang*) (China) presented a paper on a comparative study of mathematics textbooks in Shanghai, Singapore and America. Some differences could be traced to the different cultural backgrounds. The Shanghai and the Singapore textbooks reflected a typical "eastern culture" and the American textbooks a typical "western culture". In the former the teacher is dominant, the textbooks offer structured and coherent knowledge, and there is an emphasis on pure mathematics which leads to "the multi-steps, logic-based and knowledge-rich mathematics problems". The American textbooks focus on what the authors qualify as "isolated and incoherent knowledge". At the same time, the USA textbooks use context problems which convey the meaning of mathematics study.

Implementation

The session on implementation started off with a presentation by *Margaret Brown* (with *Jeremy Hodgen*, and *Dietmar Kuchemann*) (United Kingdom) of a paper on changing the grade 7 curriculum in algebra and multiplicative thinking at classroom level in response to assessment data. In this presentation, the methods and results of the project were reported. Phase 1 of the project took the form of assessment of attitude and understanding in the areas of algebra and multiplicative thinking of a nationally representative sample of students in Grades 6–8 in England. The results revealed that the majority of students in Grade 8 had an understanding of ratio, which did not extend beyond scaling up by multiplication by a small whole number, while 40 % had an understanding of algebra, which did not extend beyond that of treating letters as objects or direct evaluation. Phase 2 of the project involved working with eight teacher researchers to research the understandings of their own Grade 7 students in these areas and to explore ways of improving their students' understanding. The understanding of many students was 'patchy'. To some extent this reflected a lack of connections in the understanding of the teacher researchers. This in turn

limited the possibility of formative assessment. Analysis of the recommended schemes of work and of the most popular textbooks showed that each new topic was covered rapidly and superficially, with teachers often reducing the content to routine procedures to enable students to do the class work exercises. There was no time for deep treatment of topics, discussing the power of different models/representations, relating them to connected ideas, or discussing how they could be applied to more complex problems. In Phase 3 the project is extended to more teachers using interlinked sequences of 40 outline lessons designed by the research team.

This presentation was followed by two short paper presentations and a chat with presenters.

Ji-Won Son (United States) presented a short report on a comparative study on inquiry tasks in three senior high school mathematics textbook series in China. The purpose of the paper was to examine teachers' transformation of cognitive demand of textbook problems. A survey was carried out among 183 teachers teaching from 1st to 6th Grades, of whom eight teachers were observed. It showed that the cognitive demand of the textbook problems plays an important role in deciding the cognitive demand of the problems used by the teachers, but the teachers used lower level teacher questions. An in depth and broad analysis with respect to teachers' textbook use showed that a wide variety of factors influenced the quality of instruction.

LV Shi-hu (with *YE Bei-bei* and *CAO Chun-yan*) (China) presented a study on the implementation of the new mathematics curriculum for compulsory education in Chinese mainland. Surveys were carried out in the Gansu province, among 300 primary and middle school teachers, and 1,360 students in Grades 7–9. The surveys used both questionnaires and interviews. A comparison with earlier surveys showed that the application of the Standards had increased, and that the teachers had acquired a better understanding of the Standards, even though only 20 % said to "completely understand" the Standards. The student questionnaires revealed that different teaching methods, especially cooperative learning, exploratory learning and independent learning were used by the teachers.

The Syllabus

In the third session the syllabus was the central theme. *Tamsin Meaney* (Sweden) (with *Colleen McMurchy* and *Tony Trinick*, New Zealand) presented a paper on the contested space of Maori mathematics curriculum development in Aotearoa-New Zealand. This concerned the development of the first mathematics curriculum in te reo Māori, the Māori language, in New Zealand in the 1990s and its revision in the mid-2000s. They argued that the development of national mathematics curricula in te reo Māori involved contestation, not just around indigenous knowledge and epistemology, but also around language. The authors stressed the power relationships that existed between the various actors involved in the curriculum development process. They argued that the power embedded in the Ministry of Education

allowed it to keep a firm grip on the curriculum development process, although the process was contested and in some cases subverted by Māori because of their expectations about the use of te reo Māori. There has been a strong movement amongst some Māori communities for language revitalization and growth since the 1970s. The revision of curricula was thus done with an expectation that it would be less proscriptive, supporting a more community-developed approach to the mathematics that would actually be taught in schools. This supported Māori parents' aspirations for greater fluency in their children's Māori language and opportunities to strengthen their children's tribal identities. More of the specialist mathematical terms and grammatical structures were developed so that mathematics could be taught more easily at higher levels in Maori. The 2008 curriculum minimized the linguistic confusion that arose from the introduction of many new Māori terms in the 1990s. The revised curriculum has an emphasis on mathematical communication that has clearly been indicated by the inclusion of a Māori language strand. So this process like the earlier one has contributed to the teaching of mathematics in te reo Māori.

This presentation was followed by a series of short paper presentations, and a chat with presenters.

Anette Jahnke (Sweden) presented a paper on the process of developing a syllabus, in which she presented critical reflections from a syllabus developer. She had been involved in writing the new (2011) Swedish national syllabuses for kindergarten, elementary and upper secondary school. She observed that every tenth year politicians initiate a reform, often only in one part of the school system. Usually a small number of teachers and/or teacher educators are hired to write a draft during a very short period of time, which is then sent out a number of times for reactions. Often reforms did not result in coherent syllabus from K–12. One of the reasons of failure of syllabus reform was that teachers did not understand or even mis-understood the syllabus. This resulted in very restrictive instructions to the syllabus writers.

Tomas Hojgaard (Denmark) presented a paper on what he called "The fighting of syllabusitis". He coined the term syllabusitis as a name for a disease consisting of focusing on the mastering of individual subjects. As an alternative he suggested using a set of mathematical competencies, while using a matrix structure of the relation between subject specific competencies and subject matter. He argued that such a matrix structure has proven to be a crucial element when attempting to put the competence idea into educational practice, not least because it makes it possible for teachers to take an active part in such a project and welcome it as a developmental tool.

Math Topics

In the last session we gave attention to specific math topics. *Tomoko Yanagimoto* (with *Yuichi Hayano*) (Japan) presented a paper on the teaching and learning of knot theory in school mathematics. Knot theory is studied actively world-wide,

since, even though the basis is simple, it has many unsolved problems. Furthermore, it can be related to scientific research fields, such as Genome DNA. The members of the project have written up teaching contents for pupils from elementary school to high school as a book, "Teaching and Learning of Knot Theory in School Mathematics". Experimental teaching—based on the results of the study on teaching knot theory in elementary school and junior high school—started in public junior high school in 2009 and in elementary school in 2011. It showed that knot theory was effective in helping students improve their spatial visualization, in elementary school pupils in particular. In junior and senior high schools, knot theory led students to become more engaged in their mathematical activities. Typical for this project was that mathematicians, researchers of mathematics education and professional practitioners of education cooperated, respecting and trying to understand researches of others' professional fields while creating materials for education. An expert in knot theory could indicate the value of each teaching material from the view point of knot theory. Researchers of mathematics education could indicate the value of each teaching material from the view point of mathematics education. School teachers could realize the teaching in their classrooms based on their pupils' cognition. Officials of mathematics education society could help the teachers carry out the experimental teaching in their public school by asking the director of the school.

This presentation was followed by short paper presentations.

Qintong Hu (with *Ji-Won Son*) (United States) presented a comparative study of the initial treatment of the concept of function in selected math textbooks in the US and China. They analyzed the initial treatment of the concept of function in three curricula: a US traditional text, a US Standards-based text, and a Chinese reform text. The textbook problems were analyzed on three dimensions, contextual feature, response type, and cognitive expectation. It showed that both the US traditional textbook and the Chinese textbook were designed for teacher-centered instruction. While the reform-oriented US textbook was designed for student-centered instruction. However, the US reform-oriented textbook was more similar to the Chinese textbook in putting problems in illustrative contexts, emphasizing connections, reasoning and proof.

Linda Arnold (with *Ji-Won Son*) (United States) presented a paper on a content and problem analysis on learning opportunities related to linear relationships in USA textbooks. The textbook analysis methods included both problem and content analysis. They examined examples of four types of mathematics textbooks: (1) two different commercial texts; (2) a so-called "back-to-basics" text; (2) a reform-oriented U. S. National Science Foundation (NSF) funded text; and (4) a once commonly-used historic text, published a generation ago. There were numerous similarities between the historic textbook and present day commercial texts, suggesting that little had changed over 50 years. All of the problems of the NSF-funded text involved real world context and were geared toward extended thinking, in contrast to the back-to-basics and historic texts that showed a high degree of procedural presentation. It further showed that students using commercial texts were expected to master an especially broad array of objectives.

Sunyoung Han (Korea) presented a study on the effect of science, technology, engineering and mathematics (STEM) project based learning (PBL) on students' achievement. Even though Science, Technology, Engineering and Mathematics (STEM) are critical fields to ensure a financially sound national economy, students have been under-enrolling in STEM courses. To address this problem, STEM Project Based Learning (PBL) developed an instructional method using "ill-defined tasks". The purpose of the study was to examine how STEM PBL lessons affect students' achievement in terms of four mathematical topic areas (i.e. algebra, geometry, probability and problem solving). The participants were diverse students enrolled in a small, urban, and low socio-economic high school. The study showed STEM PBL positively influenced most mathematical topics.

The Topic Study Group meeting ended with some closing remarks by Anita Rampal and Koeno Gravemeijer. It was noted that the special time for chat with presenters' gave greater opportunity for small group intensive discussions that brought out specific issues from different cultural and country contexts.

Endnote

The design and organization of the four TSG sessions was carried out collaboratively by the organizing team. Unfortunately, Cyril Julie was eventually unable to attend the Congress and the TSG.

Assessment and Testing in Mathematics Education

Christine Suurtamm and Michael Neubrand

Introduction

The purpose of Topic Study Group 33 was to address issues related to assessment in mathematics at all levels and in a variety of forms. Assessment and evaluation play an important role in mathematics education as they often define the mathematics that is valued and worth knowing. Furthermore, sound assessment provides important feedback about students' mathematical thinking that prompts student and teacher actions to improve student learning.

Our Topic Study Group sought contributions of research in and new perspectives on assessment in mathematics education that address issues in current assessment practices. Initially we saw these issues as falling into two main strands, large-scale assessment and classroom assessment. Our original call suggested that papers might address one or more of the following topics:

Organizers Co-chairs: Christine Suurtamm (Canada), Michael Neubrand (Denmark); Team Members: Belinda Huntley (South Africa), Liv Sissel Grønmo (Norway), David C. Webb (USA), Martha Koch (Canada), Heidi Krzywacki (Finland); Liaison IPC Member: Johann Engelbrecht (South Africa).

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Large-Scale Assessment

- Issues related to the development of large-scale assessments, which might include such areas as the conceptual foundations of such assessments, designing tasks that value the complexity of mathematical thinking, etc.
- Issues related to the purposes and use of large-scale assessment in mathematics.
- Issues related to the development of large-scale assessment of mathematics teachers' mathematical and pedagogical content knowledge.

Classroom Assessment

- Issues connected to the development of teachers' professional knowledge of assessment and their use of assessment in the mathematics classroom.
- Issues and examples related to the enactment of classroom practices that reflect current thinking in assessment and mathematics education (e.g. the use of assessment for learning, as learning, and of learning in mathematics classrooms)

Broad Issues

- The development of assessment tasks that reflect the complexity of mathematical thinking, problem solving and other important competencies.
- The design of alternative modes of assessment in mathematics (e.g. online, investigations, various forms of formative assessment, etc.).

We received over 50 papers from a range of countries and continents and needed to solicit assistance from committee members and others to review the papers. All papers were reviewed by at least two reviewers. The papers presented a wide variety of issues in assessment and testing and most of the papers were accepted for plenary presentations, small group presentations or poster presentations. The difficult task for the co-chairs was to create a meaningful schedule so that all of these issues could be presented and discussed within the time frame allotted for the Topic Study Group Sessions.

The work was organized into three strands:

- Strand 1: Large-scale assessment and the implications for the development of teaching and learning
- Strand 2: Classroom assessment and developing students' and teachers' knowledge
- Strand 3: Task and test design: Various perspectives

Papers were then categorized according to these three strands and after our initial meeting to introduce the topics and structure of the group, each day consisted of the presentation of plenary papers or posters that are connected to these three strands, and then a division into three subgroups with one subgroup focused on each strand. We also had a poster session with open discussion within the TSG program, as well as posters shown only in the general poster exhibition. The following presents a summary of the main themes presented and discussed in each of the strands. We have also included the ideas from the plenary papers which typically stretched over several strands.

Strand 1: Large-Scale Assessment and the Implications for the Development of Teaching and Learning

There were over 15 papers and several plenary papers presented in this strand over several days. Numerous issues emerged through the discussion of the papers. The range of papers demonstrates that mathematics education researchers are using large-scale assessment results for many different purposes and to investigate a range of complex aspects of mathematics teaching and learning in various contexts. For instance, there are comparative studies (e.g. Wo, Sha, Wei, Li and An) and studies analyzing issues in special regions (e.g. Cheung; Fengbo; Leung; Mizumarchi), studies concentrating on specific topics (e.g. Hodgen, Brown, Coe, and Küchemann), and studies of a more experimental nature (e.g. Li). Several papers that were presented illustrate the challenge of looking for broader trends or patterns across schools and districts while being careful to acknowledge and investigate local contextual factors.

Other papers discuss the use of assessments to investigate a range of factors such as students' higher order thinking skills at different levels of schooling (e.g. Bai; Zhang), gaps in knowledge (e.g. Gersten and Woodward), teacher knowledge (e.g. Shalem, Sapire and Huntley), or teaching approaches (e.g. Thompson), and to improve connections of instruction, assessment, and learning (e.g. Paek). These papers remind us that great care must be taken to ensure that the interpretations being made from the test scores are appropriate. Paper presentations and discussion suggest that using a range of methodological approaches may help to better address the complex questions being investigated in assessment in mathematics education research. For instance, cluster analysis of large-scale data can be used to find patterns in scores but methods such as case study, think aloud protocols while students respond to test items, student and/or teacher focus groups and interviews would enrich our understanding of the patterns observed. The use of a variety of methods is helpful in making sound assertions from data from large-scale assessments.

Strand 2: Classroom Assessment and Developing Students' and Teachers' Knowledge

The papers in this strand were organized into several different categories for presentation: classroom assessment in primary grades (e.g. Makar, and Fry; Hunsader, Thompson, and Zorin), assessing conceptual thinking in the classroom, and teachers' knowledge (Grønmo, Kaarstein, and Ernest). Specific topics that arose in presentation and through discussion included:

- The teachers' role and conceptions of assessment and mathematics (e.g. Esen, Cakiroglu, and Capa-Aydin; Hoch and Amit)
- Task design to elicit students' thinking (e.g. Kim, Kim, Lee, Joen, and Park)
- The students' role and responses to open, though provoking questions (e.g. Mangulabnam)
- Development of students' self-reflection, self-assessment, and self-regulation (e.g. Teong, and Cheng)
- Developing transparency, for students in particular, in classroom assessment (e.g. Semana and Santos)
- Teachers' experiences in implementing formative assessment (e.g. Koch and Suurtamm; Krzywacki, Koistinin, and Lavonen)
- Assessing conceptual understanding through alternative assessments (e.g. Türegün)

Across all of these categories was a strong emphasis on formative assessment and at the heart of most, if not all of these papers, was the desire by either researcher and/or teachers to make sense of what students are thinking and learning. The presentations attended to various ways that students' mathematical reasoning is elicited and interpreted by teachers through classroom assessment.

There was also a great deal of discussion about initiatives in various jurisdictions to improve classroom assessment and to support teachers' use of formative assessment. These initiatives included assessment resources, collaboration, professional development, and support from Ministries of Education. It was noted that this support coupled with valuing teachers' autonomy and professional judgment seemed to provide fertile ground for sound classroom assessment practices. It was noted however that this is not occurring in all jurisdictions and we discussed the differences in teacher autonomy in different countries. International forums such as ICME provide a rich setting where these comparative discussions can occur and may prompt other jurisdictions to develop new initiatives that support strong classroom assessment.

Strand 3: Task and Test Design, Various Perspectives

The core of test design is the creation of appropriate tasks. However that is a business that requires consciousness about the various purposes tests are constructed for. Therefore, tasks and test design has aspects of conceptual and practical nature, and implementation issues are also to be considered.

Thus, the sessions in this Strand addressed a rich bundle of aspects. We started with reports on studies on teachers' knowledge (e.g. Webb). One question addressed is, how knowledge and behavior come together, and how that interplay can be measured. Studying teachers' knowledge has also internationally comparative aspects, insofar as pedagogical content knowledge for teaching has to be effectively operationalized (e.g. Kaarstein).

How specific items for goals of assessment can be constructed appropriately—closed or open as well—was the topic of the next session, with contributions of Hong and Choi; Toe and de la Torre; Kwong and Ming; Kang and Lee; Hong, Kim, Lee, and Joo. Also this question has various perspectives from elementary mathematics classrooms to college-bound students; various mathematical topics have to be attended from big ideas about measurement to the issues of learning to prove; dealing with the answers of the students is decisive and ranges from descriptions to the analysis of the competencies which can be detected in the student responses by appropriate models. All these aspects require also the discussion of methodological issues.

Finally, we also discussed some broader aspects of using tests. One topic was how teachers view and use an on-line, formative assessment system and what conclusions they can draw for their teaching (e.g. Stacey and Steinle). And even broader, was the general question as to whether entrance tests to universities are necessary (e.g. Kohanova).

Concluding Remarks

There was discussion within this topic study group as to whether it should have been two topic study groups—one for large-scale assessment and another for issues in classroom assessment. The discussion concluded by recognizing that these should not be separated as it is critically important that these two groups share their issues, ideas and practices if there is to an alignment between assessment that is ongoing, such as in a classroom and assessment that is an event, such as in large-scale assessment. The participants also found that discussions across countries pointed up many similarities in issues such as teacher professional development in assessment, transparency to students, task design to elicit student thinking, and meanings given to assessment results.

Acknowledgment The contribution of the committee members, authors and participants in TSG 33 are sincerely acknowledged.

The Role of Mathematical Competitions and Other Challenging Contexts in the Teaching and Learning of Mathematics

Mariade Losada and Ali Rejali

Statement of Purpose

The organizing group and the participants come to challenging mathematics from many different perspectives, but all firmly believe that mathematics education for the twenty-first century requires all teachers, schools and extra-curricular experiences to provide structure and support that allow and entice each student and citizen to strive to reach his or her personal best in mathematics. In the words of the discussion document for ICMI Study 16, "Mathematics is engaging, useful, and creative. What can we do to make it (engaging, useful and creative mathematics) accessible to more people?"

Aims

1. To gather teachers, mathematicians, mathematics educators, researchers and other congress participants who are interested in mathematical competitions and other challenging contexts in the teaching and learning of mathematics at all levels.

Organizers Co-chairs: Maria de Losada (Colombia), Ali Rejali (Iran); Team Members: Andy Liu (Canada), John Webb (South Africa), Jaroslav Svrcek (Czech Republic), Kyung Mi Choi (Korea); Liaison IPC Member: Petar Kenderov (Bulgaria).

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2. To present research results and reports on activities that will allow the group to make an updated sketch of the state of the art, thus further developing the aims of the 16th ICMI Study, and colouring it in by addressing new trends and developments in research and practice in mathematics competitions and other challenging contexts and their effect on mathematics teaching and learning and in pinpointing research problems of special interest to the group.

In summary, the organizing team welcomed all contributions related to mathematical challenges, the state of the art, follow-up studies and the results of studies on the impact of these activities on mathematics education. The organizing team has asked those wishing to join the study group to submit a paper of between 1500 and 2500 words in length addressing issues highlighted or others that make a significant contribution to the aims and focus of the group, and they have also invited speakers to submit their papers to the WFNMC journal (http://www.amt.edu.au/wfnmc/journal.html) for possible publication in a special issue.

Questions that Could Have Been Addressed Were

Do mathematical challenges better reflect the nature, the beauty and other characteristics of the corpus of elementary mathematics, as well as the experience of doing mathematics, than ordinary school mathematics? Does this make the mathematics involved more likely to engage the learners?

Does the widespread use of calculators and computers—marvelous tools that they are—imply that mathematics education can only justify itself (aside: in as much as it prepares the learner to use a calculator or computer in an intelligent fashion, or) in as much as it is challenging, non-routine and cannot trivially be done on a calculator or computer, that is, in as much as it provides opportunities for all learners to be engaged in challenging mathematics?

How does this last question apply to in-service and future teachers? What are the needs and characteristics of teacher education with regard to challenging mathematics?

What are the implications for more challenging assessment in mathematics—both in and beyond the classroom?

Does the involvement of teachers in challenging contexts in and outside the classroom affect their behavior in their teaching mathematics?

Does the engagement of the learners in challenging contexts affect their learning ability in mathematics?

Realizations at ICME-12

The organizing team recieved 25 contributions. Each contributed paper was reviewed by at least two referees. Finally 13 papers were accepted for presentation.

Contributions from participants from all continents addressed challenging mathematical experiences in many contexts. Unfortunately one of the contributors from India and one from Iran could not participate due to lack of funding and some contributors as well as members of the organizing team were unable to attend the Congress due to programming conflicts with IMO in Argentina, and their joint interest with other TSG's especially TSG 3 (Activities and programs for gifted students).

A joint paper, given by Emily Hobbs, Kings College London, (with David Stern and Michael Obiero Maseno University, Kenya, Zachariah Mbasu, Makhokho School, Kenya, Jeff Goodman, Lycee Francais Charles de Gaulle, UK, Tom Denton, York University, Canada) focused on the motivation challenging mathematics gives to students in Kenya to continue their studies on the university level. The talk was titled" Report on the 1st Maseno Maths Camp: a mathematics popularisation event in Kenya". It introduced a mathematics camp in Kenya, developed from the need to create a forum where mathematics could be discussed and explored at the secondary level in such a way as to show that there is more to mathematics than calculations and correct answers, mentioning that the aim of the camp was to expose young minds to new ideas in mathematics relevant to the world they live in. They reported a one-week programme which was developed for school students focusing on problem solving, promoting play, experimentation, and using computers to explore mathematical ideas. Their goal was to spark a life-long love for mathematics in students, which will both improve their performance in school and increase the chances that they will pursue mathematics and science in the longer term. They mentioned that participation of school teachers was also encouraged in order to expose them to innovative teaching methods and computer resources. The structure and content of the 1st Maseno Maths Camp 2011, the future of this camp and plans for Mini Maths Camps around schools in Kenya were explained in this presentation.

From India, we learned of challenging mathematics for students from deprived backgrounds through the paper titled "Turning Tension into Thrill (of joy), Tournament as a Tool—a case study" prepared by Arundhati Mukherjee, The New Horizon School, India. The speaker was unable to attend ICME12, but her contribution remains part of the scene sketched there.

The use of the Internet in reaching out to students with mathematical challenges was highlighted by Susana Carreira, Sciences and Technology, University of Algarve, Portugal in conjunction with Nelia Amado, of the same university and Rosa Antonia Tomas Ferreira, from the University of Coimbra, Portugal, as well as Jaime Silva, also from the University of Coimbra in their paper titled "A web-based mathematical problem solving competition in Portugal: Strategies and approaches". After each problem is posted students have two weeks to submit their answers

either through their personal e-mail or on the webpage platform. As speaker Susana Carreira mentioned some more details of the competition and discussed the results. Mark Applebaum of Kaye Academic College of Education, Israel, (in a paper prepared jointly with Margo Kondratieva, Memorial University, Canada and Viktor Freiman, University de Moncton, Canada,) gave results from the Virtual Mathematical Marathon, addressing student and family participation, and the general enthusiasm it has raised especially in Israel's Jewish community. The study, although not the presentation itself, concentrated on the following questions: what are participation patterns in an online problem-solving competition by boys and girls and how successful were participants according to the gender, and how their intermediate result related to their further participation in the event.

In a related theme, Yahya Tabesh from the Sharif University of Technology in Iran (along with Abbas Mousavi of the same institution) explained an internet resource that allows students to search and learn from ingenious solutions published on the Internet He introduced a new tool called "How to iSolve It!" which develops problem solving over the net as a smart system. He mentioned that iSolve is a reference system for problem solving which through a wiki on a social network would develop skills. He explained that the system could be referred on the net as a retrieval information system with smart algorithms which lead to more advanced results. Finally he introduced the iSolve system properly and some pilot results were also presented in his talk.

"Developing a much more challenging curriculum for all" was the theme treated by María de Losada of the Universidad Antonio Nariño, Bogotá, Colombia, a proposal to make challenging mathematics an integral part of the mathematics curriculum. In her presentation she reported on research regarding the construction of a much more challenging curriculum for students of grade six, based on her and her colleagues' own research as well as that of many other mathematics educators who have analyzed basic research, the panorama of failure and the spectrum of success.

Alexander Soifer from the University of Colorado, USA, brought the group's attention to the relationship between mathematical challenges and research in mathematics itself in his talk titled "The goal of mathematics education, including competitions, is to let students touch "real mathematics"; We ought to build that bridge". Professor Soifer maintained that as in "real" mathematics, Olympiad problems ought to include not just deductive reasoning, but also experimentation, construction of examples, synthesis in a single problem of ideas from various branches of mathematics, open ended problems, and even open problems. Olympiad problems should merit such epithets as beautiful and counter-intuitive. He explained problem creation and research subjects drawn from problems in competitions and connections between the following: mathematics olympiads, open problems, synthesis, construction, example, and mathematics, research.

From the Russian Federation the group had the experience of listening to the paper "South Mathematical Tournament: Tasks and Organization Hints" read by Daud Mamiy from Adyghe State University written jointly with Nazar Agakhanov, Moscow Institute of Physics and Technology. As the title implies an innovative

tournament nevertheless true to the traditional roots of original and challenging problem solving gave many new ideas to the participants of TSG 34. The South Mathematical Tournament has been held in Orlyonok Children's Recreational Center on the Caucasus Black Sea coast. The authors mentioned that the tournament has been administered and organized by Adyghe State University. The report showed that this tournament is a team mathematical contest structured as a series of mathematical battles between secondary school students representing various regions of Russia, that the team members are usually students who are well prepared and possess the experience of participation in competitions at various levels. In this presentation the authors claim that every year the tournament includes some of the winners and awardees of National and International Mathematics Olympiads and candidates to Russia's national mathematics team. The scheme and some of its problems were explained in detail in the presentation.

The experience of organizing a competition simulating investment strategies for university students of the administrative sciences was recounted by Yahya Tabesh in a paper titled PitGame and prepared in collaboration with Mohammad H. Ghaffari Anjadani and Farzan Masrour,also of the Sharif University of Technology, Iran. He reported that PitGame minimizes the downsides and obstacles of contests through a sort of double creativity which seems to cause a fresh environment stressing the joy of problem solving. He mentioned that contests are mainly a competitive learning activity and it is usually on an extracurricular level that creativity and problem solving skills are developed. He claimed that competitive learning could assist educators in discovering students' abilities and creativity as well as improve students' skills, and that it would support improvement of the educational system too. He explained that this problem solving contest is based on competitive learning, game theory, and role playing.

Typical of research relating into the impact of participation in high-level mathematical problem-solving competitions, Kyung-Mi Choi of Korea and the University of Iowa, USA, in a paper prepared with Laurentius Susadya also of the University of Iowa informed the TSG34 group of "Impacts of Competition Experiences on Five IMO Winners from Korea".

Conclusions

Ali Rejali as one of the group's co-chairs opened the first session and María de Losada the second co-chair made the closing remarks. One theme running throughout the wide variety of experiences and research presented is the motivational quality of challenging mathematics on all levels, allowing each student to contribute his own ideas, benefit from the ideas of his peers and "own" the mathematics being developed through the solving of original and non-routine problems. A more challenging mathematics curriculum for all is being developed. Use of the Internet is becoming more prevalent, reaching out to students everywhere and sometimes getting families involved. Much research focuses on results

analyzed by groups distinguished by gender, by level of competitivity, or by social and economic background. The heart of mathematical competitions are the problems created and posed, intimately related to the driving force in the creation of mathematics itself., but it is the students and teachers, and the transformation of their relationship to mathematics that drives the activity of this study group.

The History of the Teaching and Learning of Mathematics

Fulvia Furinghetti

History of Mathematics Teaching and Learning

History of mathematics teaching and learning is a subject that concerns two domains of research and may generate fruitful synergies between them. In 2000, during the International Symposium celebrating the centenary of the first international journal on mathematics teaching (*L'Enseignement Mathématique*), the interplay between the present educational problems in mathematics and their historical evolution through the twentieth century brought to the fore the potentialities of the field of research, "History of mathematics teaching and learning," not only for historians, but also for educators, see Coray et al. (2003). This field of research became particularly visible at ICME-10 in 2004 at Copenhagen, where a Topic Study Group (TSG 29) was dedicated to it, see Special issue (2006), Schubring and Sekiguchi (2008). History of mathematics education then became a subject of talks and workshops in various international meetings, for instance at the European Summer Universities (ESU-4 in Uppsala in 2004, ESU-5 in Prague in 2007, ESU-6 in Vienna in 2010), and at the Congresses of European Research in Mathematics Education (CERMEs). During the TSG 38 at ICME-11 in 2008 in Monterrey, research into this topic again proved its productivity, with papers presented on the history of the reform movements, on the analysis of classical textbooks, and on historical practice (inside and outside institutions), see Special issue (2009). In 2008 the celebration of the centenary of International Commission on Mathematical Instruction (ICMI), also emphasized the importance of the dialogue

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between the present and the past in mathematics education, see Menghini et al. (2008). In 2006 the first international journal devoted to this field of study, the *International Journal for the History of Mathematics Education*, was launched. Recently, specialized international research symposia took place in Iceland (2009) and in Portugal (2011), see Bjarnadóttir et al. (2009, 2012).

On the occasion of ICME-10, a first international bibliography of research in the field was prepared. The bibliography is now retrievable at the following address: http://www.icme-organisers.dk/tsg29/BiblTSG.pdf.

This bibliography outlined streams in research: transmission and socio-cultural reform movements; aspects of teaching practice (textbooks, methods, teacher professional development); cultural, social and political functions of mathematics instruction; and comparative studies.

History of Mathematics Teaching and Learning at ICME-12

Following the already established tradition of research in history of mathematics education, the International Program Committee of ICME-12 included in the scientific program a TSG 35 entitled "The history of the teaching and learning of mathematics". In the announcement of the conference the following possible themes were proposed:

- changes and roles of teachers' associations
- changes of curricula in the various countries
- changes of mathematics education as a professional independent discipline
- general trends in the organizing of the lesson
- interdisciplinarity and contexts
- methods
- policies in teacher education
- reforms movements
- the cultural and social role of mathematics
- the overall impact of digital technologies in the learning and teaching of mathematics
- the role of textbooks in the teaching and learning of mathematics
- the situation of journals on mathematics education
- treatment of particular topics (geometry, algebra, etc.)

Four timeslots of one and one-half hour each were allowed to the TSG 35. Among the submitted papers the following were selected for the presentation at ICME-12, see *ICME-12 Final Program* (2012). The full texts are reported in *ICME-12 Pre-Proceedings* (2012):

- Amy Ackerberg-Hastings (UMUC and NMAH, US). Teaching Mathematics with Objects: The Case of Protractors
- Senthil Babu (French Institute of Pondicherry, India). Learning of Mathematics in Nineteenth Century South India

- Kristín Bjarnadóttir (University of Iceland, Iceland). The Implementation of the 'New Math' and its Consequences in Iceland. Comparison to its Neighbouring Countries
- McKenzie (Ken) A. Clements, and Nerida F. Ellerton, (Illinois State University, US). Early History of School Mathematics in North America, 1607–1861
- Gregg DeYoung (The American University in Cairo, Egypt). Evangelism, Empire, Empowerment: Uses of Geometry Textbooks in 19th Century Asia
- Viktor Freiman (Université de Moncton, Canada) and Alexei Volkov (National Tsing Hua University, Taiwan). Common Fractions in L.F. Magnitskii's *Arithmetic* (1703): Interplay of Tradition and Didactical Innovations
- María Teresa González (University of Salamanca, Spain). Notebooks as a Teaching Methodology: A Glance through the Practice of Professor Cuesta (1907–1989)
- Alexander Karp (Teachers College, US). Russian Mathematics Teachers: Beginnings
- Kongxiu Kuang (Southwest University, China), Yimin Xie (Jinan University, China), Qinqiong Zhang (Wenzhou University, China), Naiqing Song (Southwest University, China) Development, Problems and Thoughts of New China (PRC)'s Mathematics Education
- Snezana Lawrence (Bath Spa University, UK). The Fortunes—Development of Mathematics Education in the Balkan Societies in the 19th Century (Distributed paper)
- Lucieli M. Trivizoli (Universidade Estadual de Maringa, Brazil). Some Aspects of Scientific Exchanges in Mathematics between USA and Brazil
- Alexei Volkov (National Tsing Hua University, Taiwan). Scholarly Treatises or School Textbooks? Mathematical Didactics in Traditional China and Vietnam

Alexander Karp presented the *Handbook on the History of Mathematics Education*, edited together with Gert Schubring (University of Bielefeld, Germany and U.F.R.J., Brazil). About 40 distinguished scholars from all over the world have agreed to participate in this major project. The publisher of the book is Springer-Verlag. This *Handbook* is a real landmark in the development of the theme in question.

It is worth mentioning other activities related to the theme of TSG 35 that enriched the panorama of the themes treated.

Regular Lecture

RL5–9, Marta Menghini (University of Rome La Sapienza, Italy). From Practical Geometry to the Laboratory Method: The Search for an Alternative to Euclid in the History of Teaching Geometry. See the text in *ICME-12 Pre-Proceedings*.

Posters and Oral Presentations

- Tanja Hamann and Barbara Schmidt-Thieme (Germany). "Macht Mengenlehre Krank?": New Math at German Primary Schools
- Sanae Fujii (Japan). Mathematics Teaching Using "Sanpou shojyo (Algorithm Girl)" for Junior High School Students
- Sung Sook Kim (Korea). Seok-Jeong Choi and Magic Squares
- Shinya Itoh (Japan). Structure of Didactical Principles in Hans Freudenthal's Didactics of Mathematics, Oral Presentation.

The abstracts are in *ICME-12 Pre-Proceedings*. The contributions cover important subjects of mathematics education:

- physical devices for teaching mathematics
- teacher professional development
- systems of instruction
- exchanges between countries
- reforms
- textbooks
- treatment of parts of mathematics
- eminent people in mathematics education.

Both specificity of national contexts and internationality of themes inherent in mathematics education were treated in the presentations and the discussions.

Final Remarks

We know that the vision and mission that inspired the journal *L'Enseignement Mathématique* and afterwards ICMI enhanced internationalization and communication in the world of mathematics education, see Furinghetti (2003). These goals were pursued throughout the ICMI's existence and, in particular, ICME conferences have been a powerful means for realizing them, see Furinghetti and Giacardi (2008). TSG 35 and the related activities are an example of internationalization and communication among researchers. All five inhabited continents have presented contributions to the history of mathematics education: Africa (Egypt), Asia (China, India, Japan, Korea, Taiwan), Europe (Germany, Iceland, Italy, Spain, UK), North America (Canada, US), and South America (Brazil).

In spite of the limitation of the scheduled time, the contributions at ICME-12 on the history of mathematics teaching and learning have allowed reflection on the double aspect of this topic. On the historical side, they showed that the present situation of mathematical education does not come out of the blue but has old roots and accompanied the growth of civilizations and societies. On the educational side, history offers to educators a different point of view for looking at educational

problems and provides insights into possible solutions. Then, really, we may see the history of mathematics education as a bridge between the past and the future.

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The Role of Ethnomathematics in Mathematics Education

Pedro Palhares and Lawrence Shirley

Report

Kay Owens (Charles Stuart University, Dubbo, NSW, Australia) presented a paper illustrating how schools can change when funds are available to assist schools and communities to implement appropriate and effective professional development, to establish partnerships between school and community, to revise teaching approaches and curriculum, to overcome disadvantage, and to value family and Aboriginal cultural heritage. She stressed that the people involved and their planning are critical for transformation. The schools were in a Smarter Stronger Learning Community so they supported each other across schools but other programs in the various schools were also important in achieving change.

Zhou Chang-jun, Shen Yu-hong, Yang Qi-xiang (Dehong Teachers' College) presented a paper about Dai ethnic mathematical culture, which is an important part of Dai ethnic culture. Mathematical elements show in their daily life. Through a research project of the Yunnan Dehong Dai people in southwest China, they collected the first-hand information, tried to do a small investigative study, and collected mathematics teaching resources that are useful to primary and secondary schools students on mathematics learning in this minority areas.

Organizers Co-chairs: Pedro Palhares, Lawrence Shirley; Team Members: Willy Alangui (Philippines), Kay Owens (Australia), Paulus Gerdes (Mozambique), Ho Kyung Ko (Korea); Liaison IPC Member: Bill Barton (New Zealand).

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Annie Savard (McGill University) discussed problems of bridging the Inuit culture of northern Canada with the official and cultural requirements of Canada's school mathematics curriculum, especially when goals seem to clash.

Igor Verner, Khayriah Massarwe and Daoud Bshouty (Technion – Israel Institute of Technology) presented a paper discussing pathways of creativity and focusing on the one going through practice in creation and analysis of useful and mathematically meaningful artifacts. They propose to involve prospective teachers in practice of construction and analysis of geometric ornaments from different cultures as well as in teaching geometry. They considered perceptions and attitudes that triggered students' creative learning behavior in this context.

Milton Rosa and Daniel Clark Orey [Universidade Federal de Ouro Preto (UFOP)] think that the application of ethnomathematical techniques and tools of modeling allows us to examine systems taken from reality and offers us an insight into forms of mathematics done in a holistic way. According to them, the pedagogical approach that connects a diversity of cultural forms of mathematics is best represented through ethnomodeling, a process of translation and elaboration of problems and the questions taken from academic systems. Seen in this context, they attempted to broaden the discussion of possibilities for the inclusion of ethnomathematics and associated ethnomodeling perspectives that respect the social diversity of distinct cultural groups with guarantees for the development of understanding different ways of doing mathematics through dialogue and respect.

Karen Francois [University Brussels (Vrije Universiteit Brussel)] and Rik Pinxten (University Ghent) started by the statement from the Vygotsky and the Cultural psychology approach (M. Cole) that 'learning is situated, socioculturally contextualized'. Learning happens in the space of background/foreground (of the learner) in his or her particular environment of experience. Math learning implies an implicit understanding, categorizing and conceptualization of reality. e.g., set theory implies intrinsically a part-whole framing of reality. They think that the tremendous dropout from math classes and the structural gap between good and bad performers (PISA) is caused by disregarding the linguistic and socioculturally formatted background/foreground of the learners. They want to use anthropological study in the classroom to know/map the child's background/foreground and adapt the entry into mathematics courses accordingly, hence their option for multimathemacy.

Maria do Carmo S. Domite (Faculty of Education, University of São Paulo) (electronically) presented an attempt to make possible an approach between ethnomathematics and the mathematics learning processes in the scholar context—however it does this from an ethnomathematician's point of view, not that of a Cognitive Psychology studious. She therefore focused on two notions of the mathematics education processes: the notion understood as the student's "prerequisite" and the notion of the teacher's "listening". She brought to the centre of discussion that the teacher should know to understand the students´ initial mathematics knowledge—how he/she uses them-, as well as know how to listen to what the students have to say—respecting the cultural and social differences in order to help them build a more critical and elaborate thinking about mathematics ideas.

Andrea V. Rohrer (Universidade Estadual Paulista) and Gert Schubring (Universidade Federal do Rio de Janeiro) started to remember that since the creation of the International Study Group on Ethnomathematics, several researchers have debated on how could or should a theory of ethnomathematics exist, and, if so, how it is to be conceptualized. So far, there exists no consensus on how this theory should be defined. During the last International Conference on Ethnomathematics (ICEM-4) in Towson, Maryland (July, 2010), Rik Pinxten emphasized on the necessity of reopening this debate. Ethnomathematics will only be acknowledged by other scientific communities if we, as ethnomathematicians, are able to establish a proper conceptualization of this field of study. They presented one possible approach to a conceptualization of a theory of ethnomathematics a theory that needs to be regarded as an interdisciplinary discipline that covers theories from both the exact and social sciences.

Alexandre Pais (Aalborg University) and Mônica Mesquita (University of Lisbon) consider that the push to marry off local and school knowledge has been a growing concern within educational sciences, particularly in mathematics education where a field of studies by the name of ethnomathematics has been producing research around the uses people do of mathematics outside school's walls. Notwithstanding the good will of educational agents in bringing to schools local knowledges, criticisms have been made on the sometimes naive way in which such a bridge is theorized and implemented. After a brief description of these criticisms, they presented the Urban Boundaries Project as an attempt to avoid the inconsistencies of schooling, and the promotion of a non-scholarized ethnomathematics.

Joana Latas (EBI/JI de Aljezur, CIEP-U. Évora) and Darlinda Moreira (Universidade Aberta) claim that the integration of cultural aspects in curricula is a means of legitimizing students' experiences and of answering to the cultural diversity in favor of a meaningful mathematical learning. (e.g. Bishop 2005; Gerdes 2007; Moreira 2008). They attempted to highlight the role of cultural mathematics in the development of the predisposition to establish mathematical connections. Such an objective was framed in a broader investigation (Latas 2011) in which a curricular project was developed, whose conceptualization followed an ethnomathematical approach. The results suggest that students: (i) appropriated cultural distinct practices through the relation that they established with their previous knowledge; (ii) gradually revealed a greater predisposition to establishing mathematical connections; (iii) deepened local and global mathematical knowledge in the interaction between both dimensions.

Roger Miarka (Universidade do Estado de Santa Catarina, Brazil) and Maria Aparecida Viggiani Bicudo (Universidade Estadual Paulista, Brazil) presented a paper, based on a PhD research, aiming to discuss the conception of mathematics, and its developments in terms of methodology, of five preeminent ethnomathematics researchers: Bill Barton (University of Auckland, New Zealand), Eduardo Sebastiani Ferreira (Universidade Estadual de Campinas, Brazil), Gelsa Knijnik (Universidade do Vale do Rio Sinos, Brazil), Paulus Gerdes (Centro Moçambicano de Investigação Etnomatemática, Mozambique) and Ubiratan D'Ambrosio (Universidade Bandeirante de São Paulo, Brazil). The research was carried out

under a phenomenological perspective, and its methodology involved an interview with each of the above-mentioned researchers. Theses interviews were analyzed hermeneutically, and through phenomenological reductions, thematic categories were articulated. In this presentation they brought the category about the presence of mathematics within ethnomathematics.

Also, there were several posters and short presentations, including reports from Portugal (Pedro Palhares), Tibet (Xiawu Cai Rang), Nepal (Bal Luitel and Amril Poudel), Philippines (Rhett Latorio), China (Xueying Ji), Mozambique (Marcos Cherinda) and Zambia (Mitsuhiro Kimura), especially on details of local mathematics and applications of local culture in school mathematics.

Marcos Cherinda made a special presentation, inviting participants (and all interested in ethnomathematics) to attend the Fifth International Conference on Ethnomathematics (ICEM-5), to be held in July 2014 (specific date to be announced), in Chidenguele, Gaza, Mozambique.

Participants

There were thirty-five participants (from twenty-two countries) in the TGS-36 sessions: Maria Aparecida Bicudo (Brazil), Bill Barton (New Zealand), Marcos Cherinda (Mozambique), Sandy Dawson (USA), Tournés Dominique (France), Cris Edmonds-Wathen (Australia), Karen François (Belgium), Kgomotso Garegae (Botswana), Kangu Hyun Jin (Korea), Jason Johnson (United Arab Emirates), Traore Kalifa (Burkina Faso), Jiyeon Kim (Korea), So Yoang Kim (Korea), Mitashiro Kimura (Japan), Ho Kyung Ko (Korea), Rhett Latonio (Philippines), Joana Latas (Portugal), Chan Gyu Lee (Korea), Bal Luitel (Nepal), Danilo Mamangon (Micronesia), Roger Miarka (Brazil), Epi Moses (Palau), Kay Owens (Australia), Alexandre Pais (Denmark), Pedro Palhares (Portugal), Amrit Poudel (Nepal), Andrea Rohrer (Brazil), Annie Savard (Canada), Lawrence Shirley (USA), Edmir Terra (Brazil), Koichi Tomita (Japan/Malaysia), Rhoda Velasques (Philippines), Igor Verner (Israel), Lim Byong Yang (Korea), Hossein Zand (United Kingdom).

Theoretical Issues in Mathematics Education: An Introduction to the Presentations

Angelika Bikner-Ahsbahs and David Clarke

Report

In the last decade, the issue of theories has been raised more than once in international conferences on mathematics education (e.g. PME 2005, 2010; CERME since 2005; ICME 11). Since 2006, a European group for the networking of theories has researched the question how mathematics education can deal with different theories (Bikner-Ahsbahs et al. 2010). All these events have shown the diversity of theories to be inherent in mathematics education. TSG 37 of ICME 12 has gathered an up-to date version of the state of the reflection on theories with respect to the theoretical questions and underpinnings of the international field, at the same time stimulating insightful exchanges and discussions crossing theoretical cultures within mathematics education. Group discussions addressed the following issues: What theories do we need in mathematics education? What do they have to cover according to the conditions, roles and functions of theory use and development, and how can we deal with the diversity of theories in a scientific fruitful way?

Radford (2008) provided a meta-theoretical frame for theories. Referring to Lotman (1990), Radford characterized the space of theory cultures as a semiosphere: a dynamically evolving space. According to Radford, theory is a dynamic way of understanding, provided and performed on the basis of a triplet (P, M, Q):

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the set P of principles of the theoretical culture, the set M of methodologies that refer to P including methods and the set Q of paradigmatic questions in the core of the theory, its P and M. The developmental dynamic is constituted by research and the exchange of research results R referring back to the triplet (P, M, Q) (Radford 2012). This understanding of the terms *theory* and *semiosphere* provides a framework for understanding the connection of theories as exchanges and dialogues between their parts. Connections can be created among the theories parts [(P, M, Q), R] and structured according to their degree of integration in the landscape of networking strategies: *understanding and making understandable*, *comparing and contrasting, combining and coordinating, and locally integrating and synthesizing* (Bikner-Ahsbahs and Prediger 2010). The first two pairs can always be done but the third and the fourth pair can only be executed if the principles of the theories are close enough. TSG 37 offered an introduction about the networking of theories, eight long presentations, five short presentations and short statements about the posters within four sessions. The first session involved questions concerning how theories from outside mathematics education might fruitfully inform the dynamics of research within mathematics education; in particular, it addressed the challenge of identifying theories suitable for use in mathematics education, contrasting the treatment of particular constructs relevant to mathematics education within two or more theories, suggesting inadequacies in the capacity of currently available theories to meet the needs of mathematics education, and recommending what developments are required. It was established early in the discussion that no single theory can claim to be comprehensive and so all theories are consequently partial and selective in their focus and the phenomena they describe and attempt to explain (Clarke 2011). Two presentations were discussed. Knijnik positions culture at the heart of teaching and learning mathematics and addresses the issue of Ethno-mathematics as an offer to think of cultural differences in grammar and logic. Her approach can be regarded as a coordination of two background theories rooted in the work of Wittgenstein and his language games and of Foucault and his work on how discourse establishes truth in the culture. Pais and Valero point to the demand of mathematics education for all and recommend the inclusion of economic considerations in the theories employed in mathematics education. According to Pais and Valero, current theories do not accommodate these concerns. The two contributions offer different perspectives: according to Pais and Valero, we have to be more open in the direction of political and economic value of mathematics, and, according to Knijnik, we must see philosophies of teaching and learning as parts of the distinct cultures from which they have developed and in which they are applied.

The second session investigated the role and function of theories in mathematics education (and mathematics education research), their capacity to provide insight into one or more different contexts or issues in mathematics education, and the methodological entailments of selecting particular theories in the process of research and design. Drawing on Vygoskian theory, Albert positioned learning mathematics as a cultural-historical activity mediated by a sign system and applied this to serve teaching practice by the use of algebra tiles. Hatfield asked how a theory could be built to capture lived mathematical experience in order to

investigate this phenomenon. He included two views, the phenomenological and the constructivist view, to start building a theory. Trninic and Kim adopt a radical position on embodied cognition. They regard learning mathematics to be cognitively embodied and employ this in the design of computer-based environments. To do so, theory and design have to co-inscribe, e.g. they mutually inform and entail each other. These three presentations accorded theory different roles and functions: (1) theory as a source of models to be applied to the practice of teaching and learning, (2) theory constructed to understand a specific phenomenon, and (3) theory as informing instruction to be co-developed with design towards a specific goal. The researcher's perspective on learning mathematics and on the aim and function of research determines what kind of theory is considered suitable. Hatfield grappled with the new idea of lived mathematical experience within learning suggesting that this focus has to be intensively theorized before it brings a theory to the fore. Trninic and Kim reconceptualised embodied cognition by situating it in design experiments and by looking at the theory in a new way. Albert used research results from a background theory with a long tradition (Vygotski's social psychology) employed in mathematics education to inform practice.

Session three discussed the question of how to deal with different theories in a scientific way, addressing the challenge of utilising the results of research studies in mathematics education undertaken using different theories. The generic term "networking" was employed to include strategies such as connecting, comparing, contrasting, combining, coordinating, integrating, and synthesising. Such strategies are intended to provide heightened insight into a complex setting. The session involved reporting examples of the networking of theories, their limits and their potential for advancing mathematics education. Three presentations addressed these issues. Even investigated the same data set from two philosophical traditions: constructivist theory, investigating learning by looking at cognitive development; and activity theory, used to investigate the teacher's participation. She showed that the object of investigation and the research questions were different according to the particular theory and therefore the results of the analysis and the answers to the research questions were also different, but complementary and mutually informative. She inferred from this that the use of more than one theory demands parallel lines of research and meta-theoretical exchanges. Trigueros et al. undertook a theoretical study to investigate the different meanings of *mathematical object* in Action-process-object-schema (APOS) theory and in the onto-semiotic approach. She showed that some concepts within one theory could be interpreted by the other, suggesting that these concepts could be associated with measures or results amenable to comparison, that is, they were commensurable, whereas results associated with other concepts might be incommensurable but compatible because they were not mutually contradictory; but could be seen as disjoint or complementary. The relationship between different theories cannot be simplistically categorised as either commensurable/incommensurable or as compatible/incompatible. Theories may partly overlap and may be mutually informative to some extent. The issue of limits was raised by the third presentation from Kidron and Monaghan, which discussed the complexity of dialogue between theories. In this presentation, Kidron showed

that dialogues as exchanges between theoretical cultures can be regarded on two levels: (i) the cultural level of theories participant in research processes—a possible mechanism to network theories; but also (ii) the individual level, where researchers work with different theories within one project and must forge connections between theories in the process of constructing their findings. Both data and results are constructed within research through methodologies that reflect the choice of theories. In this way, the networking (combination or juxtaposition) of theories might lead to uncover blind spots in the making of data and in the analyses, clarify the theories' boundaries but also advance research through enriching results.

The use of multiple theories (and associated parallel analyses) in a single research project can serve several purposes (Clarke 2011):

- By addressing different facets of the setting/s and providing a richer, more complex, more multi-perspectival portrayal of actors and actions, situations and settings;
- By offering differently-predicated explanations and differently-situated propositions;
- By increasing the authority of claims (and instructional advocacy), where findings (both explanations and emergent propositions) were coincident across analyses;
- By qualifying the nature of claims, where findings of the parallel analyses were inconsistent or contradictory (cf. Even's analyses of "the same data");
- By providing a critical perspective on the capacity of each particular theory to accommodate and/or explain data related to the same events in the same setting.

In session four, the evolving discourse could be used to discuss short presentations, reporting studies with multiple theoretical perspectives on mathematical imagination (Aralas), on mathematical visuality (Flores), and on concept formation (Rembowski). Rosa and Aparecida addressed philosophical considerations regarding mathematical technology and Kusznirczuk suggested according theories the status of organising principles for coordinating the objects that populate our discourses and our methods.

In summary: Since research questions are intimately connected with the theoretical frameworks in which they are elaborated, it may appear problematic to use different theories to answer the same research question. However, different theories may usefully address different questions about the same setting (e.g. the mathematics classroom) or even the same issue (e.g. the instructional use of representations). Researchers should draw on the expertise of the various theoretical cultures to enrich the general discourse of the mathematics education community and respond to society's major questions at an appropriate level of complexity. The discussion raised the question up to what point researchers might be able to consciously choose a theory or a theoretical paradigm for their research and it brought to the fore the criteria under which theories might be evaluated: the dichotomy true/wrong was contrasted with the useful/useless one, and the concepts of validity and viability were considered. The work carried out in this TSG has constituted a small but solid step in fighting back this danger for mathematics education.

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