# Part II
# Fixed Income Modeling

# Multi-curve Modelling Using Trees

**John Hull and Alan White**

**Abstract** Since 2008 the valuation of derivatives has evolved so that OIS discounting rather than LIBOR discounting is used. Payoffs from interest rate derivatives usually depend on LIBOR. This means that the valuation of interest rate derivatives depends on the evolution of two different term structures. The spread between OIS and LIBOR rates is often assumed to be constant or deterministic. This paper explores how this assumption can be relaxed. It shows how well-established methods used to represent one-factor interest rate models in the form of a binomial or trinomial tree can be extended so that the OIS rate and a LIBOR rate are jointly modelled in a three-dimensional tree. The procedures are illustrated with the valuation of spread options and Bermudan swap options. The tree is constructed so that LIBOR swap rates are matched.

**Keywords** OIS · LIBOR · Interest rate trees · Multi-curve modelling

## 1 Introduction

Before the 2008 credit crisis, the spread between a LIBOR rate and the corresponding OIS (overnight indexed swap) rate was typically around 10 basis points. During the crisis this spread rose dramatically. This led practitioners to review their derivatives valuation procedures. A result of this review was a switch from LIBOR discounting to OIS discounting.

Finance theory argues that derivatives can be correctly valued by estimating expected cash flows in a risk-neutral world and discounting them at the risk-free rate. The OIS rate is a better proxy for the risk-free rate than LIBOR.[1] Another argument

[1] See for example Hull and White [15].

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(appealing to many practitioners) in favor of using the OIS rate for discounting is that the interest paid on cash collateral is usually the overnight interbank rate and OIS rates are longer term rates derived from these overnight rates. The use of OIS rates therefore reflects funding costs.

Many interest rate derivatives provide payoffs dependent on LIBOR. When LIBOR discounting was used, only one rate needed to be modelled to value these derivatives. Now that OIS discounting is used, more than one rate has to be considered. The spread between OIS and LIBOR rates is often assumed to be constant or deterministic. This paper provides a way of relaxing this assumption. It describes a way in which LIBOR with a particular tenor and OIS can be modelled using a three-dimensional tree.[2] It is an extension of ideas in the many papers that have been written on how one-factor interest rate models can be represented in the form of a two-dimensional tree. These papers include Ho and Lee [9], Black, Derman, and Toy [3], Black and Karasinski [4], Kalotay, Williams, and Fabozzi [18], Hainaut and MacGilchrist [8], and Hull and White [11, 13, 14, 16].

The balance of the paper is organized as follows. We first describe how LIBOR-OIS spreads have evolved through time. Second, we describe how a three-dimensional tree can be constructed to model both OIS rates and the LIBOR-OIS spread with a particular tenor. We then illustrate the tree-building process using a simple three-step tree. We investigate the convergence of the three-dimensional tree by using it to calculate the value of options on the LIBOR-OIS spread. We then value Bermudan swap options showing that in a low-interest-rate environment, the assumption that the spread is stochastic rather than deterministic can have a non-trivial effect on valuations.

## 2 The LIBOR-OIS Spread

LIBOR quotes for maturities of one-, three-, six-, and 12-months in a variety of currencies are produced every day by the British Bankers' Association based on submissions from a panel of contributing banks. These are estimates of the unsecured rates at which AA-rated banks can borrow from other banks. The $T$-month OIS rate is the fixed rate paid on a $T$-month overnight interest rate swap. In such a swap the payment at the end of $T$-months is the difference between the fixed rate and a rate which is the geometric mean of daily overnight rates. The calculation of the payment on the floating side is designed to replicate the aggregate interest that would be earned from rolling over a sequence of daily loans at the overnight rate. (In U.S. dollars, the overnight rate used is the effective federal funds rate.) The LIBOR-OIS spread is the LIBOR rate less the corresponding OIS rate.

[2] At the end of Hull and White [17] we described an attempt to do this using a two-dimensional tree. The current procedure is better. Our earlier procedure only provides an approximate answer because the correlation between spreads at adjacent tree nodes is not fully modelled.

LIBOR-OIS spreads were markedly different during the pre-crisis (December 2001–July 2007) and post-crisis (July 2009–April 2015) periods. This is illustrated in Fig. 1. In the pre-crisis period, the spread term structure was quite flat with the 12-month spread only about 4 basis points higher than the one-month spread on average. As shown in Fig. 1a, the 12-month spread was sometimes higher and sometimes lower than one-month spread. The average one-month spread was about 10 basis points during this period. Because the term structure of spreads was on average fairly flat and quite small, it was plausible for practitioners to assume the existence of a single LIBOR zero curve and use it as a proxy for the risk-free zero curve. During the post-crisis period there has been a marked term structure of spreads. As shown in Fig. 1b, it is almost always the case that the spread curve is upward sloping. The average one-month spread continues to be about 10 basis points, but the average 12-month spread is about 62 basis points.

There are two factors that explain the difference between LIBOR rates and OIS rates. The first of these may be institutional. If a regression model is used to extrapolate the spread curve for shorter maturities, we find the one-day spread in the post-crisis period is estimated to be about 5 basis points. This is consistent with the spread between one-day LIBOR and the effective fed funds rate. Since these are both rates that a bank would pay to borrow money for 24 h, they should be the same. The 5 basis point difference must be related to institutional practices that affect the two different markets.[3]

Given that institutional differences account for about 5 basis points of spread, the balance of the spread must be attributable to credit. OIS rates are based on a continually refreshed one-day rate whereas $\tau$-maturity LIBOR is a continually refreshed $\tau$-maturity rate.[4] The difference between $\tau$-maturity LIBOR and $\tau$-maturity OIS then reflects the degree to which the credit quality of the LIBOR borrower is expected to decline over $\tau$ years.[5] In the pre-crisis period the expected decline in the borrower credit quality implied by the spreads was small but during the post-crisis period it has been much larger.

The average hazard rate over the life of a LIBOR loan with maturity $\tau$ is approximately

$$\bar{\lambda} = \frac{L(\tau)}{1-R}$$

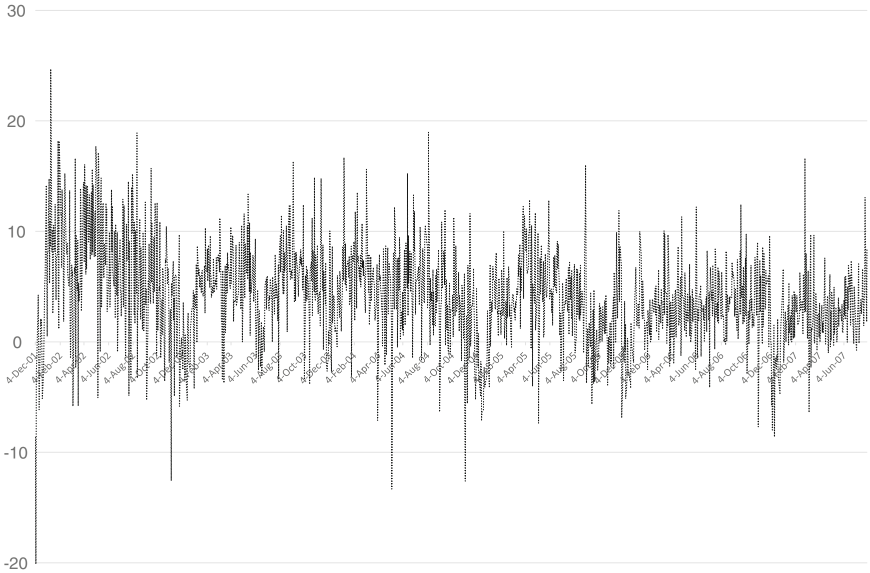
where $L(\tau)$ is the spread of LIBOR over the risk-free rate and $R$ is the recovery rate in the event of default. Let $h$ be the hazard rate for overnight loans to high quality financial institutions (those that can borrow at the effective fed funds rate). This will also be the average hazard rate associated with OIS rates.

[3] For a more detailed discussion of these issues see Hull and White [15].

[4] A continually refreshed $\tau$-maturity rate is the rate realized when a loan is made to a party with a certain specified credit rating (usually assumed in this context to be AA) for time $\tau$. At the end of the period a new $\tau$-maturity loan is made to a possibly different party with the same specified credit rating. See Collin-Dufresne and Solnik [6].

[5] It is well established that for high quality borrowers the expected credit quality declines with the passage of time.

**(a)**




**(b)**

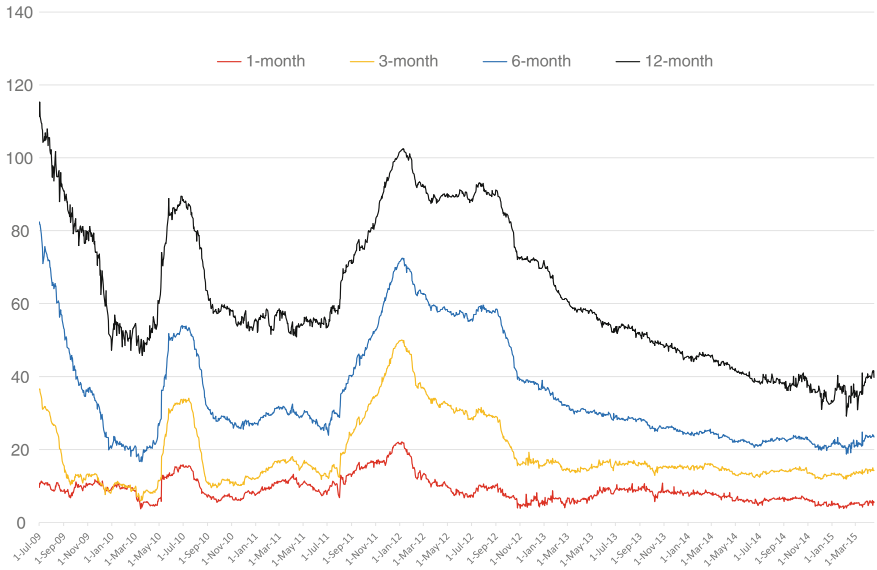



**Fig. 1** **a** Excess of 12-month LIBOR-OIS spread over one-month LIBOR-OIS spread December 4, 2001–July 31, 2007 period (basis points). Data Source: Bloomberg. **b** Post-crisis LIBOR-OIS spread for different tenors (basis points). Data Source: Bloomberg

Define $L^*(\tau)$ as the spread of LIBOR over OIS for a maturity of $\tau$ and $O(\tau)$ as the spread of OIS over the risk-free rate for this maturity. Because $L(\tau) = L^*(\tau) + O(\tau)$

$$\overline{\lambda} = \frac{L^*(\tau) + O(\tau)}{1 - R} = h + \frac{L^*(\tau)}{1 - R}$$

This shows that when we model OIS and LIBOR we are effectively modelling OIS and the difference between the LIBOR hazard rate and the OIS hazard rate.

One of the results of the post-crisis spread term structure is that a single LIBOR zero curve no longer exists. LIBOR zero curves can be constructed from swap rates, but there is a different LIBOR zero curve for each tenor. This paper shows how OIS rates and a LIBOR rate with a particular tenor can be modelled jointly using a three-dimensional tree.[6]

## 3 The Methodology

Suppose that we are interested in modelling OIS rates and the LIBOR rate with tenor of $\tau$. (Values of $\tau$ commonly used are one month, three months, six months and 12 months.) Define $r$ as the instantaneous OIS rate. We assume that some function of $r$, $x(r)$, follows the process

$$\mathrm{d}x = [\theta(t) - a_r x]\,\mathrm{d}t + \sigma_r\,\mathrm{d}z_r \tag{1}$$

This is an Ornstein–Uhlenbeck process with a time-dependent reversion level. The function $\theta(t)$ is chosen to match the initial term structure of OIS rates; $a_r$ ($\geq 0$) is the reversion rate of $x$; $\sigma_r$ ($>0$) is the volatility of $r$; and $\mathrm{d}z_r$ is a Wiener process.[7]

Define $s$ as the spread between the LIBOR rate with tenor $\tau$ and the OIS rate with tenor $\tau$ (both rates being measured with a compounding frequency corresponding to the tenor). We assume that some function of $s$, $y(s)$, follows the process:

$$\mathrm{d}y = [\phi(t) - a_s y]\,\mathrm{d}t + \sigma_s\,\mathrm{d}z_s \tag{2}$$

This is also an Ornstein–Uhlenbeck process with a time-dependent reversion level. The function $\phi(t)$ is chosen to ensure that all LIBOR FRAs and swaps that can be entered into today have a value of zero; $a_s$ ($\geq 0$) is the reversion rate of $y$; $\sigma_s$ ($>0$) is

[6] Extending the approach so that more than one LIBOR rate is modelled is not likely to be feasible as it would involve using backward induction in conjunction with a four (or more)-dimensional tree. In practice, multiple LIBOR rates are most likely to be needed for portfolios when credit and other valuation adjustments are calculated. Monte Carlo simulation is usually used in these situations.

[7] This model does not allow interest rates to become negative. Negative interest have been observed in some currencies (particularly the euro and Swiss franc). If $-e$ is the assumed minimum interest rate, this model can be adjusted so that $x = \ln(r + e)$. The choice of $e$ is somewhat arbitrary, but changes the assumptions made about the behavior of interest rates in a non-trivial way.

the volatility of $s$; and $dz_s$ is a Wiener process. The correlation between $dz_r$ and $dz_s$ will be denoted by $\rho$.

We will use a three-dimensional tree to model $x$ and $y$. A tree is a discrete time, discrete space approximation of a continuous stochastic process for a variable. The tree is constructed so that the mean and standard deviation of the variable is matched over each time step. Results in Ames [1] show that in the limit the tree converges to the continuous time process. At each node of the tree, $r$ and $s$ can be calculated using the inverse of the functions $x$ and $y$.

We will first outline a step-by-step approach to constructing the three-dimensional tree and then provide more details in the context of a numerical example in Sect. 4.[8] The steps in the construction of the tree are as follows:

1. Model the instantaneous OIS rate using a tree. We assume that the process for $r$ is defined by Eq. (1) and that a trinomial tree is constructed as described in Hull and White [11, 13] or Hull [10]. However, the method we describe can be used in conjunction with other binomial and trinomial tree-building procedures such as those in Ho and Lee [9], Black, Derman and Toy [3], Black and Karasinski [4], Kalotay, Williams and Fabozzi [18] and Hull and White [14, 16]. Tree building procedures are also discussed in a number of texts.[9] If the tree has steps of length $\Delta t$, the interest rate at each node of the tree is an OIS rate with maturity $\Delta t$. We assume the tree can be constructed so that both the LIBOR tenor, $\tau$, and all potential payment times for the instrument being valued are multiples of $\Delta t$. If this is not possible, a tree with varying time steps can be constructed.[10]
2. Use backward induction to calculate at each node of the tree the price of an OIS zero-coupon bond with a life of $\tau$. For a node at time $t$ this involves valuing a bond that has a value of \$1 at time $t + \tau$. The value of the bond at nodes earlier than $t + \tau$ is found by discounting through the tree. For each node at time $t + \tau - \Delta t$ the price of the bond is $e^{-r\Delta t}$ where $r$ is the ($\Delta t$-maturity) OIS rate at the node. For each node at time $t + \tau - 2\Delta t$ the price is $e^{-r\Delta t}$ times a probability-weighted average of prices at the nodes at time $t + \tau - \Delta t$ which can be reached from that node, and so on. The calculations are illustrated in the next section. Based on the bond price calculated in this way, $P$, the $\tau$-maturity OIS rate, expressed with a compounding period of $\tau$, is[11]
$$\frac{1/P - 1}{\tau}$$
3. Construct a trinomial tree for the process for the spread function, $y$, in Eq. (2) when the function $\phi(t)$ is set equal to zero and the initial value of $y$ is set equal to

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[8] Readers who have worked with interest rate trees will be able to follow our step-by-step approach. Other readers may prefer to follow the numerical example.

[9] See for example Brigo and Mercurio [5] or Hull [10].

[10] See for example Hull and White [14].

[11] The $r$-tree shows the evolution of the $\Delta t$-maturity OIS rate. Since we are interested in modelling the $\tau$-maturity LIBOR-OIS spread, it is necessary to determine the evolution of the $\tau$-maturity OIS rate.

zero.[12] We will refer to this as the "preliminary tree". When interest rate trees are built, the expected value of the short rate at each time step is chosen so that the initial term structure is matched. The adjustment to the expected rate at time $t$ is achieved by adding some constant, $\alpha_t$, to the value of $x$ at each node at that step.[13] The expected value of the spread at each step of the spread tree that is eventually constructed will similarly be chosen to match forward LIBOR rates. The current preliminary tree is a first step toward the construction of the final spread tree.

4. Create a three-dimensional tree from the OIS tree and the preliminary spread tree assuming zero correlation between the OIS rate and the spread. The probabilities on the branches of this three-dimensional tree are the product of the probabilities on the corresponding branches of the underlying two-dimensional trees.
5. Build in correlation between the OIS rate and the spread by adjusting the probabilities on the branches of the three-dimensional tree. The way of doing this is described in Hull and White [12] and will be explained in more detail later in this paper.
6. Using an iterative procedure, adjust the expected spread at each of the times considered by the tree. For the nodes at time $t$, we consider a receive-fixed forward rate agreement (FRA) applicable to the period between $t$ and $t + \tau$.[14] The fixed rate, $F$, equals the forward rate at time zero. The value of the FRA at a node, where the $\tau$-maturity OIS rate is $w$ and the $\tau$-maturity LIBOR-OIS spread is $s$, is[15]

$$\frac{F - (w + s)}{1 + w\tau}$$

The value of the FRA is calculated for all nodes at time $t$ and the values are discounted back through the three-dimensional tree to find the present value.[16] As discussed in step 3, the expected spread (i.e., the amount by which nodes are shifted from their positions in the preliminary tree) is chosen so that this present value is zero.

[12] As in the case of the tree for the interest rate function, $x$, the method can be generalized to accommodate a variety of two-dimensional and three-dimensional tree-building procedures.

[13] This is equivalent to determining the time varying drift parameter, $\theta(t)$, that is consistent with the current term structure.

[14] A forward rate agreement (FRA) is one leg of a fixed for floating interest rate swap. Typically, the forward rates underlying some FRAs can be observed in the market. Others can be bootstrapped from the fixed rates exchanged in interest rate swaps.

[15] $F$, $w$, and $s$ are expressed with a compounding period of $\tau$.

[16] Calculations are simplified by calculating Arrow–Debreu prices, first at all nodes of the two-dimensional OIS tree and then at all nodes of the three-dimensional tree. The latter can be calculated at the end of the fifth step as they do not depend on spread values. This is explained in more detail and illustrated numerically in Sect. 4.

## 4 A Simple Three-Step Example

We now present a simple example to illustrate the implementation of our procedure. We assume that the LIBOR maturity of interest is 12 months ($\tau = 1$). We assume that $x = \ln(r)$ with $x$ following the process in Eq. (1). Similarly we assume that $y = \ln(s)$ with $y$ following the process in Eq. (2). We assume that the initial OIS zero rates and 12 month LIBOR forward rates are those shown in Table 1. We will build a 1.5-year tree where the time step, $\Delta t$, equals 0.5 years. We assume that the reversion rate and volatility parameters are as shown in Table 2.

As explained in Hull and White [11, 13] we first build a tree for $x$ assuming that $\theta(t) = 0$. We set the spacing of the $x$ nodes, $\Delta x$, equal to $\sigma_r\sqrt{3\Delta t} = 0.3062$. Define node $(i, j)$ as the node at time $i\Delta t$ for which $x = j\Delta x$. (The middle node at each time has $j = 0$.) The normal branching process in the tree is from $(i, j)$ to one of $(i+1, j+1)$, $(i+1, j)$, and $(i+1, j-1)$. The transition probabilities to these three nodes are $p_u$, $p_m$, and $p_d$ and are chosen to match the mean and standard deviation

**Table 1** Percentage interest rates for the examples

| Maturity (years) | OIS zero rate | Forward 12-month LIBOR rate | Forward 12-month OIS rate | Forward Spread: 12-month LIBOR less 12-month OIS |
|---|---|---|---|---|
| 0 | 3.000 | 3.300 | 3.149 | 0.151 |
| 0.5 | 3.050 | 3.410 | 3.252 | 0.158 |
| 1.0 | 3.100 | 3.520 | 3.355 | 0.165 |
| 1.5 | 3.150 | 3.630 | 3.458 | 0.172 |
| 2.0 | 3.200 | 3.740 | 3.562 | 0.178 |
| 2.5 | 3.250 | 3.850 | 3.666 | 0.184 |
| 3.0 | 3.300 | 3.960 | 3.769 | 0.191 |
| 4.0 | 3.400 | 4.180 | 3.977 | 0.203 |
| 5.0 | 3.500 | 4.400 | 4.185 | 0.215 |
| 7.0 | 3.700 | | | |

The OIS zero rates are expressed with continuous compounding while all forward and forward spread rates are expressed with annual compounding. The OIS zero rates and LIBOR forward rates are exact. OIS zero rates and LIBOR forward rates for maturities other than those given are determined using linear interpolation. The rates in the final two columns are rounded values calculated from the given OIS zero rates and LIBOR forward rates

**Table 2** Reversion rates, volatilities, and correlation for the examples

| | |
|---|---|
| OIS reversion rate, $a_r$ | 0.22 |
| OIS volatility, $\sigma_r$ | 0.25 |
| Spread reversion rate, $a_s$ | 0.10 |
| Spread volatility, $\sigma_s$ | 0.20 |
| Correlation between OIS and spread, $\rho$ | 0.05 |

of changes in time $\Delta t$[17]

$$
\begin{aligned}
p_u &= \frac{1}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 - a_r j \Delta t) \\
p_m &= \frac{2}{3} - a_r^2 j^2 \Delta t^2 \\
p_d &= \frac{1}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 + a_r j \Delta t)
\end{aligned}
$$

As soon as $j > 0.184/(a_r \Delta t)$, the branching process is changed so that $(i, j)$ leads to one of $(i+1, j)$, $(i+1, j-1)$, and $(i+1, j-2)$. The transition probabilities to these three nodes are

$$
\begin{aligned}
p_u &= \frac{7}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 - 3a_r j \Delta t) \\
p_m &= -\frac{1}{3} - a_r^2 j^2 \Delta t^2 + 2a_r j \Delta t \\
p_d &= \frac{1}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 - a_r j \Delta t)
\end{aligned}
$$

Similarly, as soon as $j < -0.184/(a_r \Delta t)$ the branching process is changed so that $(i, j)$ leads to one of $(i+1, j+2)$, $(i+1, j+1)$, and $(i+1, j)$. The transition probabilities to these three nodes are

$$
\begin{aligned}
p_u &= \frac{1}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 + a_r j \Delta t) \\
p_m &= -\frac{1}{3} - a_r^2 j^2 \Delta t^2 - 2a_r j \Delta t \\
p_d &= \frac{7}{6} + \frac{1}{2}(a_r^2 j^2 \Delta t^2 + 3a_r j \Delta t)
\end{aligned}
$$

We then use an iterative procedure to calculate in succession the amount that the $x$-nodes at each time step must be shifted, $\alpha_0, \alpha_{\Delta t}, \alpha_{2\Delta t}, \ldots$, so that the OIS term structure is matched. The first value, $\alpha_0$, is chosen so that the tree correctly prices a discount bond maturing $\Delta t$. The second value, $\alpha_{\Delta t}$, is chosen so that the tree correctly prices a discount bond maturing $2\Delta t$, and so on.

Arrow–Debreu prices facilitate the calculation. The Arrow–Debreu price for a node is the price of a security that pays off \$1 if the node is reached and zero otherwise. Define $A_{i,j}$ as the Arrow–Debreu price for node $(i, j)$ and define $r_{i,j}$ as the $\Delta t$-maturity interest rate at node $(i, j)$. The value of $\alpha_{i\Delta t}$ can be calculated using an iterative search procedure from the $A_{i,j}$ and the price at time zero, $P_{i+1}$, of a bond maturing at time $(i+1)\Delta t$ using

[17] See for example Hull ([10], p. 725).

$$P_{i+1} = \sum_j A_{i,j} \exp(-r_{i,j}\Delta t) \tag{3}$$

in conjunction with

$$r_{i,j} = \exp(\alpha_{i\Delta t} + j\Delta x) \tag{4}$$

where the summation in Eq. (3) is over all $j$ at time $i\Delta t$. The Arrow–Debreu prices can then be updated using

$$A_{i+1,k} = \sum_j A_{i,j} p_{j,k} \exp(-r_{i,j}\Delta t) \tag{5}$$

where $p(j,k)$ is the probability of branching from $(i, j)$ to $(i+1, k)$, and the summation is over all $j$ at time $i\Delta t$. The Arrow–Debreu price at the base of the tree, $A_{0,0}$, is one. From this $\alpha_0$ can be calculated using Eqs. (3) and (4). The $A_{1,k}$ can then be calculated using Eqs. (4) and (5). After that $\alpha_{\Delta t}$ can be calculated using Eqs. (3) and (4), and so on.

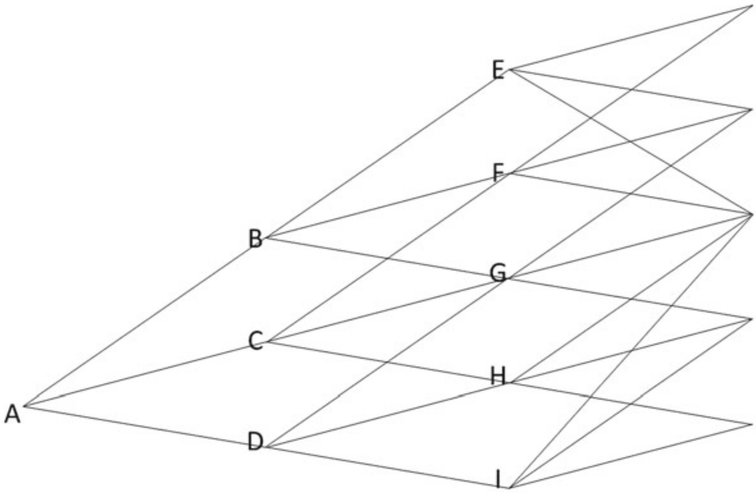
It is then necessary to calculate the value of the 12-month OIS rate at each node (step 2 in the previous section). As the tree has six-month time steps, a two-period roll back is required in the case of our simple example. It is necessary to build a four-step tree. The value at the $j$th node at time $4\Delta t (= 2)$ of a discount bond that pays \$1 at time $5\Delta t (= 2.5)$ is $\exp(-r_{4,j}\Delta t)$.

Discounting these values back to time $3\Delta t (= 1.5)$ gives the price of a one-year discount bond at each node at $3\Delta t$ from which the bond's yield can be determined. This is repeated for a bond that pays \$1 at time $4\Delta t$ resulting in the one-year yields at time $2\Delta t$, and so on. The tree constructed so far and the values calculated are shown in Fig. 2.[18]

The next stage (step 3 in the previous section) is to construct a tree for the spread assuming that the expected future spread is zero (the preliminary tree). As in the case of the OIS tree, $\Delta t = 0.5$ and $\Delta y = \sigma_s\sqrt{3\Delta t} = 0.2449$. The branching process and probabilities are calculated as for the OIS tree (with $a_r$ replaced by $a_s$).

A three-dimensional tree is then created (step 4 in the previous section) by combining the spread tree and the OIS tree assuming zero correlation. We denote the node at time $i\Delta t$ where $x = j\Delta x$ and $y = k\Delta y$ by node $(i, j, k)$. Consider for example node $(2, -2, 2)$. This corresponds to node $(2, -2)$ in the OIS tree, node $I$ in Fig. 2, and node $(2, 2)$ in the spread tree. The probabilities for the OIS tree are $p_u = 0.0809$, $p_m = 0.0583$, $p_d = 0.8609$ and the branching process is to nodes where $j = 0$, $j = -1$, and $j = -2$. The probabilities for the spread tree are $p_u = 0.1217$, $p_m = 0.6567$, $p_d = 0.2217$ and the branching process is to nodes where $k = 1$, $k = 2$, and $k = 3$. Denote $p_{uu}$ as the probability of the highest move in the OIS tree being combined with the highest move in the spread tree; $p_{um}$ as the probability of the highest move in the OIS tree being combined with the middle move in the spread tree; and so on. The probability, $p_{uu}$ of moving from node $(2, -2, 2)$ to

[18] More details on the construction of the tree can be found in Hull [10].




| Node | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| $x$-value | −3.490 | −3.167 | −3.473 | −3.779 | −2.841 | −3.147 | −3.454 | −3.760 | −4.066 |
| $r$-value | 3.050% | 4.213% | 3.102% | 2.284% | 5.835% | 4.296% | 3.163% | 2.329% | 1.715% |
| 12-month rate (ann. comp.) | 3.149% | 4.306% | 3.207% | 2.393% | 5.910% | 4.397% | 3.275% | 2.443% | 1.828% |
| $p_u$ | 0.1667 | 0.1177 | 0.1667 | 0.2277 | 0.8609 | 0.1177 | 0.1667 | 0.2277 | 0.0809 |
| $p_m$ | 0.6667 | 0.6546 | 0.6667 | 0.6546 | 0.0582 | 0.6546 | 0.6667 | 0.6546 | 0.0582 |
| $p_d$ | 0.1667 | 0.2277 | 0.1667 | 0.1177 | 0.0809 | 0.2277 | 0.1667 | 0.1177 | 0.8609 |
| Arrow-Debreu price | 1.0000 | 0.1641 | 0.6566 | 0.1641 | 0.0189 | 0.2129 | 0.5045 | 0.2140 | 0.0191 |

**Fig. 2** Tree for OIS rates in three-step example

node (3, 0, 3) is therefore $0.0809 \times 0.1217$ or 0.0098; the probability, $p_{um}$ of moving from node $(2, -2, 2)$ to node (3, 0, 2) is $0.0809 \times 0.6567$ or 0.0531 and so on. These (unadjusted) branching probabilities at node $(2, -2, 2)$ are shown in Table 4a.

The next stage (step 5 in the previous section) is to adjust the probabilities to build in correlation between the OIS rate and the spread (i.e., the correlation between $\mathrm{d}z_r$ and $\mathrm{d}z_s$). As explained in Hull and White [12], probabilities are changed as indicated in Table 3.[19] This leaves the marginal distributions unchanged. The resulting adjusted probabilities at node $(2, -2, 2)$ are shown in Table 4b. In the example we are currently considering the adjusted probabilities are never negative. In practice negative probabilities do occur, but disappear as $\Delta t$ tends zero. They tend to occur only on the edges of the tree where the non-standard branching process is used and do not interfere with convergence. Our approach when negative probabilities are encountered at a node is to change the correlation at that node to the greatest (positive or negative) correlation that is consistent with non-negative probabilities.

[19]The procedure described in Hull and White [12] applies to trinomial trees. For binomial trees the analogous procedure is to increase $p_{uu}$ and $p_{dd}$ by $\varepsilon$ while decreasing $p_{ud}$ and $p_{du}$ by $\varepsilon$ where $\varepsilon = \rho/4$.

**Table 3** Adjustments to probabilities to reflect correlation in a three-dimensional trinomial tree

| Probability | Change when $\rho > 0$ | Change when $\rho < 0$ |
|---|---|---|
| $p_{uu}$ | $+5e$ | $+e$ |
| $p_{um}$ | $-4e$ | $+4e$ |
| $p_{ud}$ | $-e$ | $-5e$ |
| $p_{mu}$ | $-4e$ | $+4e$ |
| $p_{mm}$ | $+8e$ | $-8e$ |
| $p_{md}$ | $-4e$ | $+4e$ |
| $p_{du}$ | $-e$ | $-5e$ |
| $p_{dm}$ | $-4e$ | $+4e$ |
| $p_{dd}$ | $+5e$ | $+e$ |

($e = \rho/36$ where $\rho$ is the correlation)

**Table 4** (a) The unadjusted branching probabilities at node $(2, -2, 2)$. The probabilities on the edge of the table are the branching probabilities at node $(2, -2)$ of the $r$-tree and $(2, 2)$ of the $s$-tree. (b) The adjusted branching probabilities at node $(2, -2, 2)$. The probabilities on the edge of the table are the branching probabilities at node $(2, -2)$ of the $r$-tree and $(2, 2)$ of the $s$-tree. The adjustment is based on a correlation of 0.05 so $e = 0.00139$

*a*

| | | | $r$-tree | | |
|---|---|---|---|---|---|
| | | | $p_u$ | $p_m$ | $p_d$ |
| | | | 0.0809 | 0.0583 | 0.8609 |
| $s$-tree | $p_u$ | 0.1217 | 0.0098 | 0.0071 | 0.1047 |
| | $p_m$ | 0.6567 | 0.0531 | 0.0383 | 0.5653 |
| | $p_d$ | 0.2217 | 0.0179 | 0.0129 | 0.1908 |

*b*

| | | | $r$-tree | | |
|---|---|---|---|---|---|
| | | | $p_u$ | $p_m$ | $p_d$ |
| | | | 0.0809 | 0.0583 | 0.8609 |
| $s$-tree | $p_u$ | 0.1217 | 0.0168 | 0.0015 | 0.1033 |
| | $p_m$ | 0.6567 | 0.0475 | 0.0494 | 0.5597 |
| | $p_d$ | 0.2217 | 0.0165 | 0.0074 | 0.1978 |

The tree constructed so far reflects actual OIS movements and artificial spread movements where the initial spread and expected future spread are zero. We are now in a position to calculate Arrow–Debreu prices for each node of the three-dimensional tree. These Arrow–Debreu prices remain the same when the positions of the spread nodes are changed because the Arrow–Debreu price for a node depends only on OIS rates and the probability of the node being reached. They are shown in Table 5.

The final stage involves shifting the position of the spread nodes so that the prices of all LIBOR FRAs with a fixed rate equal to the initial forward LIBOR rate are zero. An iterative procedure is used to calculate the adjustment to the values of $y$

**Table 5** Arrow–Debreu prices for simple three-step example

| $i = 1$ | $k = -1$ | $k = 0$ | $k = 1$ | | | | |
|---|---|---|---|---|---|---|---|
| $j = 1$ | 0.0260 | 0.1040 | 0.0342 | | | | |
| $j = 0$ | 0.1040 | 0.4487 | 0.1040 | | | | |
| $j = -1$ | 0.0342 | 0.1040 | 0.0260 | | | | |
| $i = 2$ | $k = -2$ | $k = -1$ | $k = 0$ | $k = 1$ | $k = 2$ | | |
| $j = 2$ | 0.0004 | 0.0037 | 0.0089 | 0.0051 | 0.0008 | | |
| $j = 1$ | 0.0045 | 0.0443 | 0.1064 | 0.0516 | 0.0061 | | |
| $j = 0$ | 0.0112 | 0.1100 | 0.2620 | 0.1100 | 0.0112 | | |
| $j = -1$ | 0.0061 | 0.0518 | 0.1070 | 0.0445 | 0.0046 | | |
| $j = -2$ | 0.0008 | 0.0052 | 0.0090 | 0.0037 | 0.0004 | | |
| $i = 3$ | $k = -3$ | $k = -2$ | $k = -1$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ |
| $j = 2$ | 0.0001 | 0.0016 | 0.0085 | 0.0163 | 0.0109 | 0.0027 | 0.0002 |
| $j = 1$ | 0.0005 | 0.0094 | 0.0496 | 0.0932 | 0.0551 | 0.0116 | 0.0007 |
| $j = 0$ | 0.0012 | 0.0197 | 0.1016 | 0.1849 | 0.1016 | 0.0197 | 0.0012 |
| $j = -1$ | 0.0008 | 0.0117 | 0.0557 | 0.0941 | 0.0501 | 0.0095 | 0.0005 |
| $j = -2$ | 0.0002 | 0.0028 | 0.0111 | 0.0167 | 0.0087 | 0.0017 | 0.0001 |

at each node at each time step, $\beta_0$, $\beta_{\Delta t}$, $\beta_{2\Delta t}$, ..., so that the FRAs have a value of zero. Given that Arrow–Debreu prices have already been calculated this is a fairly straightforward search. When the $\alpha_{j\Delta t}$ are determined it is necessary to first consider $j = 0$, then $j = 1$, then $j = 2$, and so on because the $\alpha$-value at a particular time depends on the $\alpha$-values at earlier times. The $\beta$-values however are independent of each other and can be determined in any order, or as needed. In the case of our example, $\beta_0 = -6.493$, $\beta_{\Delta t} = -6.459$, $\beta_{2\Delta t} = -6.426$, $\beta_{3\Delta t} = -6.395$.

## 5 Valuation of a Spread Option

To illustrate convergence, we use the tree to calculate the value of a European call option that pays off 100 times $\max(s - 0.002, 0)$ at time $T$ where $s$ is the spread. First, we let $T = 1.5$ years and use the three-step tree developed in the previous section. At the third step of the tree we calculate the spread at each node. The spread at node $(3, j, k)$ is $\exp[\phi(3\Delta t) + k\Delta y]$. These values are shown in the second line of Table 6. Once the spread values have been determined the option payoffs, 100 times $\max(s - 0.002, 0)$, at each node are calculated. These values are shown in the rest of Table 6. The option value is found by multiplying each option payoff by the corresponding Arrow–Debreu price in Table 5 and summing the values. The resulting option value is 0.00670. Table 7 shows how, for a 1.5- and 5-year spread option, the value converges as the number of time steps per year is increased.

**Table 6** Spread and spread option payoff at time 1.5 years when spread option is evaluated using a three-step tree

| $i = 3$ | $k = -3$ | $k = -2$ | $k = -1$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ |
|---|---|---|---|---|---|---|---|
| Spread | 0.0008 | 0.0010 | 0.0013 | 0.0017 | 0.0021 | 0.0027 | 0.0035 |
| $j = 2$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0133 | 0.0725 | 0.1482 |
| $j = 1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0133 | 0.0725 | 0.1482 |
| $j = 0$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0133 | 0.0725 | 0.1482 |
| $j = -1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0133 | 0.0725 | 0.1482 |
| $j = -2$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0133 | 0.0725 | 0.1482 |

**Table 7** Value of a European spread option paying off 100 times the greater of the spread less 0.002 and zero

| Time steps per year | 1.5-year option | 5-year option |
|---|---|---|
| 2 | 0.00670 | 0.0310 |
| 4 | 0.00564 | 0.0312 |
| 8 | 0.00621 | 0.0313 |
| 16 | 0.00592 | 0.0313 |
| 32 | 0.00596 | 0.0313 |

The market data used to build the tree is given in Tables 1 and 2

**Table 8** Value of a five-year European spread option paying off 100 times the greater of the spread less 0.002 and zero

| Spread volatility | Spread/OIS correlation | | | | | | |
|---|---|---|---|---|---|---|---|
| | –0.75 | –0.50 | –0.25 | 0 | 0.25 | 0.5 | 0.75 |
| 0.05 | 0.0141 | 0.0142 | 0.0142 | 0.0143 | 0.0143 | 0.0144 | 0.0144 |
| 0.10 | 0.0193 | 0.0194 | 0.0195 | 0.0195 | 0.0196 | 0.0196 | 0.0197 |
| 0.15 | 0.0250 | 0.0252 | 0.0253 | 0.0254 | 0.0254 | 0.0255 | 0.0256 |
| 0.20 | 0.0308 | 0.0309 | 0.0311 | 0.0313 | 0.0314 | 0.0316 | 0.0317 |
| 0.25 | 0.0367 | 0.0369 | 0.0371 | 0.0373 | 0.0374 | 0.0376 | 0.0377 |

The market data used to build the tree are given in Tables 1 and 2 except that the volatility of the spread and the correlation between the spread and the OIS rate are as given in this table. The number of time steps is 32 per year

Table 8 shows how the spread option price is affected by the assumed correlation and the volatility of the spread. All of the input parameters are as given in Tables 1 and 2 except that correlations between $-0.75$ and $0.75$, and spread volatilities between 0.05 and 0.25 are considered. As might be expected the spread option price is very sensitive to the spread volatility. However, it is not very sensitive to the correlation. The reason for this is that changing the correlation primarily affects the Arrow–Debreu prices and leaves the option payoffs almost unchanged. Increasing the correlation increases the Arrow–Debreu prices on one diagonal of the final nodes and decreases them on the other diagonal. For example, in the three-step tree used to evaluate the option, the Arrow–Debreu price for nodes (3, 2, 3) and (3, −2, −3) increase while those for nodes (3, −2, 3) and (3, 2, −3) decrease. Since the option payoffs at nodes (3, 2, 3) and (3, −2, 3) are the same, the changes on the Arrow–Debreu prices offset one another resulting in only a small correlation effect.

## 6 Bermudan Swap Option

We now consider how the valuation of a Bermudan swap option is affected by a stochastic spread in a low-interest-rate environment such as that experienced in the years following 2009. Bermudan swap options are popular instruments where the holder has the right to enter into a particular swap on a number of different swap payment dates.

The valuation procedure involves rolling back through the tree calculating both the swap price and (where appropriate) the option price. The swap's value is set equal to zero at the nodes on the swap's maturity date. The value at earlier nodes is calculated by rolling back adding in the present value of the next payment on each reset date. The option's value is set equal to $\max(S, 0)$ where $S$ is the swap value at the option's maturity. It is then set equal to $\max(S, V)$ for nodes on exercise dates where $S$ is the swap value and $V$ is the value of the option given by the roll back procedure.

We assume an OIS term structure that increases linearly from 15 basis points at time zero to 250 basis points at time 10 years. The OIS zero rate for maturity $t$ is therefore

$$0.0015 + \frac{0.0235t}{10}$$

The process followed by the instantaneous OIS rate was similar to that derived by Deguillaume, Rebonato and Pogodin [7], and Hull and White [16]. For short rates between 0 and 1.5 %, changes in the rate are assumed to be lognormal with a volatility of 100 %. Between 1.5 % and 6 % changes in the short rate are assumed to be normal with the standard deviation of rate moves in time $\Delta t$ being $0.015\sqrt{\Delta t}$. Above 6 % rate moves were assumed to be lognormal with volatility 25 %. This pattern of the short rate's variability is shown in Fig. 3.

The spread between the forward 12-month OIS and the forward 12-month LIBOR was assumed to be 50 basis points for all maturities. The process assumed for the 12-month LIBOR-OIS spread, $s$, is that used in the example in Sects. 4 and 5

$$d\ln(s) = a_s[\phi(t) - \ln(s)] + \sigma_s\, dz_s$$

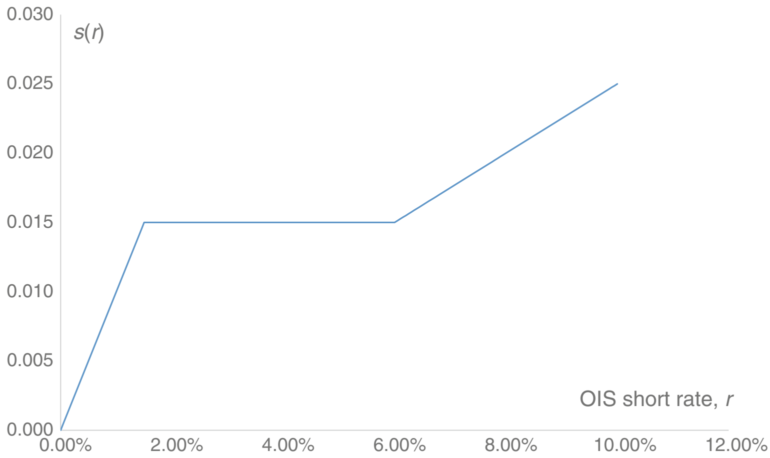



**Fig. 3** Variability assumed for short OIS rate, $r$, in Bermudan swap option valuation. The standard deviation of the short rate in time $\Delta t$ is $s(r)\sqrt{\Delta t}$

**Table 9** (a) Value in a low-interest rate environment, of a receive-fixed Bermudan swap option on a 5-year annual-pay swap where the notional principal is 100 and the option can be exercised at times 1, 2, and 3 years. The swap rate is 1.5 %. (b) Value in a low-interest-rate environment of a received-fixed Bermudan swap option on a 10-year annual-pay swap where the notional principal is 100 and the option can be exercised at times 1, 2, 3, 4, and 5 years. The swap rate is 3.0 %

| Spread volatility | Spread/OIS correlation | | | | | | |
|---|---|---|---|---|---|---|---|
| *a* | | | | | | | |
| | –0.5 | –0.25 | –0.1 | 0 | 0.1 | 0.25 | 0.5 |
| 0 | 0.398 | 0.398 | 0.398 | 0.398 | 0.398 | 0.398 | 0.398 |
| 0.3 | 0.333 | 0.371 | 0.393 | 0.407 | 0.421 | 0.441 | 0.473 |
| 0.5 | 0.310 | 0.373 | 0.407 | 0.429 | 0.449 | 0.480 | 0.527 |
| 0.7 | 0.309 | 0.389 | 0.432 | 0.459 | 0.485 | 0.522 | 0.580 |
| *b* | | | | | | | |
| | –0.5 | –0.25 | –0.1 | 0 | 0.1 | 0.25 | 0.5 |
| 0 | 2.217 | 2.218 | 2.218 | 2.218 | 2.218 | 2.218 | 2.218 |
| 0.3 | 2.100 | 2.164 | 2.201 | 2.225 | 2.248 | 2.283 | 2.339 |
| 0.5 | 2.031 | 2.141 | 2.203 | 2.242 | 2.280 | 2.335 | 2.421 |
| 0.7 | 1.980 | 2.134 | 2.218 | 2.271 | 2.321 | 2.392 | 2.503 |

A maximum likelihood analysis of data on the 12-month LIBOR-OIS spread over the 2012 to 2014 period indicates that the behavior of the spread can be approximately described by a high volatility in conjunction with a high reversion rate. We set $a_s$ equal to 0.4 and considered values of $\sigma_s$ equal to 0.30, 0.50, and 0.70. A number of alternative correlations between the spread process and the OIS process were also considered. We find that correlation of about $-0.1$ between one month OIS and the 12-month LIBOR OIS spread is indicated by the data.[20]

We consider two cases:

1. A $3 \times 5$ swap option. The underlying swap lasts 5 years and involves 12-month LIBOR being paid and a fixed rate of 1.5 % being received. The option to enter into the swap can be exercised at the end of years 1, 2, and 3.
2. A $5 \times 10$ swap option. The underlying swap lasts 10 years and involves 12-month LIBOR being paid and a fixed rate of 3.0 % being received. The option to enter into the swap can be exercised at the end of years 1, 2, 3, 4, and 5.

Table 9a shows results for the $3 \times 5$ swap option. In this case, even when the correlation between the spread rate and the OIS rate is relatively small, a stochastic spread is liable to change the price by 5–10 %. Table 9b shows results for the $5 \times 10$ swap option. In this case, the percentage impact of a stochastic spread is smaller. This is because the spread, as a proportion of the average of the relevant forward OIS rates, is lower. The results in both tables are based on 32 time steps per year. As the level of OIS rates increases the impact of a stochastic spread becomes smaller in both Table 9a, b.

Comparing Tables 8 and 9, we see that the correlation between the OIS rate and the spread has a much bigger effect on the valuation of a Bermudan swap option than on the valuation of a spread option. For a spread option we argued that option payoffs for high Arrow–Debreu prices tend to offset those for low Arrow–Debreu prices. This is not the case for a Bermudan swap option because the payoff depends on the LIBOR rate, which depends on the OIS rate as well as the spread.

## 7 Conclusions

For investment grade companies it is well known that the hazard rate is an increasing function of time. This means that the credit spread applicable to borrowing by AA-rated banks from other banks is an increasing function of maturity. Since 2008, markets have recognized this with the result that the LIBOR-OIS spread has been an increasing function of tenor.

Since 2008, practitioners have also switched from LIBOR discounting to OIS discounting. This means that two zero curves have to be modelled when most interest rate derivatives are valued. Many practitioners assume that the relevant LIBOR-OIS spread is either constant or deterministic. Our research shows that this is liable to lead to inaccurate pricing, particularly in the current low interest rate environment.

The tree approach we have presented provides an alternative to Monte Carlo simulation for simultaneously modelling spreads and OIS rates. It can be regarded as

[20]Because of the way LIBOR is calculated, daily LIBOR changes can be less volatile than the corresponding daily OIS changes (particularly if the Fed is not targeting a particular overnight rate). In some circumstances, it may be appropriate to consider changes over periods longer than one day when estimating the correlation.

an extension of the explicit finite difference method and is particularly useful when American-style derivatives are valued. It avoids the need to use techniques such as those suggested by Longstaff and Schwartz [19] and Andersen (2000) for handling early exercise within a Monte Carlo simulation.

Implying all the model parameters from market data is not likely to be feasible. One reasonable approach is to use historical data to determine the spread process and its correlation with the OIS process so that only the parameters driving the OIS process are implied from the market. The model can then be used in the same way that two-dimensional tree models for LIBOR were used pre-crisis.

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# Derivative Pricing for a Multi-curve Extension of the Gaussian, Exponentially Quadratic Short Rate Model

**Zorana Grbac, Laura Meneghello and Wolfgang J. Runggaldier**

**Abstract** The recent financial crisis has led to so-called multi-curve models for the term structure. Here we study a multi-curve extension of short rate models where, in addition to the short rate itself, we introduce short rate spreads. In particular, we consider a Gaussian factor model where the short rate and the spreads are second order polynomials of Gaussian factor processes. This leads to an exponentially quadratic model class that is less well known than the exponentially affine class. In the latter class the factors enter linearly and for positivity one considers square root factor processes. While the square root factors in the affine class have more involved distributions, in the quadratic class the factors remain Gaussian and this leads to various advantages, in particular for derivative pricing. After some preliminaries on martingale modeling in the multi-curve setup, we concentrate on pricing of linear and optional derivatives. For linear derivatives, we exhibit an adjustment factor that allows one to pass from pre-crisis single curve values to the corresponding post-crisis multi-curve values.

**Keywords** Multi-curve models · Short rate models · Short rate spreads · Gaussian exponentially quadratic models · Pricing of linear and optional interest rate derivatives · Riccati equations · Adjustment factors

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## 1 Introduction

The recent financial crisis has heavily impacted the financial market and the fixed income markets in particular. Key features put forward by the crisis are counterparty and liquidity/funding risk. In interest rate derivatives the underlying rates are typically Libor/Euribor. These are determined by a panel of banks and thus reflect various risks in the interbank market, in particular counterparty and liquidity risk. The standard no-arbitrage relations between Libor rates of different maturities have broken down and significant spreads have been observed between Libor rates of different tenors, as well as between Libor and OIS swap rates, where OIS stands for Overnight Indexed Swap. For more details on this issue see Eqs. (5)–(7) and the paragraph following them, as well as the paper by Bormetti et al. [1] and a corresponding version in this volume. This has led practitioners and academics alike to construct multi-curve models where future cash flows are generated through curves associated to the underlying rates (typically the Libor, one for each tenor structure), but are discounted by another curve.

For the pre-crisis single-curve setup various interest rate models have been proposed. Some of the standard model classes are: the short rate models; the instantaneous forward rate models in an Heath–Jarrow–Morton (HJM) setup; the market forward rate models (Libor market models). In this paper we consider a possible multi-curve extension of the short rate model class that, with respect to the other model classes, has in particular the advantage of leading more easily to a Markovian structure. Other multi-curve extensions of short rate models have appeared in the literature such as Kijima et al. [22], Kenyon [20], Filipović and Trolle [14], Morino and Runggaldier [27]. The present paper considers an exponentially quadratic model, whereas the models in the mentioned papers concern mainly the exponentially affine framework, except for [22] in which the exponentially quadratic models are mentioned. More details on the difference between the exponentially affine and exponentially quadratic short rate models will be provided below.

Inspired by a credit risk analogy, but also by a common practice of deriving multi-curve quantities by adding a spread over the corresponding single-curve risk-free quantities, we shall consider, next to the short rate itself, a short rate spread to be added to the short rate, one for each possible tenor structure. Notice that these spreads are added from the outset.

To discuss the basic ideas in an as simple as possible way, we consider just a two-curve model, namely with one curve for discounting and one for generating future cash flows; in other words, we shall consider a single tenor structure. We shall thus concentrate on the short rate $r_t$ and a single short rate spread $s_t$ and, for their dynamics, introduce a factor model. In the pre-crisis single-curve setting there are two basic factor model classes for the short rate: the exponentially affine and the exponentially quadratic model classes. Here we shall concentrate on the less common quadratic class with Gaussian factors. In the exponentially affine class where, to guarantee positivity of rates and spreads, one considers generally square root models for the factors, the distribution of the factors is $\chi^2$. In the exponentially quadratic class the factors have a more convenient Gaussian distribution.

The paper is structured as follows. In the preliminary Sect. 2 we mainly discuss issues related to martingale modeling. In Sect. 3 we introduce the multi-curve Gaussian, exponentially quadratic model class. In Sect. 4 we deal with pricing of linear interest rate derivatives and, finally, in Sect. 5 with nonlinear/optional interest rate derivatives.

## 2 Preliminaries

### 2.1 Discount Curve and Collateralization

In the presence of multiple curves, the choice of the curve for discounting the future cash flows, and a related choice of the numeraire for the standard martingale measure used for pricing, in other words, the question of absence of arbitrage, becomes nontrivial (see e.g. the discussion in Kijima and Muromachi [21]). To avoid issues of arbitrage, one should possibly have a common discount curve to be applied to all future cash flows independently of the tenor. A choice, which has been widely accepted and became practically standard, is given by the OIS-curve $T \mapsto p(t, T) = p^{OIS}(t, T)$ that can be stripped from OIS rates, namely the fair rates in an OIS. The arguments justifying this choice and which are typically evoked in practice, are the fact that the majority of the traded interest rate derivatives are nowadays being collateralized and the rate used for remuneration of the collateral is exactly the overnight rate, which is the rate the OIS are based on. Moreover, the overnight rate bears very little risk due to its short maturity and therefore can be considered relatively risk-free. In this context we also point out that prices, corresponding to fully collateralized transactions, are considered as clean prices (this terminology was first introduced by Crépey [6] and Crépey et al. [9]). Since collateralization is by now applied in the majority of cases, one may thus ignore counterparty and liquidity risk between individual parties when pricing interest rate derivatives, but cannot ignore the counterparty and liquidity risk in the interbank market as a whole. These risks are often jointly referred to as interbank risk and they are main drivers of the multi-curve phenomenon, as documented in the literature (see e.g. Crépey and Douady [7], Filipović and Trolle [14], and Gallitschke et al. [15]). We shall thus consider only *clean valuation* formulas, which take into account the multi-curve issue. Possible ways to account for counterparty risk and funding issues between individual counterparties in a contract are, among others, to follow a global valuation approach that leads to nonlinear derivative valuation (see Brigo et al. [3, 4] and other references therein, and in particular Pallavicini and Brigo [28] for a global valuation approach applied specifically to interest rate modeling), or to consider various valuation adjustments that are generally computed on top of the clean prices (see Crépey [6]). A fully nonlinear valuation is preferable, but is more difficult to achieve. On the other hand, valuation adjustments are more consolidated and also used in practice and this gives a further justification to still look for clean prices. Concerning the explicit role of collateral in the pricing of interest rate derivatives, we refer to the above-mentioned paper by Pallavicini and Brigo [28].

## 2.2 Martingale Measures

The fundamental theorem of asset pricing links the economic principle of absence of arbitrage with the notion of a martingale measure. As it is well known, this is a measure, under which the traded asset prices, expressed in units of a same numeraire, are local martingales. Models for interest rate markets are typically incomplete so that absence of arbitrage admits many martingale measures. A common approach in interest rate modeling is to perform martingale modeling, namely to model the quantities of interest directly under a generic martingale measure; one has then to perform a calibration in order to single out the specific martingale measure of interest. The modeling under a martingale measure now imposes some conditions on the model and, in interest rate theory, a typical such condition is the Heath–Jarrow–Morton (HJM) drift condition.

Starting from the OIS bonds, we shall first derive a suitable numeraire and then consider as martingale measure a measure $Q$ under which not only the OIS bonds, but also the FRA contracts seen as basic quantities in the bond market, are local martingales when expressed in units of the given numeraire. To this basic market one can then add various derivatives imposing that their prices, expressed in units of the numeraire, are local martingales under $Q$.

Having made the choice of the OIS curve $T \mapsto p(t, T)$ as the discount curve, consider the instantaneous forward rates $f(t, T) := -\frac{\partial}{\partial T} \log p(t, T)$ and let $r_t = f(t, t)$ be the corresponding short rate at the generic time $t$. Define the OIS bank account as

$$B_t = \exp\left(\int_0^t r_s ds\right) \tag{1}$$

and, as usual, the standard martingale measure $Q$ as the measure, equivalent to the physical measure $P$, that is associated to the bank account $B_t$ as numeraire. Hence the arbitrage-free prices of all assets, discounted by $B_t$, have to be local martingales with respect to $Q$. For derivative pricing, among them also FRA pricing, it is often more convenient to use, equivalently, the forward measure $Q^T$ associated to the OIS bond $p(t, T)$ as numeraire. The two measures $Q$ and $Q^T$ are related by their Radon–Nikodym density process

$$\frac{dQ^T}{dQ}\bigg|_{\mathscr{F}_t} = \frac{p(t, T)}{B_t p(0, T)} \qquad 0 \leq t \leq T. \tag{2}$$

As already mentioned, we shall follow the traditional *martingale modeling*, whereby the model dynamics are assigned under the martingale measure $Q$. This leads to defining the OIS bond prices according to

$$p(t,T) = E^Q\left\{\exp\left[-\int_t^T r_u du\right] \mid \mathscr{F}_t\right\} \tag{3}$$

after having specified the $Q-$dynamics of $r$.

Coming now to the FRA contracts, recall that they concern a forward rate agreement, established at a time $t$ for a future interval $[T, T+\Delta]$, where at time $T+\Delta$ the interest corresponding to a floating rate is received in exchange for the interest corresponding to a fixed rate $R$. There exist various possible conventions concerning the timing of the payments. Here we choose payment in arrears, which in this case means at time $T+\Delta$. Typically, the floating rate is given by the Libor rate and, having assumed payments in arrears, we also assume that the rate is fixed at the beginning of the interval of interest, here at $T$. Recall that for expository simplicity we had reduced ourselves to a two-curve setup involving just a single Libor for a given tenor $\Delta$. The floating rate received at $T+\Delta$ is therefore the rate $L(T; T, T+\Delta)$, fixed at the inception time $T$. For a unitary notional, and using the $(T+\Delta)$-forward measure $Q^{T+\Delta}$ as the pricing measure, the arbitrage-free price at $t \le T$ of the FRA contract is then

$$P^{FRA}(t; T, T+\Delta, R) = \Delta p(t, T+\Delta) E^{T+\Delta}\left\{L(T; T, T+\Delta) - R \mid \mathscr{F}_t\right\}, \tag{4}$$

where $E^{T+\Delta}$ denotes the expectation with respect to the measure $Q^{T+\Delta}$. From this expression it follows that the value of the fixed rate $R$ that makes the contract fair at time $t$ is given by

$$R_t = E^{T+\Delta}\left\{L(T; T, T+\Delta) \mid \mathscr{F}_t\right\} := L(t; T, T+\Delta) \tag{5}$$

and we shall call $L(t; T, T+\Delta)$ the *forward Libor rate*. Note that $L(\cdot; T, T+\Delta)$ is a $Q^{T+\Delta}-$martingale by construction.

In view of developing a model for $L(T; T, T+\Delta)$, recall that, by absence of arbitrage arguments, the classical discrete compounding forward rate at time $t$ for the future time interval $[T, T+\Delta]$ is given by

$$F(t; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right),$$

where $p(t,T)$ represents here the price of a risk-free zero coupon bond. This expression can be justified also by the fact that it represents the fair fixed rate in a forward rate agreement, where the floating rate received at $T+\Delta$ is

$$F(T; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{1}{p(T,T+\Delta)} - 1\right) \tag{6}$$

and we have

$$F(t;T,T+\Delta) = E^{T+\Delta}\left\{F(T;T,T+\Delta) \mid \mathscr{F}_t\right\}. \tag{7}$$

This makes the forward rate coherent with the risk-free bond prices, where the latter represent the expectation of the market concerning the future value of money.

Before the financial crisis, $L(T;T,T+\Delta)$ was assumed to be equal to $F(T;T,T+\Delta)$, an assumption that allowed for various simplifications in the determination of derivative prices. After the crisis $L(T;T,T+\Delta)$ is no longer equal to $F(T;T,T+\Delta)$ and what one considers for $F(T;T,T+\Delta)$ is in fact the *OIS discretely compounded rate*, which is based on the OIS bonds, even though the OIS bonds are not necessarily equal to the risk-free bonds (see Sects. 1.3.1 and 1.3.2 of Grbac and Runggaldier [18] for more details on this issue). In particular, the Libor rate $L(T;T,T+\Delta)$ cannot be expressed by the right-hand side of (6). The fact that $L(T;T,T+\Delta) \neq F(T;T,T+\Delta)$ implies by (5) and (7) that also $L(t;T,T+\Delta) \neq F(t;T,T+\Delta)$ for all $t \leq T$ and this leads to a *Libor-OIS spread* $L(t;T,T+\Delta) - F(t;T,T+\Delta)$.

Following some of the recent literature (see e.g. Kijima et al. [22], Crépey et al. [8], Filipović and Trolle [14]), one possibility is now to keep the classical relationship (6) also for $L(T;T,T+\Delta)$ thereby replacing however the bonds $p(t,T)$ by fictitious risky ones $\bar{p}(t,T)$ that are assumed to be affected by the same factors as the Libor rates. Such a bond can be seen as an average bond issued by a representative bank from the Libor group and it is therefore sometimes referred to in the literature as a *Libor bond*. This leads to

$$L(T;T,T+\Delta) = \frac{1}{\Delta}\left(\frac{1}{\bar{p}(T,T+\Delta)} - 1\right). \tag{8}$$

Recall that, for simplicity of exposition, we consider a single Libor for a single tenor $\Delta$ and so also a single fictitious bond. In general, one has one Libor and one fictitious bond for each tenor, i.e. $L^{\Delta}(T;T,T+\Delta)$ and $\bar{p}^{\Delta}(T,T+\Delta)$. Note that we shall model the bond prices $\bar{p}(t,T)$, for all $t$ and $T$ with $t \leq T$, even though only the prices $\bar{p}(T,T+\Delta)$, for all $T$, are needed in relation (8). Moreover, keeping in mind that the bonds $\bar{p}(t,T)$ are fictitious, they do not have to satisfy the boundary condition $\bar{p}(T,T) = 1$, but we still assume this condition in order to simplify the modeling.

To derive a dynamic model for $L(t;T,T+\Delta)$, we may now derive a dynamic model for $\bar{p}(t,T+\Delta)$, where we have to keep in mind that the latter is not a traded quantity. Inspired by a credit-risk analogy, but also by a common practice of deriving multi-curve quantities by adding a spread over the corresponding single-curve (risk-free) quantities, which in this case is the short rate $r_t$, let us define then the Libor (risky) bond prices as

$$\bar{p}(t,T) = E^Q\left\{\exp\left[-\int_t^T (r_u + s_u)du\right] \mid \mathscr{F}_t\right\}, \tag{9}$$

with $s_t$ representing the short rate spread. In case of default risk alone, $s_t$ corresponds to the hazard rate/default intensity, but here it corresponds more generally to all the factors affecting the Libor rate, namely besides credit risk, also liquidity risk, etc. Notice also that the spread is introduced here from the outset. Having for simplicity considered a single tenor $\Delta$ and thus a single $\bar{p}(t, T)$, we shall also consider only a single spread $s_t$. In general, however, one has a spread $s_t^{\Delta}$ for each tenor $\Delta$.

We need now a dynamical model for both $r_t$ and $s_t$ and we shall define this model directly under the martingale measure $Q$ (*martingale modeling*).

# 3 Short Rate Model

## 3.1 The Model

As mentioned, we shall consider a dynamical model for $r_t$ and the single spread $s_t$ under the martingale measure $Q$ that, in practice, has to be calibrated to the market. For this purpose we shall consider a factor model with several factors driving $r_t$ and $s_t$.

The two basic factor model classes for the short rate in the pre-crisis single-curve setup, namely the exponentially affine and the exponentially quadratic model classes, both allow for flexibility and analytical tractability and this in turn allows for closed or semi-closed formulas for linear and optional interest rate derivatives. The former class is usually better known than the latter, but the latter has its own advantages. In fact, for the exponentially affine class one would consider $r_t$ and $s_t$ as given by a linear combination of the factors and so, in order to obtain positivity, one has to consider a square root model for the factors. On the other hand, in the Gaussian exponentially quadratic class, one considers mean reverting Gaussian factor models, but at least some of the factors in the linear combination for $r_t$ and $s_t$ appear as a square. In this way the distribution of the factors remains always Gaussian; in a square-root model it is a non-central $\chi^2-$distribution. Notice also that the exponentially quadratic models can be seen as dual to the square root exponentially affine models.

In the pre-crisis single-curve setting, the exponentially quadratic models have been considered, e.g. in El Karoui et al. [12], Pelsser [29], Gombani and Runggaldier [17], Leippold and Wu [24], Chen et al. [5], and Gaspar [16]. However, since the pre-crisis exponentially affine models are more common, there have also been more attempts to extend them to a post-crisis multi-curve setting (for an overview and details see e.g. Grbac and Runggaldier [18]). A first extension of exponentially quadratic models to a multi-curve setting can be found in Kijima et al. [22] and the present paper is devoted to a possibly full extension.

Let us now present the model for $r_t$ and $s_t$, where we consider not only the short rate $r_t$ itself, but also its spread $s_t$ to be given by a linear combination of the factors, where at least some of the factors appear as a square. To keep the presentation simple, we shall consider a small number of factors and, in order to model also a possible correlation between $r_t$ and $s_t$, the minimal number of factors is three. It also follows from some of the econometric literature that a small number of factors may suffice to adequately model most situations (see also Duffee [10] and Duffie and Gârleanu [11]).

Given three independent affine factor processes $\Psi_t^i,\ i = 1, 2, 3$, having under $Q$ the Gaussian dynamics

$$d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i\, dw_t^i, \quad i = 1, 2, 3, \tag{10}$$

with $b_i, \sigma_i > 0$ and $w_t^i$, $i = 1, 2, 3$, independent $Q-$Wiener processes, we let

$$\begin{cases} r_t = \Psi_t^1 + (\Psi_t^2)^2 \\ s_t = \kappa\Psi_t^1 + (\Psi_t^3)^2 \end{cases}, \tag{11}$$

where $\Psi_t^1$ is the common systematic factor allowing for instantaneous correlation between $r_t$ and $s_t$ with correlation intensity $\kappa$ and $\Psi_t^2$ and $\Psi_t^3$ are the idiosyncratic factors. Other factors may be added to drive $s_t$, but the minimal model containing common and idiosyncratic components requires three factors, as explained above. The common factor is particularly important because we want to take into account the realistic feature of non-zero correlation between $r_t$ and $s_t$ in the model.

*Remark 3.1* The zero mean-reversion level is here considered only for convenience of simpler formulas, but can be easily taken to be positive, so that short rates and spreads can become negative only with small probability (see Kijima and Muromachi [21] for an alternative representation of the spreads in terms of Gaussian factors that guarantee the spreads to remain nonnegative and still allows for correlation between $r_t$ and $s_t$). Note, however, that given the current market situation where the observed interest rates are very close to zero and sometimes also negative, even models with negative mean-reversion level have been considered, as well as models allowing for regime-switching in the mean reversion parameter.

*Remark 3.2* For the short rate itself one could also consider the model $r_t = \phi_t + \Psi_t^1 + (\Psi_t^2)^2$ where $\phi_t$ is a *deterministic shift extension* (see Brigo and Mercurio [2]) that allows for a good fit to the initial term structure in short rate models even with constant model parameters.

In the model (11) we have included a linear term $\Psi_t^1$ which may lead to negative values of rates and spreads, although only with small probability in the case of models of the type (10) with a positive mean reversion level. The advantage of including this linear term is more generality and flexibility in the model. Moreover, it allows to express $\bar{p}(t, T)$ in terms of $p(t, T)$ multiplied by a factor. This property will lead to an *adjustment factor* by which one can express post-crisis quantities in terms of corresponding pre-crisis quantities, see Morino and Runggaldier [27] in which this idea has been first proposed in the context of exponentially affine short rate models for multiple curves.

### 3.2 Bond Prices (OIS and Libor Bonds)

In this subsection we derive explicit pricing formulas for the OIS bonds $p(t, T)$ as defined in (3) and the fictitious Libor bonds $\bar{p}(t, T)$ as defined in (9). Thereby, $r_t$ and $s_t$ are supposed to be given by (11) with the factor processes $\Psi_t^i$ evolving under the standard martingale measure $Q$ according to (10). Defining the matrices

$$F = \begin{bmatrix} -b^1 & 0 & 0 \\ 0 & -b^2 & 0 \\ 0 & 0 & -b^3 \end{bmatrix}, \; D = \begin{bmatrix} \sigma^1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^3 \end{bmatrix} \tag{12}$$

and considering the vector factor process $\Psi_t := [\Psi_t^1, \Psi_t^2, \Psi_t^3]'$ as well as the multivariate Wiener process $W_t := [w_t^1, w_t^2, w_t^3]'$, where $'$ denotes transposition, the dynamics (10) can be rewritten in synthetic form as

$$d\Psi_t = F\Psi_t dt + DdW_t. \tag{13}$$

Using results on exponential quadratic term structures (see Gombani and Runggaldier [17], Filipović [13]), we have

$$\begin{aligned} p(t, T) &= E^Q\left\{e^{-\int_t^T r_u du}\middle|\mathscr{F}_t\right\} = E^Q\left\{e^{-\int_t^T (\Psi_u^1 + (\Psi_u^2)^2)du}\middle|\mathscr{F}_t\right\} \\ &= \exp\Big[-A(t, T) - B'(t, T)\Psi_t - \Psi_t' C(t, T)\Psi_t\Big] \end{aligned} \tag{14}$$

and, setting $R_t := r_t + s_t$,

$$\begin{aligned} \bar{p}(t, T) &= E^Q\left\{e^{-\int_t^T R_u du}\middle|\mathscr{F}_t\right\} = E^Q\left\{e^{-\int_t^T ((1+\kappa)\Psi_u^1 + (\Psi_u^2)^2 + (\Psi_u^3)^2)du}\middle|\mathscr{F}_t\right\} \\ &= \exp\Big[-\bar{A}(t, T) - \bar{B}'(t, T)\Psi_t - \Psi_t' \bar{C}(t, T)\Psi_t\Big], \end{aligned} \tag{15}$$

where $A(t, T)$, $\bar{A}(t, T)$, $B(t, T)$, $\bar{B}(t, T)$, $C(t, T)$ and $\bar{C}(t, T)$ are scalar, vector, and matrix-valued deterministic functions to be determined.

For this purpose we recall the Heath–Jarrow–Morton (HJM) approach for the case when $p(t, T)$ in (14) represents the price of a risk-free zero coupon bond. The HJM approach leads to the so-called HJM drift conditions that impose conditions on the coefficients in (14) so that the resulting prices $p(t, T)$ do not imply arbitrage possibilities. Since the risk-free bonds are traded, the no-arbitrage condition is expressed by requiring $\frac{p(t,T)}{B_t}$ to be a $Q-$martingale for $B_t$ defined as in (1) and it is exactly this martingality property to yield the drift condition. In our case, $p(t, T)$ is the price of an OIS bond that is not necessarily traded and in general does not coincide with the price of a risk-free bond. However, whether the OIS bond is traded or not, $\frac{p(t,T)}{B_t}$ is a $Q-$martingale by the very definition of $p(t, T)$ in (14) (see the first equality in (14)) and so we can follow the same HJM approach to obtain conditions on the coefficients in (14) also in our case.

For what concerns, on the other hand, the coefficients in (15), recall that $\bar{p}(t,T)$ is a fictitious asset that is not traded and thus is not subject to any no-arbitrage condition. Notice, however, that by analogy to $p(t,T)$ in (14), by its very definition given in the first equality in (15), $\frac{\bar{p}(t,T)}{\bar{B}_t}$ is a $Q-$martingale for $\bar{B}_t$ given by $\bar{B}_t := \exp \int_0^t R_u du$. The two cases $p(t,T)$ and $\bar{p}(t,T)$ can thus be treated in complete analogy provided that we use for $\bar{p}(t,T)$ the numeraire $\bar{B}_t$.

We shall next derive from the $Q-$martingality of $\frac{p(t,T)}{B_t}$ and $\frac{\bar{p}(t,T)}{\bar{B}_t}$ conditions on the coefficients in (14) and (15) that correspond to the classical HJM drift condition and lead thus to ODEs for these coefficients. For this purpose we shall proceed by analogy to Sect. 2 in [17], in particular to the proof of Proposition 2.1 therein, to which we also refer for more detail.

Introducing the "instantaneous forward rates" $f(t,T) := -\frac{\partial}{\partial T} \log p(t,T)$ and $\bar{f}(t,T) := -\frac{\partial}{\partial T} \log \bar{p}(t,T)$, and setting

$$a(t,T) := \frac{\partial}{\partial T} A(t,T)\,, \quad b(t,T) := \frac{\partial}{\partial T} B(t,T)\,, \quad c(t,T) := \frac{\partial}{\partial T} C(t,T) \tag{16}$$

and analogously for $\bar{a}(t,T), \bar{b}(t,T), \bar{c}(t,T)$, from (14) and (15) we obtain

$$f(t,T) = a(t,T) + b'(t,T)\Psi_t + \Psi_t' c(t,T)\Psi_t, \tag{17}$$

$$\bar{f}(t,T) = \bar{a}(t,T) + \bar{b}'(t,T)\Psi_t + \Psi_t' \bar{c}(t,T)\Psi_t. \tag{18}$$

Recalling that $r_t = f(t,t)$ and $R_t = \bar{f}(t,t)$, this implies, with $a(t) := a(t,t)$, $b(t) := b(t,t)$, $c(t) := c(t,t)$ and analogously for the corresponding quantities with a bar, that

$$r_t = a(t) + b'(t)\Psi_t + \Psi_t' c(t)\Psi_t \tag{19}$$

and

$$R_t = r_t + s_t = \bar{a}(t) + \bar{b}'(t)\Psi_t + \Psi_t' \bar{c}(t)\Psi_t. \tag{20}$$

Comparing (19) and (20) with (11), we obtain the following conditions where $i, j = 1, 2, 3$, namely

$$\begin{cases} a(t) = 0 \\ b^i(t) = \mathbf{1}_{\{i=1\}} \\ c^{ij}(t) = \mathbf{1}_{\{i=j=2\}} \end{cases} \qquad \begin{cases} \bar{a}(t) = 0 \\ \bar{b}^i(t) = (1+\kappa)\mathbf{1}_{\{i=1\}} \\ \bar{c}^{ij}(t) = \mathbf{1}_{\{i=j=2\}\cup\{i=j=3\}}. \end{cases}$$

Using next the fact that

$$p(t,T) = \exp\left[-\int_t^T f(t,s)ds\right], \quad \bar{p}(t,T) = \exp\left[-\int_t^T \bar{f}(t,s)ds\right],$$

and imposing $\frac{p(t,T)}{B_t}$ and $\frac{\bar{p}(t,T)}{\bar{B}_t}$ to be $Q-$martingales, one obtains ordinary differential equations to be satisfied by $c(t,T)$, $b(t,T)$, $a(t,T)$ and analogously for the quantities with a bar. Integrating these ODEs with respect to the second variable and recalling (16) one obtains (for the details see the proof of Proposition 2.1 in [17])

$$\begin{cases} C_t(t,T) + 2FC(t,T) - 2C(t,T)DDC(t,T) + c(t) = 0, & C(T,T) = 0 \\ \bar{C}_t(t,T) + 2F\bar{C}(t,T) - 2\bar{C}(t,T)DD\bar{C}(t,T) + \bar{c}(t) = 0, & \bar{C}(T,T) = 0 \end{cases} \tag{21}$$

with

$$c(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{c}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

The special forms of $F, D, c(t)$ and $\bar{c}(t)$ together with boundary conditions $C(T,T) = 0$ and $\bar{C}(T,T) = 0$ imply that only $C^{22}$, $\bar{C}^{22}$, $\bar{C}^{33}$ are non-zero and satisfy

$$\begin{cases} C_t^{22}(t,T) - 2b^2C^{22}(t,T) - 2(\sigma^2)^2(C^{22}(t,T))^2 + 1 = 0, & C^{22}(T,T) = 0 \\ \bar{C}_t^{22}(t,T) - 2b^2\bar{C}^{22}(t,T) - 2(\sigma^2)^2(\bar{C}^{22}(t,T))^2 + 1 = 0, & \bar{C}^{22}(T,T) = 0 \\ \bar{C}_t^{33}(t,T) - 2b^3\bar{C}^{33}(t,T) - 2(\sigma^3)^2(\bar{C}^{33}(t,T))^2 + 1 = 0, & \bar{C}^{33}(T,T) = 0 \end{cases} \tag{23}$$

that can be shown to have as solution

$$\begin{cases} C^{22}(t,T) = \bar{C}^{22}(t,T) = \frac{2(e^{(T-t)h^2}-1)}{2h^2+(2b^2+h^2)(e^{(T-t)h^2}-1)} \\ \bar{C}^{33}(t,T) = \frac{2(e^{(T-t)h^3}-1)}{2h^3+(2b^3+h^3)(e^{(T-t)h^3}-1)} \end{cases} \tag{24}$$

with $h^i = \sqrt{4(b^i)^2 + 8(\sigma^i)^2} > 0,\ i = 2, 3.$

Next, always by analogy to the proof of Proposition 2.1 in [17], the vectors of coefficients $B(t,T)$ and $\bar{B}(t,T)$ of the first order terms can be seen to satisfy the following system

$$\begin{cases} B_t(t,T) + B(t,T)F - 2B(t,T)DDC(t,T) + b(t) = 0, & B(T,T) = 0 \\ \bar{B}_t(t,T) + \bar{B}(t,T)F - 2\bar{B}(t,T)DD\bar{C}(t,T) + \bar{b}(t) = 0, & \bar{B}(T,T) = 0 \end{cases} \tag{25}$$

with

$$b(t) = [1, 0, 0] \quad \bar{b}(t) = [(1+\kappa), 0, 0].$$

Noticing similarly as above that only $B^1(t,T), \bar{B}^1(t,T)$ are non-zero, system (25) becomes

$$\begin{cases} B_t^1(t,T) - b^1B^1(t,T) + 1 = 0 & B^1(T,T) = 0 \\ \bar{B}_t^1(t,T) - b^1\bar{B}^1(t,T) + (1+\kappa) = 0 & \bar{B}^1(T,T) = 0 \end{cases} \tag{26}$$

leading to the explicit solution

$$\begin{cases} B^1(t,T) = \frac{1}{b^1}\Big(1 - e^{-b^1(T-t)}\Big) \\ \bar{B}^1(t,T) = \frac{1+\kappa}{b^1}\Big(1 - e^{-b^1(T-t)}\Big) = (1+\kappa)B^1(t,T). \end{cases} \tag{27}$$

Finally, $A(t,T)$ and $\bar{A}(t,T)$ have to satisfy

$$\begin{cases} A_t(t,T) + (\sigma^2)^2 C^{22}(t,T) - \frac{1}{2}(\sigma^1)^2(B^1(t,T))^2 = 0, \\ \bar{A}_t(t,T) + (\sigma^2)^2 \bar{C}^{22}(t,T) + (\sigma^3)^2 \bar{C}^{33}(t,T) - \frac{1}{2}(\sigma^1)^2(\bar{B}^1(t,T))^2 = 0 \end{cases} \tag{28}$$

with boundary conditions $A(T,T) = 0, \bar{A}(T,T) = 0$. The explicit expressions can be obtained simply by integrating the above equations.

Summarizing, we have proved the following:

**Proposition 3.1** *Assume that the OIS short rate $r$ and the spread $s$ are given by (11) with the factor processes $\Psi_t^i$, $i = 1, 2, 3$, evolving according to (10) under the standard martingale measure $Q$. The time-$t$ price of the OIS bond $p(t,T)$, as defined in (3), is given by*

$$p(t,T) = \exp[-A(t,T) - B^1(t,T)\Psi_t^1 - C^{22}(t,T)(\Psi_t^2)^2], \tag{29}$$

*and the time-$t$ price of the fictitious Libor bond $\bar{p}(t,T)$, as defined in (9), by*

$$\begin{aligned} \bar{p}(t,T) &= \exp[-\bar{A}(t,T) - (\kappa+1)B^1(t,T)\Psi_t^1 - C^{22}(t,T)(\Psi_t^2)^2 - \bar{C}^{33}(t,T)(\Psi_t^3)^2] \\ &= p(t,T)\exp[-\tilde{A}(t,T) - \kappa B^1(t,T)\Psi_t^1 - \bar{C}^{33}(t,T)(\Psi_t^3)^2], \end{aligned} \tag{30}$$

*where $\tilde{A}(t,T) := \bar{A}(t,T) - A(t,T)$ with $A(t,T)$ and $\bar{A}(t,T)$ given by (28), $B^1(t,T)$ given by (27) and $C^{22}(t,T)$ and $C^{33}(t,T)$ given by (24).*

In particular, expression (30) gives $\bar{p}(t,T)$ in terms of $p(t,T)$. Based on this we shall derive in the following section the announced *adjustment factor* allowing to pass from pre-crisis quantities to the corresponding post-crisis quantities.

## *3.3 Forward Measure*

The underlying factor model was defined in (10) under the standard martingale measure $Q$. For derivative prices, which we shall determine in the following two sections, it will be convenient to work under forward measures, for which, given the single tenor $\Delta$, we shall consider a generic $(T+\Delta)$-forward measure. The density process to change the measure from $Q$ to $Q^{T+\Delta}$ is

$$\mathscr{L}_t := \frac{d\,Q^{T+\Delta}}{d\,Q}\Big|_{\mathscr{F}_t} = \frac{p(t,T+\Delta)}{p(0,T+\Delta)}\,\frac{1}{B_t} \tag{31}$$

from which it follows by (29) and the martingale property of $\left(\frac{p(t,T+\Delta)}{B_t}\right)_{t\le T+\Delta}$ that

$$d\mathscr{L}_t = \mathscr{L}_t\,\left(-B^1(t,T+\Delta)\sigma^1 dw_t^1 - 2C^{22}(t,T+\Delta)\Psi_t^2\sigma^2 dw_t^2\right).$$

This implies by Girsanov's theorem that

$$\begin{cases} dw_t^{1,T+\Delta} = dw_t^1 + \sigma^1 B^1(t,T+\Delta)dt \\ dw_t^{2,T+\Delta} = dw_t^2 + 2C^{22}(t,T+\Delta)\Psi_t^2\sigma^2 dt \\ dw_t^{3,T+\Delta} = dw_t^3 \end{cases} \tag{32}$$

are $Q^{T+\Delta}-$Wiener processes. From the $Q-$dynamics (10) we then obtain the following $Q^{T+\Delta}-$dynamics for the factors

$$\begin{aligned} d\Psi_t^1 &= -\left[b^1\Psi_t^1 + (\sigma^1)^2 B^1(t,T+\Delta)\right]dt + \sigma^1 dw_t^{1,T+\Delta} \\ d\Psi_t^2 &= -\left[b^2\Psi_t^2 + 2(\sigma^2)^2 C^{22}(t,T+\Delta)\Psi_t^2\right]dt + \sigma^2 dw_t^{2,T+\Delta} \\ d\Psi_t^3 &= -b^3\Psi_t^3 dt + \sigma^3 dw_t^{3,T+\Delta}. \end{aligned} \tag{33}$$

*Remark 3.3* While in the dynamics (10) for $\Psi_t^i$, $(i=1,2,3)$ under $Q$ we had for simplicity assumed a zero mean-reversion level, under the $(T+\Delta)$-forward measure the mean-reversion level is for $\Psi_t^1$ now different from zero due to the measure transformation.

**Lemma 3.1** *Analogously to the case when $p(t,T)$ represents the price of a risk-free zero coupon bond, also for $p(t,T)$ viewed as OIS bond we have that $\frac{p(t,T)}{p(t,T+\Delta)}$ is a $Q^{T+\Delta}-$martingale.*

*Proof* We have seen that also for OIS bonds as defined in (3) we have that, with $B_t$ as in (1), the ratio $\frac{p(t,T)}{B_t}$ is a $Q-$martingale. From Bayes' formula we then have

$$E^{T+\Delta}\left\{\frac{p(T,T)}{p(T,T+\Delta)}\mid\mathscr{F}_t\right\} = \frac{E^Q\left\{\frac{1}{p(0,T+\Delta)}\,\frac{1}{B_{T+\Delta}}\,\frac{p(T,T)}{p(T,T+\Delta)}|\mathscr{F}_t\right\}}{E^Q\left\{\frac{1}{p(0,T+\Delta)}\,\frac{1}{B_{T+\Delta}}|\mathscr{F}_t\right\}}$$

$$= \frac{E^Q\left\{\frac{p(T,T)}{p(T,T+\Delta)}E^Q\left\{\frac{1}{B_{T+\Delta}}|\mathscr{F}_T\right\}|\mathscr{F}_t\right\}}{\frac{p(t,T+\Delta)}{B_t}} = \frac{B_t E^Q\left\{\frac{p(T,T)}{p(T,T+\Delta)}\,\frac{p(T,T+\Delta)}{B_T}|\mathscr{F}_t\right\}}{p(t,T+\Delta)}$$

$$= \frac{B_t E^Q\left\{\frac{p(T,T)}{B_T}|\mathscr{F}_t\right\}}{p(t,T+\Delta)} = \frac{p(t,T)}{p(t,T+\Delta)},$$

thus proving the statement of the lemma. □

We recall that we denote the expectation with respect to the measure $Q^{T+\Delta}$ by $E^{T+\Delta}\{\cdot\}$. The dynamics in (33) lead to Gaussian distributions for $\Psi_t^i,\ i=1,2,3$ that, given $B^1(\cdot)$ and $C^{22}(\cdot)$, have mean and variance

$$E^{T+\Delta}\{\Psi_t^i\} = \bar{\alpha}_t^i = \bar{\alpha}_t^i(b^i,\sigma^i) \quad , \quad Var^{T+\Delta}\{\Psi_t^i\} = \bar{\beta}_t^i = \bar{\beta}_t^i(b^i,\sigma^i),$$

which can be explicitly computed. More precisely, we have

$$\begin{cases} \bar{\alpha}_t^1 & = e^{-b^1 t}\Big[\Psi_0^1 - \frac{(\sigma^1)^2}{2(b^1)^2} e^{-b^1(T+\Delta)}(1-e^{2b^1 t}) - \frac{(\sigma^1)^2}{(b^1)^2}(1-e^{b^1 t})\Big] \\ \bar{\beta}_t^1 & = e^{-2b^1 t}(e^{2b^1 t}-1)\frac{(\sigma^1)^2}{2(b^1)} \\ \bar{\alpha}_t^2 & = e^{-(b^2 t + 2(\sigma^2)^2 \tilde{C}^{22}(t,T+\Delta))}\Psi_0^2 \\ \bar{\beta}_t^2 & = e^{-(2b^2 t + 4(\sigma^2)^2 \tilde{C}^{22}(t,T+\Delta))}\int_0^t e^{2b^2 s + 4(\sigma^2)^2 \tilde{C}^{22}(s,T+\Delta)}(\sigma^2)^2 ds \\ \bar{\alpha}_t^3 & = e^{-b^3 t}\Psi_0^3 \\ \bar{\beta}_t^3 & = e^{-2b^3 t}\frac{(\sigma^3)^2}{2b^3}(e^{2b^3 t}-1), \end{cases} \tag{34}$$

with

$$\tilde{C}^{22}(t,T+\Delta) = \frac{2(2\log(2b^2(e^{(T+\Delta-t)h^2}-1) + h^2(e^{(T+\Delta-t)h^2}+1)) + t(2b^2+h^2))}{(2b^2+h^2)(2b^2-h^2)} - \frac{2(2\log(2b^2(e^{(T+\Delta)h^2}-1) + h^2(e^{(T+\Delta)h^2}+1))}{(2b^2+h^2)(2b^2-h^2)} \tag{35}$$

and $h^2 = \sqrt{(2b^2)^2 + 8(\sigma^2)^2}$, and where we have assumed deterministic initial values $\Psi_0^1$, $\Psi_0^2$ and $\Psi_0^3$. For details of the above computation see the proof of Corollary 4.1.3. in Meneghello [25].

## 4 Pricing of Linear Interest Rate Derivatives

We have discussed in Sect. 3.2 the pricing of OIS and Libor bonds in the Gaussian, exponentially quadratic short rate model introduced in Sect. 3.1. In the remaining part of the paper we shall be concerned with the pricing of interest rate derivatives, namely with derivatives having the Libor rate as underlying rate. In the present section we shall deal with the basic linear derivatives, namely FRAs and interest rate swaps, while nonlinear derivatives will then be dealt with in the following Sect. 5. For the FRA rates discussed in the next Sect. 4.1 we shall in Sect. 4.1.1 exhibit an *adjustment factor* allowing to pass from the single-curve FRA rate to the multi-curve FRA rate.

## 4.1 FRAs

We start by recalling the definition of a standard forward rate agreement. We emphasize that we use a text-book definition which differs slightly from a market definition, see Mercurio [26].

**Definition 4.1** Given the time points $0 \leq t \leq T < T + \Delta$, a *forward rate agreement* (FRA) is an OTC derivative that allows the holder to lock in at the generic date $t \leq T$ the interest rate between the inception date $T$ and the maturity $T + \Delta$ at a fixed value $R$. At maturity $T + \Delta$ a payment based on the interest rate $R$, applied to a notional amount of $N$, is made and the one based on the relevant floating rate *(generally the spot Libor rate* $L(T; T, T + \Delta)$*)* is received.

Recalling that for the Libor rate we had postulated the relation (8) to hold at the inception time $T$, namely

$$L(T; T, T + \Delta) = \frac{1}{\Delta}\left(\frac{1}{\bar{p}(T, T + \Delta)} - 1\right),$$

the price, at $t \leq T$, of the FRA with fixed rate $R$ and notional $N$ can be computed under the $(T + \Delta)$-forward measure as

$$\begin{aligned} &P^{FRA}(t; T, T + \Delta, R, N) \\ &= N\Delta p(t, T + \Delta)E^{T+\Delta}\left\{L(T; T, T + \Delta) - R \mid \mathscr{F}_t\right\} \\ &= Np(t, T + \Delta)E^{T+\Delta}\left\{\frac{1}{\bar{p}(T, T + \Delta)} - (1 + \Delta R) \mid \mathscr{F}_t\right\}, \end{aligned} \tag{36}$$

Defining

$$\bar{\nu}_{t,T} := E^{T+\Delta}\left\{\frac{1}{\bar{p}(T, T + \Delta)} \mid \mathscr{F}_t\right\}, \tag{37}$$

it is easily seen from (36) that the *fair rate of the FRA, namely the FRA rate*, is given by

$$\bar{R}_t = \frac{1}{\Delta}\left(\bar{\nu}_{t,T} - 1\right). \tag{38}$$

In the *single-curve case* we have instead

$$R_t = \frac{1}{\Delta}\left(\nu_{t,T} - 1\right), \tag{39}$$

where, given that $\frac{p(\cdot,T)}{p(\cdot,T+\Delta)}$ is a $Q^{T+\Delta}-$martingale (see Lemma 3.1),

$$\nu_{t,T} := E^{T+\Delta}\left\{\frac{1}{p(T, T + \Delta)} \mid \mathscr{F}_t\right\} = \frac{p(t, T)}{p(t, T + \Delta)}, \tag{40}$$

which is the classical expression for the FRA rate in the single-curve case. Notice that, contrary to (37), the expression in (40) can be explicitly computed on the basis of bond price data without requiring an interest rate model.

#### 4.1.1 Adjustment Factor

We shall show here the following:

**Proposition 4.1** *We have the relationship*

$$\bar{\nu}_{t,T} = \nu_{t,T} \cdot Ad_t^{T,\Delta} \cdot Res_t^{T,\Delta} \tag{41}$$

*with*

$$\begin{aligned} Ad_t^{T,\Delta} &:= E^Q \left\{ \frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \mid \mathscr{F}_t \right\} = E^Q \Big\{ \exp\Big[ \tilde{A}(T,T+\Delta) \\ &\quad + \kappa B^1(T,T+\Delta)\Psi_T^1 + \bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2 \Big] \mid \mathscr{F}_t \Big\} \end{aligned} \tag{42}$$

*and*

$$Res_t^{T,\Delta} = \exp\Big[ -\kappa \frac{(\sigma^1)^2}{2(b^1)^3} \left(1 - e^{-b^1\Delta}\right) \left(1 - e^{-b^1(T-t)}\right)^2 \Big], \tag{43}$$

*where* $\tilde{A}(t,T)$ *is defined after (30),* $B^1(t,T)$ *in (27) and* $\bar{C}^{33}(t,T)$ *in (24).*

*Proof* Firstly, from (30) we obtain

$$\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} = e^{\tilde{A}(T,T+\Delta)+\kappa B^1(T,T+\Delta)\Psi_T^1+\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2}. \tag{44}$$

In (37) we now change back from the $(T+\Delta)$-forward measure to the standard martingale measure using the density process $\mathscr{L}_t$ given in (31). Using furthermore the above expression for the ratio of the OIS and the Libor bond prices and taking into account the definition of the short rate $r_t$ in terms of the factor processes, we obtain

$$\begin{aligned} \bar{\nu}_{t,T} &= E^{T+\Delta} \left\{ \frac{1}{\bar{p}(T,T+\Delta)} \middle| \mathscr{F}_t \right\} = \mathscr{L}_t^{-1} E^Q \left\{ \frac{\mathscr{L}_T}{\bar{p}(T,T+\Delta)} \middle| \mathscr{F}_t \right\} \\ &= \frac{1}{p(t,T+\Delta)} E^Q \left\{ \exp\Big( -\int_t^T r_u du \Big) \frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \middle| \mathscr{F}_t \right\} \\ &= \frac{1}{p(t,T+\Delta)} \exp[\tilde{A}(T,T+\Delta)] E^Q \left\{ e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2} \middle| \mathscr{F}_t \right\} \end{aligned}$$

$$
\begin{aligned}
&\cdot E^Q\left\{e^{-\int_t^T(\Psi_u^1+(\Psi_u^2)^2)du}e^{\kappa B^1(T,T+\Delta)\Psi_T^1}\Big|\mathscr{F}_t\right\}\\
&=\frac{1}{p(t,T+\Delta)}\exp[\tilde{A}(T,T+\Delta)]E^Q\left\{e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2}\Big|\mathscr{F}_t\right\}\\
&\cdot E^Q\left\{e^{-\int_t^T\Psi_u^1du}e^{\kappa B^1(T,T+\Delta)\Psi_T^1}\Big|\mathscr{F}_t\right\}E^Q\left\{e^{-\int_t^T(\Psi_u^2)^2du}\Big|\mathscr{F}_t\right\},
\end{aligned}
\tag{45}
$$

where we have used the independence of the factors $\Psi^i,\ i=1,2,3$ under $Q$.

Recall now from the theory of affine processes (see e.g. Lemma 2.1 in Grbac and Runggaldier [18]) that, for a process $\Psi_t^1$ satisfying (10), we have for all $\delta, K\in\mathbb{R}$

$$
E^Q\left\{\exp\left[-\int_t^T\delta\Psi_u^1du-K\Psi_T^1\right]\mid\mathscr{F}_t\right\}=\exp[\alpha^1(t,T)-\beta^1(t,T)\Psi_t^1],
\tag{46}
$$

where

$$
\begin{cases}
\beta^1(t,T)=Ke^{-b^1(T-t)}-\frac{\delta}{b^1}\left(e^{-b^1(T-t)}-1\right)\\
\alpha^1(t,T)=\frac{(\sigma^1)^2}{2}\int_t^T(\beta^1(u,T))^2du.
\end{cases}
$$

Setting $K=-\kappa B^1(T,T+\Delta)$ and $\delta=1$, and recalling from (27) that $B^1(t,T)=\frac{1}{b^1}\left(1-e^{-b^1(T-t)}\right)$, this leads to

$$
\begin{aligned}
&E^Q\left\{e^{-\int_t^T\Psi_u^1du}e^{\kappa B^1(T,T+\Delta)\Psi_T^1}\Big|\mathscr{F}_t\right\}\\
&=\exp\left[\frac{(\sigma^1)^2}{2}(\kappa B^1(T,T+\Delta))^2\int_t^T e^{-2b^1(T-u)}du\right.\\
&-\kappa B^1(T,T+\Delta)(\sigma^1)^2\int_t^T B^1(u,T)e^{-b^1(T-u)}du+\frac{(\sigma^1)^2}{2}\int_t^T(B^1(u,T))^2du\\
&\left.+\left(\kappa B^1(T,T+\Delta)e^{-b^1(T-t)}-B^1(t,T)\right)\Psi_t^1\right].
\end{aligned}
\tag{47}
$$

On the other hand, from the results of Sect. 3.2 we also have that, for a process $\Psi_t^2$ satisfying (10),

$$
E^Q\left\{\exp\left[-\int_t^T(\Psi_u^2)^2du\right]\mid\mathscr{F}_t\right\}=\exp\left[-\alpha^2(t,T)-C^{22}(t,T)(\Psi_t^2)^2\right],
$$

where $C^{22}(t,T)$ corresponds to (24) and (see (28))

$$
\alpha^2(t,T)=(\sigma^2)^2\int_t^T C^{22}(u,T)du.
$$

This implies that

$$
\begin{aligned}
&E^Q\left\{\exp\left[-\int_t^T (\Psi_u^2)^2 du\right] \mid \mathscr{F}_t\right\} \\
&\quad = \exp\left[-(\sigma^2)^2\int_t^T C^{22}(u,T)du - C^{22}(t,T)\left(\Psi_t^2\right)^2\right]. \qquad (48)
\end{aligned}
$$

Replacing (47) and (48) into (45), and recalling the expression for $p(t,T)$ in (29) with $A(\cdot)$, $B^1(\cdot)$, $C^{22}(\cdot)$ according to (28), (27) and (24) respectively, we obtain

$$
\begin{aligned}
\bar{\nu}_{t,T} = &\frac{p(t,T)}{p(t,T+\Delta)} e^{\tilde{A}(T,T+\Delta)} E^Q\left[e^{\bar{C}^{33}(T,T+\Delta)(\Psi_T^3)^2} | \mathscr{F}_t\right] \\
&\cdot \exp\left[\frac{(\sigma^1)^2}{2}(\kappa B^1(T,T+\Delta))^2 \int_t^T e^{-2b^1(T-u)}du + \kappa B^1(T,T+\Delta)e^{-b^1(T-t)}\Psi_t^1\right] \\
&\cdot \exp\left[-\kappa B^1(T,T+\Delta)(\sigma^1)^2\int_t^T B^1(u,T)e^{-b^1(T-u)}du\right]. \qquad (49)
\end{aligned}
$$

We recall the expression (44) for $\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)}$ and the fact that, according to (46), we have

$$
\begin{aligned}
&E^Q\left\{e^{\kappa B^1(T,T+\Delta)\Psi_T^1} | \mathscr{F}_t\right\} \\
&\quad = \exp\left[\frac{(\sigma^1)^2}{2}(\kappa B^1(T,T+\Delta))^2\int_t^T e^{-2b^1(T-u)}du + \kappa B^1(T,T+\Delta)e^{-b^1(T-t)}\Psi_t^1\right].
\end{aligned}
$$

Inserting these expressions into (49) we obtain the result, namely

$$
\begin{aligned}
\bar{\nu}_{t,T} &= \tfrac{p(t,T)}{p(t,T+\Delta)} E^Q\left\{\tfrac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} | \mathscr{F}_t\right\} \\
&\quad \cdot \exp\left[-\kappa B^1(T,T+\Delta)(\sigma^1)^2\int_t^T B^1(u,T)e^{-b^1(T-u)}du\right] \\
&= \tfrac{p(t,T)}{p(t,T+\Delta)} E^Q\left\{\tfrac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} | \mathscr{F}_t\right\} \\
&\quad \cdot \exp\left[-\tfrac{\kappa}{b^1}(e^{-b^1\Delta}-1)(\sigma^1)^2\left(\tfrac{1}{2(b^1)^2}(1-e^{-2b^1(T-t)}) - \tfrac{1}{(b^1)^2}(1-e^{-b^1(T-t)})\right)\right], \qquad (50)
\end{aligned}
$$

where we have also used the fact that

$$
\begin{aligned}
\int_t^T B^1(u,T)e^{-b^1(T-u)}du &= \int_t^T \frac{1}{b^1}\left(1-e^{-b^1(T-u)}\right)e^{-b^1(T-u)}du \\
&= -\frac{1}{2(b^1)^2}\left(1-e^{-2b^1(T-t)}\right) + \frac{1}{(b^1)^2}\left(1-e^{-b^1(T-t)}\right).
\end{aligned}
$$

□

*Remark 4.1* The adjustment factor $Ad_t^{T,\Delta}$ allows for some intuitive interpretations. Here we mention only the easiest one for the case when $\kappa = 0$ (independence of $r_t$ and $s_t$). In this case we have $r_t + s_t > r_t$ implying that $\bar{p}(T, T + \Delta) < p(T, T + \Delta)$ so that $Ad_t^{T,\Delta} \geq 1$. Furthermore, always for $\kappa = 0$, the residual factor has value $Res_t^{T,\Delta} = 1$. All this in turn implies $\bar{\nu}_{t,T} \geq \nu_{t,T}$ and with it $\bar{R}_t \geq R_t$, which is what one would expect to be the case.

*Remark 4.2* (*Calibration to the initial term structure*). The parameters in the model (10) for the factors $\Psi_t^i$ and thus also in the model (11) for the short rate $r_t$ and the spread $s_t$ are the coefficients $b^i$ and $\sigma^i$ for $i = 1, 2, 3$. From (14) notice that, for $i = 1, 2$, these coefficients enter the expressions for the OIS bond prices $p(t, T)$ that can be assumed to be observable since they can be bootstrapped from the market quotes for the OIS swap rates. We may thus assume that these coefficients, i.e. $b^i$ and $\sigma^i$ for $i = 1, 2$, can be calibrated as in the pre-crisis single-curve short rate models. It remains to calibrate $b^3$, $\sigma^3$ and, possibly the correlation coefficient $\kappa$. Via (15) they affect the prices of the fictitious Libor bonds $\bar{p}(t, T)$ that are, however, not observable. One may observe though the FRA rates $R_t$ and $\bar{R}_t$ and thus also $\nu_{t,T}$, as well as $\bar{\nu}_{t,T}$. Via (41) this would then allow one to calibrate also the remaining parameters. This task would turn out to be even simpler if one would have access to the value of $\kappa$ by other means.

We emphasize that in order to ensure a good fit to the initial bond term structure, a deterministic shift extension of the model or time-dependent coefficients $b^i$ could be considered. We recall also that we have assumed the mean-reversion level equal to zero for simplicity; in practice it would be one more coefficient to be calibrated for each factor $\Psi_t^i$.

## 4.2 Interest Rate Swaps

We first recall the notion of a *(payer) interest rate swap*. Given a collection of dates $0 \leq T_0 < T_1 < \cdots < T_n$ with $\gamma \equiv \gamma_k := T_k - T_{k-1}$ $(k = 1, \cdots, n)$, as well as a notional amount $N$, a payer swap is a financial contract, where a stream of interest payments on the notional $N$ is made at a fixed rate $R$ in exchange for receiving an analogous stream corresponding to the Libor rate. Among the various possible conventions concerning the fixing for the Libor and the payment dates, we choose here the one where, for each interval $[T_{k-1}, T_k]$, the Libor rates are fixed in advance and the payments are made in arrears. The swap is thus initiated at $T_0$ and the first payment is made at $T_1$. A *receiver swap* is completely symmetric with the interest at the fixed rate being received; here we concentrate on payer swaps.

The arbitrage-free price of the swap, evaluated at $t \leq T_0$, is given by the following expression where, analogously to $E^{T+\Delta}\{\cdot\}$, we denote by $E^{T_k}\{\cdot\}$ the expectation with respect to the forward measure $Q^{T_k}$ $(k = 1, \cdots, n)$

$$P^{Sw}(t;T_0,T_n,R) = \gamma\sum_{k=1}^{n} p(t,T_k)E^{T_k}\left\{L(T_{k-1};T_{k-1},T_k) - R|\mathscr{F}_t\right\}$$
$$= \gamma\sum_{k=1}^{n} p(t,T_k)\left(L(t;T_{k-1},T_k) - R\right). \tag{51}$$

For easier notation we have assumed the notional to be 1, i.e. $N = 1$.

We shall next obtain an explicit expression for $P^{Sw}(t;T_0,T_n,R)$ starting from the first equality in (51). To this effect, recalling from (24) that $C^{22}(t,T) = \bar{C}^{22}(t,T)$, introduce again some shorthand notation, namely

$$\begin{aligned} A_k &:= \bar{A}(T_{k-1},T_k), B_k^1 := B^1(T_{k-1},T_k),\\ C_k^{22} &:= C^{22}(T_{k-1},T_k) = \bar{C}^{22}(T_{k-1},T_k),\ \bar{C}_k^{33} := \bar{C}^{33}(T_{k-1},T_k). \end{aligned} \tag{52}$$

The crucial quantity to be computed in (51) is the following one

$$E^{T_k}\{\gamma L(T_{k-1};T_{k-1},T_k)|\mathscr{F}_t\} = E^{T_k}\left\{\frac{1}{\bar{p}(T_{k-1},T_k)}|\mathscr{F}_t\right\} - 1$$
$$= e^{A_k}E^{T_k}\{\exp((\kappa+1)B_k^1\Psi_{T_{k-1}}^1 + C_k^{22}(\Psi_{T_{k-1}}^2)^2 + \bar{C}_k^{33}(\Psi_{T_{k-1}}^3)^2)|\mathscr{F}_t\} - 1, \tag{53}$$

where we have used the first relation on the right in (30). The expectations in (53) have to be computed under the measures $Q^{T_k}$, under which, by analogy to (33), the factors have the dynamics

$$\begin{aligned} d\Psi_t^1 &= -\left[b^1\Psi_t^1 + (\sigma^1)^2B^1(t,T_k)\right]dt + \sigma^1 dw_t^{1,k}\\ d\Psi_t^2 &= -\left[b^2\Psi_t^2 + 2(\sigma^2)^2C^{22}(t,T_k)\Psi_t^2\right]dt + \sigma^2 dw_t^{2,k}\\ d\Psi_t^3 &= -b^3\Psi_t^3dt + \sigma^3 dw_t^{3,k}. \end{aligned} \tag{54}$$

where $w^{i,k}$, $i = 1,2,3$, are independent Wiener processes with respect to $Q^{T_k}$. A straightforward generalization of (46) to the case where the factor process $\Psi_t^1$ satisfies the following affine Hull–White model

$$d\Psi_t^1 = (a^1(t) - b^1\Psi_t^1)dt + \sigma^1 dw_t$$

can be obtained as follows

$$E^Q\left\{\exp\left[-\int_t^T \delta\Psi_u^1du - K\Psi_T^1\right] \mid \mathscr{F}_t\right\} = \exp[\alpha^1(t,T) - \beta^1(t,T)\Psi_t^1], \tag{55}$$

with

$$\begin{cases} \beta^1(t,T) = Ke^{-b^1(T-t)} - \frac{\delta}{b^1}\left(e^{-b^1(T-t)} - 1\right)\\ \alpha^1(t,T) = \frac{(\sigma^1)^2}{2}\int_t^T(\beta^1(u,T))^2du - \int_t^T a^1(u)\beta^1(u,T)du. \end{cases} \tag{56}$$

We apply this result to our situation where under $Q^{T_k}$ the process $\Psi_t^1$ satisfies the first SDE in (54) and thus corresponds to the above dynamics with $a^1(t) = -(\sigma^1)^2 B^1(t, T_k)$. Furthermore, setting $K = -(\kappa+1)\, B_k^1$ and $\delta = 0$, we obtain for the first expectation in the second line of (53)

$$E^{T_k}\{\exp((\kappa+1)B_k^1 \Psi^1_{T_{k-1}} | \mathscr{F}_t\} = \exp[\Gamma^1(t, T_k) - \rho^1(t, T_k)\, \Psi_t^1], \tag{57}$$

with

$$\begin{cases} \rho^1(t, T_k) &= -(\kappa+1) B_k^1 e^{-b^1(T_k - t)} \\ \Gamma^1(t, T_k) &= \frac{(\sigma^1)^2}{2} \int_t^{T_k} \left(\rho^1(u, T_k)\right)^2 du + (\sigma^1)^2 \int_t^{T_k} B^1(u, T_k) \rho^1(u, T_k) du. \end{cases} \tag{58}$$

For the remaining two expectations in the second line of (53) we shall use the following:

**Lemma 4.1** *Let a generic process $\Psi_t$ satisfy the dynamics*

$$d\Psi_t = b(t)\Psi_t dt + \sigma\, dw_t \tag{59}$$

*with $w_t$ a Wiener process. Then, for all $C \in \mathbb{R}$ such that $E^Q\left\{\exp\left[C\,(\Psi_T)^2\right]\right\} < \infty$, we have*

$$E^Q\left\{\exp\left[C\,(\Psi_T)^2\right] \mid \mathscr{F}_t\right\} = \exp\left[\Gamma(t, T) - \rho(t, T)\,(\Psi_t)^2\right] \tag{60}$$

*with $\rho(t, T)$ and $\Gamma(t, T)$ satisfying*

$$\begin{cases} \rho_t(t, T) + 2b(t)\rho(t, T) - 2(\sigma)^2\,(\rho(t, T))^2 = 0\,; \quad \rho(T, T) = -C \\ \Gamma_t(t, T) = (\sigma)^2 \rho(t, T). \end{cases} \tag{61}$$

*Proof* An application of Itô's formula yields that the nonnegative process $\Phi_t := (\Psi_t)^2$ satisfies the following SDE

$$d\Phi_t = \left((\sigma)^2 + 2b(t)\,\Phi_t\right) dt + 2\sigma \sqrt{\Phi_t}\, dw_t.$$

We recall that a process $\Phi_t$ given in general form by

$$d\Phi_t = (a + \lambda(t)\Phi_t)dt + \eta \sqrt{\Phi_t}\, dw_t,$$

with $a, \eta > 0$ and $\lambda(t)$ a deterministic function, is a CIR process. Thus, $(\Psi_t)^2$ is equivalent in distribution to a CIR process with coefficients given by

$$\lambda(t) = 2b(t) \quad , \quad \eta = 2\sigma \quad , \quad a = (\sigma)^2.$$

From the theory of affine term structure models (see e.g. Lamberton and Lapeyre [23], or Lemma 2.2 in Grbac and Runggaldier [18]) it now follows that

$$\begin{aligned} E^Q\left\{\exp\left[C\,(\Psi_T)^2\right] \mid \mathscr{F}_t\right\} &= E^Q\left\{\exp\left[C\,\Phi_T\right] \mid \mathscr{F}_t\right\} = \exp\left[\Gamma(t,T) - \rho(t,T)\,\Phi_t\right] \\ &= \exp\left[\Gamma(t,T) - \rho(t,T)\,(\Psi_t)^2\right] \end{aligned}$$

with $\rho(t,T)$ and $\Gamma(t,T)$ satisfying (61).

**Corollary 4.1** *When $b(t)$ is constant with respect to time, i.e. $b(t) \equiv b$, so that also $\lambda(t) \equiv \lambda$, then the equations for $\rho(t,T)$ and $\Gamma(t,T)$ in (61) admit an explicit solution given by*

$$\begin{cases} \rho(t,T) = \frac{4bhe^{2b(T-t)}}{4(\sigma)^2 he^{2b(T-t)}-1} \quad \text{with} \quad h := \frac{C}{4(\sigma)^2 C + 4b} \\ \Gamma(t,T) = -(\sigma)^2 \int_t^T \rho(u,T)du. \end{cases} \tag{62}$$

Coming now to the second expectation in the second line of (53) and using the second equation in (54), we set

$$b(t) := -\left[b^2 + 2(\sigma^2)^2 C^{22}(t,T_k)\right],\ \sigma := \sigma^2,\ C = C_k^{22}$$

and apply Lemma 4.1, provided that the parameters $b^2$ and $\sigma^2$ of the process $\Psi^2$ are such that $C = C_k^{22}$ satisfies the assumption from the lemma. We thus obtain

$$E^{T_k}\{\exp(C_k^{22}(\Psi^2_{T_{k-1}})^2)|\mathscr{F}_t\} = \exp[\Gamma^2(t,T_k) - \rho^2(t,T_k)(\Psi_t^2)^2], \tag{63}$$

with $\rho^2(t,T)$, $\Gamma^2(t,T)$ satisfying

$$\begin{cases} \rho_t^2(t,T) - 2\left[b^2 + 2(\sigma^2)^2 C^{22}(t,T_k)\right]\rho^2(t,T) - 2(\sigma^2)^2(\rho^2(t,T))^2 = 0 \\ \rho^2(T_k,T_k) = -C_k^{22} \\ \Gamma^2(t,T) = -(\sigma^2)^2 \int_t^T \rho^2(u,T)du. \end{cases} \tag{64}$$

Finally, for the third expectation in the second line of (53), we may take advantage of the fact that the dynamics of $\Psi_t^3$ do not change when passing from the measure $Q$ to the forward measure $Q^{T_k}$. We can then apply Lemma 4.1, this time with (see the third equation in (54))

$$b(t) := -b^3,\ \sigma := \sigma^3,\ C = \bar{C}_k^{33}$$

and ensuring that the parameters $b^3$ and $\sigma^3$ of the process $\Psi^3$ are such that $C = \bar{C}_k^{33}$ satisfies the assumption from the lemma. Since $b(t)$ is constant with respect to time, also Corollary 4.1 applies and we obtain

$$E^{T_k}\{\exp(\bar{C}_k^{33}(\Psi^3_{T_{k-1}})^2)|\mathscr{F}_t\} = \exp[\Gamma^3(t,T_k) - \rho^3(t,T_k)(\Psi_t^3)^2],$$

where

$$\begin{cases} \rho^3(t,T_k) = \frac{-4b^3 h_k^3 e^{-2b^3(T_k-t)}}{4(\sigma^3)^2 h_k^3 e^{-2b^3(T_k-t)}-1} \quad \text{with} \quad h_k^3 = \frac{\bar{C}_k^{33}}{4(\sigma^3)^2\bar{C}_k^{33}-4b^3} \\ \Gamma^3(t,T_k) = -(\sigma^3)^2 \int_t^{T_k} \rho^3(u,T_k)du. \end{cases} \tag{65}$$

With the use of the explicit expressions for the expectations in (53), and taking also into account the expression for $p(t,T)$ in (29), it follows immediately that the arbitrage-free swap price in (51) can be expressed according to the following

**Proposition 4.2** *The price of a payer interest rate swap at $t \leq T_0$ is given by*

$$\begin{aligned} P^{Sw}(t;T_0,T_n,R) &= \gamma \sum_{k=1}^n p(t,T_k) E^{T_k}\left\{L(T_{k-1};T_{k-1},T_k) - R|\mathscr{F}_t\right\} \\ &= \sum_{k=1}^n p(t,T_k)\Big(D_{t,k} e^{-\rho^1(t,T_k)\Psi_t^1 - \rho^2(t,T_k)(\Psi_t^2)^2 - \rho^3(t,T_k)(\Psi_t^3)^2} - (R\gamma+1)\Big) \\ &= \sum_{k=1}^n \Big(D_{t,k} e^{-A_{t,k}} e^{-\tilde{B}_{t,k}^1\Psi_t^1 - \tilde{C}_{t,k}^{22}(\Psi_t^2)^2 - \tilde{C}_{t,k}^{33}(\Psi_t^3)^2} \\ &\qquad - (R\gamma+1)e^{-A_{t,k}} e^{-B_{t,k}^1\Psi_t^1 - C_{t,k}^{22}(\Psi_t^2)^2}\Big), \end{aligned} \tag{66}$$

*where*

$$\begin{aligned} &A_{t,k} := A(t,T_k),\ B_{t,k}^1 := B^1(t,T_k),\ C_{t,k}^{22} := C^{22}(t,T_k) \\ &\tilde{B}_{t,k}^1 := B_{t,k}^1 + \rho^1(t,T_k),\ \tilde{C}_{t,k}^{22} := C_{t,k}^{22} + \rho^2(t,T_k),\ \tilde{C}_{t,k}^{33} := \rho^3(t,T_k) \\ &D_{t,k} := e^{A_k}\exp[\Gamma^1(t,T_k) + \Gamma^2(t,T_k) + \Gamma^3(t,T_k)], \end{aligned} \tag{67}$$

*with $\rho^i(t,T_k)$, $\Gamma^i(t,T_k)$ $(i=1,2,3)$ determined according to (58), (64), and (65) respectively and with $A_k$ as in (52).*

## 5 Nonlinear/optional Interest Rate Derivatives

In this section we consider the main nonlinear interest rate derivatives with the Libor rate as underlying. They are also called *optional derivatives* since they have the form of an option. In Sect. 5.1 we shall consider the case of caps and, symmetrically, that of floors. In the subsequent Sect. 5.2 we shall then concentrate on swaptions as options on a payer swap of the type discussed in Sect. 4.2.

### *5.1 Caps and Floors*

Since floors can be treated in a completely symmetric way to the caps simply by interchanging the roles of the fixed rate and the Libor rate, we shall concentrate here on caps. Furthermore, to keep the presentation simple, we consider here just a single caplet for the time interval $[T, T+\Delta]$ and for a fixed rate $R$ (recall also that we consider just one tenor $\Delta$). The payoff of the caplet at time $T+\Delta$ is thus $\Delta(L(T;T,T+\Delta)-R)^+$, assuming the notional $N=1$, and its time-$t$ price $P^{Cpl}(t;T+\Delta,R)$ is given by the following risk-neutral pricing formula under the forward measure $Q^{T+\Delta}$

$$P^{Cpl}(t;T+\Delta,R) = \Delta\, p(t,T+\Delta)E^{T+\Delta}\left\{(L(T;T,T+\Delta)-R)^+ \mid \mathscr{F}_t\right\}.$$

In view of deriving pricing formulas, recall from Sect. 3.3 that, under the $(T+\Delta)-$forward measure, at time $T$ the factors $\Psi_T^i$ have independent Gaussian distributions (see (34)) with mean and variance given, for $i=1,2,3$, by

$$E^{T+\Delta}\{\Psi_t^i\} = \bar{\alpha}_t^i = \bar{\alpha}_t^i(b^i,\sigma^i), \qquad Var^{T+\Delta}\{\Psi_t^i\} = \bar{\beta}_t^i = \bar{\beta}_t^i(b^i,\sigma^i).$$

In the formulas below we shall consider the joint probability density function of $(\Psi_T^1, \Psi_T^2, \Psi_T^3)$ under the $T+\Delta$ forward measure, namely, using the independence of the processes $\Psi_t^i$, $(i=1,2,3)$,

$$f_{(\Psi_T^1,\Psi_T^2,\Psi_T^3)}(x_1,x_2,x_3) = \prod_{i=1}^{3} f_{\Psi_T^i}(x_i) = \prod_{i=1}^{3} \mathscr{N}(x_i, \bar{\alpha}_T^i, \bar{\beta}_T^i), \tag{68}$$

and use the shorthand notation $f_i(\cdot)$ for $f_{\Psi_T^i}(\cdot)$ in the sequel. We shall also write $\bar{A}, B^1, C^{22}, \bar{C}^{33}$ for the corresponding functions evaluated at $(T, T+\Delta)$ and given in (28), (27) and (24) respectively.

Setting $\tilde{R} := 1+\Delta R$, and recalling the first equality in (30), the time-0 price of the caplet can be expressed as

$$\begin{aligned}
P^{Cpl}(0;T+\Delta,R) &= \Delta\, p(0,T+\Delta)E^{T+\Delta}\left\{(L(T;T,T+\Delta)-R)^+\right\} \\
&= p(0,T+\Delta)E^{T+\Delta}\left\{\left(\frac{1}{\bar{p}(T,T+\Delta)} - \tilde{R}\right)^+\right\} \\
&= p(0,T+\Delta)E^{T+\Delta}\left\{\left(e^{\bar{A}+(\kappa+1)B^1\Psi_T^1 + C^{22}(\Psi_T^2)^2 + \bar{C}^{33}(\Psi_T^3)^2} - \tilde{R}\right)^+\right\} \\
&= p(0,T+\Delta)\int_{\mathbb{R}^3}\left(e^{\bar{A}+(\kappa+1)B^1 x + C^{22}y^2 + \bar{C}^{33}z^2} - \tilde{R}\right)^+ \\
&\qquad \cdot f_{(\Psi_T^1,\Psi_T^2,\Psi_T^3)}(x,y,z)\,d(x,y,z).
\end{aligned} \tag{69}$$

To proceed, we extend to the multi-curve context an idea suggested in Jamshidian [19] (where it is applied to the pricing of coupon bonds) by considering the function

$$g(x,y,z) := \exp[\bar{A} + (\kappa+1)B^1 x + C^{22}y^2 + \bar{C}^{33}z^2]. \tag{70}$$

Noticing that $\bar{C}^{33}(T, T + \Delta) > 0$ (see (24) together with the fact that $h^3 > 0$ and $2b^3 + h^3 > 0$), for fixed $x, y$ the function $g(x, y, z)$ can be seen to be continuous and increasing for $z \geq 0$ and decreasing for $z < 0$ with $\lim_{z\to\pm\infty} g(x, y, z) = +\infty$. It will now be convenient to introduce some objects according to the following:

**Definition 5.1** Let a set $M \subset \mathbb{R}^2$ be given by

$$M := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq \tilde{R}\} \tag{71}$$

and let $M^c$ be its complement. Furthermore, for $(x, y) \in M$ let

$$\bar{z}^1 = \bar{z}^1(x, y), \quad \bar{z}^2 = \bar{z}^2(x, y)$$

be the solutions of $g(x, y, z) = \tilde{R}$. They satisfy $\bar{z}^1 \leq 0 \leq \bar{z}^2$.

Notice that, for $z \leq \bar{z}^1 \leq 0$ and $z \geq \bar{z}^2 \geq 0$, we have $g(x, y, z) \geq g(x, y, \bar{z}^k) = \tilde{R}$, and for $z \in (\bar{z}^1, \bar{z}^2)$, we have $g(x, y, z) < \tilde{R}$. In $M^c$ we have $g(x, y, z) \geq g(x, y, 0) > \tilde{R}$ and thus no solution of the equation $g(x, y, z) = \tilde{R}$.

In view of the main result of this subsection, given in Proposition 5.1 below, we prove as a preliminary the following:

**Lemma 5.1** *Assuming that the (nonnegative) coefficients $b^3, \sigma^3$ in the dynamics (10) of the factor $\Psi_t^3$ satisfy the condition*

$$b^3 \geq \frac{\sigma^3}{\sqrt{2}}, \tag{72}$$

*we have that $1 - 2\bar{\beta}_T^3 \bar{C}^{33} > 0$, where $\bar{C}^{33} = \bar{C}^{33}(T, T + \Delta)$ is given by (24) and where $\bar{\beta}_T^3 = \frac{(\sigma^3)^2}{2b^3}(1 - e^{-2b^3 T})$ according to (34).*

*Proof* From the definitions of $\bar{\beta}_T^3$ and $\bar{C}^{33}$ we may write

$$1 - 2\bar{\beta}_T^3 \bar{C}^{33} = 1 - \left(1 - e^{-2b^3 T}\right) \frac{2\left(e^{\Delta h^3} - 1\right)}{2\frac{b^3 h^3}{(\sigma^3)^2} + \frac{b^3}{(\sigma^3)^2}(2b^3 + h^3)\left(e^{\Delta h^3} - 1\right)}. \tag{73}$$

Notice next that $b^3 > 0$ implies that $1 - e^{-2b^3 T} \in (0, 1)$ and that $\frac{b^3 h^3}{(\sigma^3)^2} \geq 0$. From (73) it then follows that a sufficient condition for $1 - 2\bar{\beta}_T^3 \bar{C}^{33} > 0$ to hold is that

$$2 \leq \frac{b^3}{(\sigma^3)^2}(2b^3 + h^3). \tag{74}$$

Given that, see definition after (24), $h^3 = 2\sqrt{(b^3)^2 + 2(\sigma^3)^2} \geq 2b^3$, the condition (74) is satisfied under our assumption (72). □

**Proposition 5.1** *Under assumption (72) we have that the time-0 price of the caplet for the time interval $[T, T+\Delta]$ and with fixed rate $R$ is given by*

$$P^{Cpl}(0; T+\Delta, R) = p(0, T+\Delta)\Bigg[\int_M e^{\bar{A}+(\kappa+1)B^1x+C^{22}(y)^2} \cdot \Big[\gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33})\Big(\Phi(d^1(x,y)) + \Phi(-d^2(x,y))\Big) - e^{\bar{C}^{33}(\bar{z}^1(x,y))^2}\Phi(d^3(x,y)) + e^{\bar{C}^{33}(\bar{z}^2(x,y))^2}\Phi(-d^4(x,y))\Big] \times f_1(x)f_2(y)dxdy + \gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33})\int_{M^c} e^{\bar{A}+(\kappa+1)B^1x+C^{22}(y)^2} \times f_1(x)f_2(y)dxdy - \tilde{R}\, Q^{T+\Delta}\Big\{(\Psi_T^1, \Psi_T^2) \in M^c\Big\}\Bigg], \tag{75}$$

*where $\Phi(\cdot)$ is the cumulative standard Gaussian distribution function, $M$ and $M^c$ are as in Definition 5.1,*

$$\begin{cases} d^1(x,y) := \frac{\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}\bar{z}^1(x,y)-(\bar{\alpha}_T^3-\theta\bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}} \\ d^2(x,y) := \frac{\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}\bar{z}^2(x,y)-(\bar{\alpha}_T^3-\theta\bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}} \\ d^3(x,y) := \frac{\bar{z}^1(x,y)-\bar{\alpha}_T^3}{\sqrt{\bar{\beta}_T^3}} \\ d^4(x,y) := \frac{\bar{z}^2(x,y)-\bar{\alpha}_T^3}{\sqrt{\bar{\beta}_T^3}} \end{cases} \tag{76}$$

*with $\theta := \frac{\bar{\alpha}_T^3\left(1-1/\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}\right)}{\bar{\beta}_T^3}$, which by Lemma 5.1 is well defined under the given assumption (72), and with $\gamma(\bar{\alpha}_T^3, \bar{\beta}_T^3, \bar{C}^{33}) := \frac{e^{(\frac{1}{2}(\theta)^2\bar{\beta}_T^3-\bar{\alpha}_T^3\theta)}}{\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}}$.*

*Remark 5.1* Notice that, once the set $M$ and its complement $M^c$ from Definition 5.1 are made explicit, the integrals, as well as the probability in (75), can be computed explicitly.

*Proof* On the basis of the sets $M$ and $M^c$ we can continue (69) as

$$\begin{aligned} P^{Cpl}(0; T+\Delta, R) &= p(0, T+\Delta)\int_{\mathbb{R}^3}\left(e^{\bar{A}+(\kappa+1)B^1x+C^{22}y^2+\bar{C}^{33}z^2} - \tilde{R}\right)^+ \\ &\quad \cdot f_{(\Psi_T^1,\Psi_T^2,\Psi_T^3)}(x,y,z)d(x,y,z) \\ &= p(0, T+\Delta)\int_{M\times\mathbb{R}}\left(e^{\bar{A}+(\kappa+1)B^1x+C^{22}y^2+\bar{C}^{33}z^2} - \tilde{R}\right)^+ \\ &\quad \cdot f_{(\Psi_T^1,\Psi_T^2,\Psi_T^3)}(x,y,z)d(x,y,z) \end{aligned}$$

$$
\begin{aligned}
&+ p(0, T+\Delta) \int_{M^c \times \mathbb{R}} \left( e^{\bar{A}+(\kappa+1)B^1 x + C^{22} y^2 + \bar{C}^{33} z^2} - \tilde{R} \right)^+ \\
&\cdot f_{(\Psi_T^1, \Psi_T^2, \Psi_T^3)}(x, y, z) d(x, y, z) \\
=:& \, P^1(0; T+\Delta) + P^2(0; T+\Delta). \qquad (77)
\end{aligned}
$$

We shall next compute separately the two terms in the last equality in (77) distinguishing between two cases according to whether $(x, y) \in M$ or $(x, y) \in M^c$.

*Case (i):* For $(x, y) \in M$ we have from Definition 5.1 that there exist $\bar{z}^1(x, y) \le 0$ and $\bar{z}^2(x, y) \ge 0$ so that for $z \in [\bar{z}^1, \bar{z}^2]$ we have $g(x, y, z) \le g(x, y, \bar{z}^k) = \tilde{R}$. For $P^1(0; T+\Delta)$ we now obtain

$$
\begin{aligned}
P^1(0; T+\Delta) &= p(0, T+\Delta) \\
&\cdot \int_M e^{\bar{A}+(\kappa+1)B^1 x + C^{22} y^2} \left( \int_{-\infty}^{\bar{z}^1(x,y)} (e^{\bar{C}^{33} z^2} - e^{\bar{C}^{33} (\bar{z}^1)^2}) f_3(z) dz \right. \\
&\left. + \int_{\bar{z}^2(x,y)}^{+\infty} (e^{\bar{C}^{33} z^2} - e^{\bar{C}^{33} (\bar{z}^2)^2}) f_3(z) dz \right) f_2(y) f_1(x) dy dx. \qquad (78)
\end{aligned}
$$

Next, using the results of Sect. 3.3 concerning the Gaussian distribution $f_3(\cdot) = f_{\Psi_T^3}(\cdot)$, we obtain the calculations in (79) below, where, recalling Lemma 5.1, we make successively the following changes of variables: $\zeta := \sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} z$, $\theta := \frac{\bar{\alpha}_T^3 (1 - 1/\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}})}{\bar{\beta}_T^3}$, $s := \frac{\zeta - (\bar{\alpha}_T^3 - \theta \bar{\beta}_T^3)}{\sqrt{\bar{\beta}_T^3}}$ and where $d^i(x, y),\ i = 1, \cdots, 4$ are as defined in (76)

$$
\begin{aligned}
\int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33} z^2} f_3(z) dz &= \int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33} z^2} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(z - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} dz \\
&= \int_{-\infty}^{\bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} z - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\frac{\bar{\alpha}_T^3 (\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} - 1)}{\bar{\beta}_T^3} z} dz \\
&= \int_{-\infty}^{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\zeta - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\frac{\bar{\alpha}_T^3 (1 - 1/\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}})}{\bar{\beta}_T^3} \zeta} \frac{1}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} d\zeta \\
&= \frac{1}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \int_{-\infty}^{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}} \bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi \bar{\beta}_T^3}} e^{-\frac{1}{2} \frac{(\zeta - \bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} e^{-\theta \zeta} d\zeta \\
&= \frac{e^{(\frac{1}{2} (\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)}}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \int_{-\infty}^{d^1(x,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds = \frac{e^{(\frac{1}{2} (\theta)^2 \bar{\beta}_T^3 - \bar{\alpha}_T^3 \theta)}}{\sqrt{1 - 2\bar{\beta}_T^3 \bar{C}^{33}}} \Phi(d^1(x, y)). \qquad (79)
\end{aligned}
$$

On the other hand, always using the results of Sect. 3.3 concerning the Gaussian distribution $f_3(\cdot) = f_{\Psi_T^3}(\cdot)$ and making this time the change of variables $\zeta := \frac{(z-\bar{\alpha}_T^3)}{\sqrt{\bar{\beta}_T^3}}$, we obtain

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{\bar{C}^{33}(\bar{z}^1)^2} f_3(z)dz = e^{\bar{C}^{33}(\bar{z}^1)^2} \int_{-\infty}^{\bar{z}^1(x,y)} \frac{1}{\sqrt{2\pi\bar{\beta}_T^3}} e^{-\frac{1}{2}\frac{(z-\bar{\alpha}_T^3)^2}{\bar{\beta}_T^3}} dz$$

$$= e^{\bar{C}^{33}(\bar{z}^1)^2} \int_{-\infty}^{d^3(x,y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2} d\zeta = e^{\bar{C}^{33}(\bar{z}^1)^2} \Phi(d^3(x,y)). \tag{80}$$

Similarly, we have

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{\bar{C}^{33}z^2} f_3(z)dz = \frac{1}{\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}} e^{(\frac{1}{2}(\theta)^2\bar{\beta}_T^3 - \bar{\alpha}_T^3\theta)} \Phi(-d^2(x,y)) \tag{81}$$

$$\int_{\bar{z}^2(x,y)}^{+\infty} e^{\bar{C}^{33}(\bar{z}^1)^2} f_3(z)dz = e^{\bar{C}^{33}(\bar{z}^2)^2} \Phi(-d^4(x,y)).$$

*Case (ii):* We come next to the case $(x, y) \in M^c$, for which $g(x, y, z) \geq g(x, y, 0) > \tilde{R}$. For $P^2(0; T + \Delta)$ we obtain

$$\begin{aligned}
P^2(0; T+\Delta) &= p(0, T+\Delta) \int_{M^c\times\mathbb{R}} \left( e^{\bar{A}+(\kappa+1)B^1x + C^{22}y^2 + \bar{C}^{33}z^2} - \tilde{R} \right) \\
&\quad \cdot f_3(z) f_2(y) f_1(x) dz dy dx \\
&= p(0, T+\Delta) \Big( e^{\bar{A}} \int_{M^c} e^{(\kappa+1)B^1x + C^{22}y^2} f_1(x) f_2(y) dx dy \int_{\mathbb{R}} e^{\bar{C}^{33}z^2} f_3(z) dz \\
&\quad - \tilde{R} Q^{T+\Delta}[(\Psi_T^1, \Psi_T^2) \in M^c] \Big) \\
&= p(0, T+\Delta) \Big( e^{\bar{A}} \Big[ \int_{M^c} e^{(\kappa+1)B^1x + C^{22}y^2} f_1(x) f_2(y) dx dy \Big] \frac{e^{(\frac{1}{2}(\theta^3)^2\bar{\beta}_T^3 - \bar{\alpha}_T^3\theta^3)}}{\sqrt{1-2\bar{\beta}_T^3\bar{C}^{33}}} \\
&\quad - \tilde{R} Q^{T+\Delta}[(\Psi_T^1, \Psi_T^2) \in M^c] \Big),
\end{aligned} \tag{82}$$

where we computed the integral over $\mathbb{R}$ analogously to (79).

Adding the two expressions derived for Cases (i) and (ii), we obtain the statement of the proposition. □

## 5.2 Swaptions

We start by recalling some of the most relevant aspects of a (payer) swaption. Considering a swap (see Sect. 4.2) for a given collection of dates $0 \leq T_0 < T_1 < \cdots < T_n$, a swaption is an option to enter the swap at a pre-specified initiation date $T \leq T_0$, which is thus also the maturity of the swaption and that, for simplicity of notation, we assume to coincide with $T_0$, i.e. $T = T_0$. The arbitrage-free swaption price at $t \leq T_0$ can be computed as

$$P^{Swn}(t; T_0, T_n, R) = p(t, T_0)E^{T_0}\left\{\left(P^{Sw}(T_0; T_n, R)\right)^+ |\mathscr{F}_t\right\}, \tag{83}$$

where we have used the shorthand notation $P^{Sw}(T_0; T_n, R) = P^{Sw}(T_0; T_0, T_n, R)$.

We first state the next Lemma, that follows immediately from the expression for $\rho^3(t, T_k)$ and the corresponding expression for $h_k^3$ in (65).

**Lemma 5.2** *We have the equivalence*

$$\rho^3(t, T_k) > 0 \Leftrightarrow h_k^3 \in \left(0, \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}\right). \tag{84}$$

This lemma prompts us to split the swaption pricing problem into two cases:

$$\begin{array}{ll} \textbf{Case(1)}: & h_k^3 < 0 \text{ or } h_k^3 > \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}} \\ \textbf{Case(2)}: & 0 < h_k^3 < \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}. \end{array} \tag{85}$$

Note from the definition of $\rho^3(t, T_k)$ that $h_k^3 \neq \frac{1}{4(\sigma^3)^2 e^{-2b^3(T_k-t)}}$ and that $h_k^3 = 0$ would imply $\bar{C}_k^{33} = 0$ which corresponds to a trivial case in which the factor $\Psi^3$ is not present in the dynamics of the spread $s$, hence the inequalities in Case (1) and Case (2) above are indeed strict.

To proceed, we shall introduce some more notation. In particular, instead of only one function $g(x, y, z)$ as in (70), we shall consider also a function $h(x, y)$, more precisely, we shall define here the continuous functions

$$g(x, y, z) := \sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} e^{-\tilde{B}_{0,k}^1 x - \tilde{C}_{0,k}^{22} y^2 - \tilde{C}_{0,k}^{33} z^2} \tag{86}$$

$$h(x, y) := \sum_{k=1}^{n} (R\gamma + 1) e^{-A_{0,k}} e^{-B_{0,k}^1 x - C_{0,k}^{22} y^2}, \tag{87}$$

with the coefficients given by (67) for $t = T_0$. Note that by a slight abuse of notation we write $D_{0,k}$ for $D_{T_0,k}$ and similarly for other coefficients above, always meaning $t = T_0$ in (67). We distinguish the two cases specified in (85):

For *Case (1)* we have (see (67) and Lemma 5.2) that $\tilde{C}^{33}_{0,k} = \rho^3(T_0, T_k) < 0$ for all $k = 1, \cdots, n$, and so the function $g(x, y, z)$ in (86) is, for given $(x, y)$, monotonically increasing for $z \geq 0$ and decreasing for $z < 0$ with

$$\lim_{z \to \pm\infty} g(x, y, z) = +\infty.$$

For *Case (2)* we have instead that $\tilde{C}^{33}_{0,k} = \rho^3(T_0, T_k) > 0$ for all $k = 1, \cdots, n$ and so the nonnegative function $g(x, y, z)$ in (86) is, for given $(x, y)$, monotonically decreasing for $z \geq 0$ and increasing for $z < 0$ with

$$\lim_{z \to \pm\infty} g(x, y, z) = 0.$$

Analogously to Definition 5.1 we next introduce the following objects:

**Definition 5.2** Let a set $\bar{M} \subset \mathbb{R}^2$ be given by

$$\bar{M} := \{(x, y) \in \mathbb{R}^2 \mid g(x, y, 0) \leq h(x, y)\}. \tag{88}$$

Since $g(x, y, z)$ and $h(x, y)$ are continuous, $\bar{M}$ is closed, measurable and connected. Let $\bar{M}^c$ be its complement. Furthermore, we define two functions $\bar{z}^1(x, y)$ and $\bar{z}^2(x, y)$ distinguishing between the two Cases (1) and (2) as specified in (85).

Case (1) If $(x, y) \in \bar{M}$, we have $g(x, y, 0) \leq h(x, y)$ and so there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ for which, for $i = 1, 2$,

$$\begin{aligned} g(x, y, \bar{z}^i) &= \sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} e^{-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2 - \tilde{C}^{33}_{0,k} (\bar{z}^i)^2} \\ &= \sum_{k=1}^{n} (R\gamma + 1) e^{-A_{0,k}} e^{-B^1_{0,k} x - C^{22}_{0,k} y^2} = h(x, y) \end{aligned} \tag{89}$$

and, for $z \notin [\bar{z}^1, \bar{z}^2]$, one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$.
If $(x, y) \in \bar{M}^c$, we have $g(x, y, 0) > h(x, y)$ so that $g(x, y, z) \geq g(x, y, 0) > h(x, y)$ for all $z$ and we have no points corresponding to $\bar{z}^1(x, y)$ and $\bar{z}^2(x, y)$ above.

Case (2) If $(x, y) \in \bar{M}$, we have, as for Case (1), $g(x, y, 0) \leq h(x, y)$ and so there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ for which, for $i = 1, 2$, (89) holds. However, this time it is for $z \in [\bar{z}^1, \bar{z}^2]$ that one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$.
If $(x, y) \in M^c$, then we are in the same situation as for Case (1).

Starting from (83) combined with (66) and taking into account the set $\bar{M}$ according to Definition 5.2, we can obtain the following expression for the swaption price at $t = 0$. As for the caps, here too we consider the joint Gaussian distribution $f_{(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})}(x, y, z)$ of the factors under the $T_0$−forward measure $Q^{T_0}$ and we have

$$
\begin{aligned}
P^{Swn}(0; T_0, T_n, R) &= p(0, T_0) E^{T_0} \left\{ \left( P^{Sw}(T_0; T_n, R) \right)^+ | \mathscr{F}_0 \right\} \\
&= p(0, T_0) \int_{\mathbb{R}^3} \Big[ \sum_{k=1}^n D_{0,k} e^{-A_{0,k}} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2 - \tilde{C}^{33}_{0,k} z^2) \\
&\quad - \sum_{k=1}^n (R\gamma + 1) e^{-A_{0,k}} \exp(-B^1_{0,k} x - C^{22}_{0,k} y^2) \Big]^+ f_{(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})}(x, y, z) dx dy dz \\
&= p(0, T_0) \int_{\bar{M} \times \mathbb{R}} \Big[ \sum_{k=1}^n D_{0,k} e^{-A_{0,k}} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2 - \tilde{C}^{33}_{0,k} z^2) \\
&\quad - \sum_{k=1}^n (R\gamma + 1) e^{-A_{0,k}} \exp(-B^1_{0,k} x - C^{22}_{0,k} y^2) \Big]^+ f_{(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})}(x, y, z) dx dy dz \\
&\quad + p(0, T_0) \int_{\bar{M}^c \times \mathbb{R}} \Big[ \sum_{k=1}^n D_{0,k} e^{-A_{0,k}} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2 - \tilde{C}^{33}_{0,k} z^2) \\
&\quad - \sum_{k=1}^n (R\gamma + 1) e^{-A_{0,k}} \exp(-B^1_{0,k} x - C^{22}_{0,k} y^2) \Big]^+ f_{(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})}(x, y, z) dx dy dz \\
&=: P^1(0; T_0, T_n, R) + P^2(0; T_0, T_n, R). \qquad (90)
\end{aligned}
$$

We can now state and prove the main result of this subsection consisting in a pricing formula for swaptions for the Gaussian exponentially quadratic model of this paper. We have

**Proposition 5.2** *Assume that the parameters in the model are such that, if $h^3_k$ belongs to Case (1) in (85) and $h^3_k > 0$, then $h^3_k > \frac{1}{4(\sigma^3)^2 e^{-2b^3 T_k}}$. The arbitrage-free price at $t = 0$ of the swaption with payment dates $T_1 < \cdots < T_n$ such that $\gamma = \gamma_k := T_k - T_{k-1}$ ($k = 1, \cdots, n$), with a given fixed rate $R$ and a notional $N = 1$, can be computed as follows where we distinguish between the Cases (1) and (2) specified in Definition 5.2.*

**Case (1)** *We have*

$$
\begin{aligned}
P^{Swn}(0; T_0, T_n, R) = p(0, T_0) \Bigg\{ \sum_{k=1}^n e^{-A_{0,k}} \Big[ \int_{\bar{M}} & D_{0,k} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2) \\
& \cdot \left( \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}^3_{T_0} - \bar{\alpha}^3_{T_0} \theta_k)}}{\sqrt{1 + 2\bar{\beta}^3_{T_0} \tilde{C}^{33}_{0,k}}} \Phi(d^1_k(x, y)) - e^{-\tilde{C}^{33}_{0,k} (\bar{z}^1)^2} \Phi(d^2_k(x, y)) \right. \\
& \left. + \frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}^3_{T_0} - \bar{\alpha}^3_{T_0} \theta_k)}}{\sqrt{1 + 2\bar{\beta}^3_{T_0} \tilde{C}^{33}_{0,k}}} \Phi(-d^3_k(x, y)) - e^{-\tilde{C}^{33}_{0,k} (\bar{z}^2)^2} \Phi(-d^4_k(x, y)) \right)
\end{aligned}
$$

$$\times f_2(y)f_1(x)dydx + \int_{\bar{M}^c}\Big(D_{0,k}e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2}\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}$$
$$-(R\gamma+1)e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}\Big)f_2(y)f_1(x)dydx\Big]\Big\}. \quad (91)$$

**Case (2)** *We have*

$$P^{Swn}(0;T_0,T_n,R) = p(0,T_0)\Big\{\sum_{k=1}^{n} e^{-A_{0,k}}$$
$$\Big[\int_{\bar{M}} D_{0,k}\exp(-\tilde{B}^1_{0,k}x - \tilde{C}^{22}_{0,k}y^2)\Big(\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}$$
$$\times\Big[\Phi(d^3_k(x,y)) - \Phi(d^1_k(x,y))\Big] - e^{-\tilde{C}^{33}_{0,k}(\bar{z}^1)^2}\Big[\Phi(d^4_k(x,y))$$
$$-\Phi(d^2_k(x,y))\Big]\Big)f_2(y)f_1(x)dydx$$
$$+\int_{\bar{M}^c}\Big(D_{0,k}e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2}\frac{e^{(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}$$
$$-(R\gamma+1)e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}\Big)f_2(y)f_1(x)dydx\Big]\Big\}. \quad (92)$$

*The coefficients in these formulas are as specified in (67) for $t = T_0$, $f_1(x)$, $f_2(x)$ are the Gaussian densities corresponding to (68) for $T = T_0$ and the functions $d^i_k(x,y)$, for $i = 1,\dots,4$ and $k = 1,\dots,n$, are given by*

$$\begin{cases} d^1_k(x,y) := \frac{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\bar{z}^1(x,y)-(\bar{\alpha}^3_{T_0}-\theta_k\bar{\beta}^3_{T_0})}{\sqrt{\bar{\beta}^3_{T_0}}} \\ d^2_k(x,y) := \frac{\bar{z}^1(x,y)-\bar{\alpha}^3_{T_0}}{\sqrt{\bar{\beta}^3_{T_0}}} \\ d^3_k(x,y) := \frac{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\bar{z}^2(x,y)-(\bar{\alpha}^3_{T_0}-\theta_k\bar{\beta}^3_{T_0})}{\sqrt{\bar{\beta}^3_{T_0}}} \\ d^4_k(x,y) := \frac{\bar{z}^2(x,y)-\bar{\alpha}^3_{T_0}}{\sqrt{\bar{\beta}^3_{T_0}}} \end{cases} \quad (93)$$

*with $\theta_k := \frac{\bar{\alpha}^3_{T_0}\left(1-1/\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}\right)}{\bar{\beta}^3_{T_0}}$, for $k = 1,\dots,n$, and where $\bar{z}^1 = \bar{z}^1(x,y)$, $\bar{z}^2 = \bar{z}^2(x,y)$ are solutions in $z$ of the equation $g(x,y,z) = h(x,y)$.*

*In addition, the mean and variance values for the Gaussian factors $(\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0})$ are here given by*

$$\begin{cases} \bar{\alpha}^1_{T_0} = e^{-b^1 T_0}\Psi^1_0 - \frac{(\sigma^1)^2}{2(b^1)^2}e^{-b^1 T_0}(1-e^{2b^1 T_0}) - \frac{(\sigma^1)^2}{(b^1)^2}(1-e^{b^1 T_0})\Big] \\ \bar{\beta}^1_{T_0} = e^{-2b^1 T_0}(e^{2b^1 T_0}-1)\frac{(\sigma^1)^2}{2(b^1)} \\ \bar{\alpha}^2_{T_0} = e^{-b^2 T_0}\Psi^2_0 \\ \bar{\beta}^2_{T_0} = e^{-2b^2 T_0}\displaystyle\int_0^{T_0} e^{2b^2 u + 4(\sigma^2)^2 \bar{C}^{22}(u,T_0)}(\sigma^2)^2 du \\ \bar{\alpha}^3_{T_0} = e^{-b^3 T_0}\Psi^3_0 \\ \bar{\beta}^3_{T_0} = e^{-2b^3 T_0}\frac{(\sigma^3)^2}{2b^3}(e^{2b^3 T_0}-1). \end{cases} \tag{94}$$

*Remark 5.2* A remark analogous to Remark 5.1 applies here too concerning the sets $\bar{M}$ and $\bar{M}^c$.

*Proof* First of all notice that, when $h^3_k < 0$ or $h^3_k > \frac{1}{4(\sigma^3)^2 e^{-2b^3 T_k}}$ in Case (1), this implies $1 + 2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k} \geq 0$ (in Case (2) we always have $1 + 2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k} \geq 0$). Hence, the square-root of the latter expression in the various formulas of the statement of the proposition is well-defined. This can be checked, similarly as in the proof of Lemma 5.1, by direct computation taking into account the definitions of $\bar{\beta}^3_{T_0}$ in (94) and of $\tilde{C}^{33}_{0,k}$ in (67) and (65) for $t = T_0$.

We come now to the statement for:

**Case 1.** We distinguish between whether $(x, y) \in \bar{M}$ or $(x, y) \in \bar{M}^c$ and compute separately the two terms in the last equality in (90).

**(i)** For $(x, y) \in \bar{M}$ we have from Definition 5.2 that there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ so that, for $z \notin [\bar{z}^1, \bar{z}^2]$, one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$. Taking into account that, under $Q^{T_0}$, the random variables $\Psi^1_{T_0}, \Psi^2_{T_0}, \Psi^3_{T_0}$ are independent, so that we shall write $f_{(\Psi^1_{T_0},\Psi^2_{T_0},\Psi^3_{T_0})}(x, y, z) = f_1(x)f_2(y)f_3(z)$ (see also (68) and the line following it), we obtain

$$\begin{aligned} P^1(0; T_0, T_n, R) = p(0, T_0)\Big[\sum_{k=1}^n D_{0,k}e^{-A_{0,k}}\int_M \exp(-\tilde{B}^1_{0,k}x - \tilde{C}^{22}_{0,k}y^2) \\ \cdot\Big(\int_{-\infty}^{\bar{z}^1(x,y)} \exp(-\tilde{C}^{33}_{0,k}z^2)f_3(z)dz \\ - \int_{-\infty}^{\bar{z}^1(x,y)} \exp(-\tilde{C}^{33}_{0,k}(\bar{z}^1)^2)f_3(z)dz \\ + \int_{\bar{z}^2(x,y)}^{+\infty} \exp(-\tilde{C}^{33}_{0,k}z^2)f_3(z)dz \\ - \int_{\bar{z}^2(x,y)}^{+\infty} \exp(-\tilde{C}^{33}_{0,k}(\bar{z}^2)^2)f_3(z)dz\Big)f_2(y)f_1(x)dydx\Big]. \end{aligned} \tag{95}$$

By means of calculations that are completely analogous to those in the proof of Proposition 5.1, we obtain, corresponding to (79)–(81) respectively and with the same meaning of the symbols, the following explicit expressions for the integrals in the last four lines of (95), namely

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{-\tilde{C}^{33}_{0,k}z^2} f_3(z)dz = \frac{e^{\left(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k\right)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}\Phi(d^1_k(x,y)), \tag{96}$$

$$\int_{-\infty}^{\bar{z}^1(x,y)} e^{-\tilde{C}^{33}_{0,k}(\bar{z}^1)^2} f_3(z)dz = e^{-\tilde{C}^{33}_{0,k}(\bar{z}^1)^2}\Phi(d^2_k(x,y)), \tag{97}$$

and, similarly,

$$\begin{aligned}\int_{\bar{z}^2(x,y)}^{+\infty} e^{-\tilde{C}^{33}_{0,k}z^2} f_3(z)dz &= \frac{e^{\left(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k\right)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}\Phi(-d^3_k(x,y)),\\ \int_{\bar{z}^2(x,y)}^{+\infty} e^{-\tilde{C}^{33}_{0,k}(\bar{z}^2)^2} f_3(z)dz &= e^{-\tilde{C}^{33}_{0,k}(\bar{z}^2)^2}\Phi(-d^4_k(x,y)),\end{aligned} \tag{98}$$

where the $d^i_k(x,y)$, for $i=1,\ldots,4$ and $k=1,\ldots,n$, are as specified in (93).

**(ii)** If $(x,y)\in\bar{M}^c$ then, according to Definition 5.2 we have $g(x,y,z)\geq g(x,y,0)>h(x,y)$ for all $z$. Noticing that, analogously to (96),

$$\int_{\mathbb{R}} e^{-\tilde{C}^{33}_{0,k}\zeta^2} f_3(\zeta)d\zeta = \frac{e^{\left(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k\right)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}$$

we obtain the following expression

$$\begin{aligned}P^2(0;T_0,T_n,R) &= p(0,T_0)\sum_{k=1}^n e^{-A_{0,k}}\Big[\int_{\bar{M}^c\times\mathbb{R}}\Big(D_{0,k}e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2-\tilde{C}^{33}_{0,k}z^2}\\ &\quad -(R\gamma+1)e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}\Big)f_3(z)f_2(y)f_1(x)dzdydx\Big]\\ &= p(0,T_0)\sum_{k=1}^n e^{-A_{0,k}}\Big[D_{0,k}\Big(\int_{M^c} e^{-\tilde{B}^1_{0,k}x-\tilde{C}^{22}_{0,k}y^2}f_2(y)f_1(x)dydx\Big)\\ &\quad\times\frac{e^{\left(\frac{1}{2}(\theta_k)^2\bar{\beta}^3_{T_0}-\bar{\alpha}^3_{T_0}\theta_k\right)}}{\sqrt{1+2\bar{\beta}^3_{T_0}\tilde{C}^{33}_{0,k}}}-(R\gamma+1)\Big(\int_{\bar{M}^c} e^{-B^1_{0,k}x-C^{22}_{0,k}y^2}f_2(y)f_1(x)dydx\Big)\Big].\end{aligned} \tag{99}$$

Adding the two expressions in (i) and (ii) we obtain the statement for Case 1.

**Case (2).** Also for this case we distinguish between whether $(x,y)\in\bar{M}$ or $(x,y)\in\bar{M}^c$ and, again, compute separately the two terms in the last equality in (90).

**(i)** For $(x, y) \in \bar{M}$ we have that there exist $\bar{z}^1(x, y) \leq 0$ and $\bar{z}^2(x, y) \geq 0$ so that, contrary to Case 1), one has $g(x, y, z) \geq g(x, y, \bar{z}^i)$ when $z \in [\bar{z}^1, \bar{z}^2]$. It follows that

$$\begin{aligned} P^1(0; T_0, T_n, R) &= p(0, T_0)\Bigg[\sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} \int_{\bar{M}} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2) \\ &\quad \cdot \Bigg(\int_{\bar{z}^1(x,y)}^{\bar{z}^2(x,y)} \exp(-\tilde{C}^{33}_{0,k} z^2) f_3(z) dz \\ &\quad - \int_{\bar{z}^1(x,y)}^{\bar{z}^2(x,y)} \exp(-\tilde{C}^{33}_{0,k} (\bar{z}^1)^2) f_3(z) dz\Bigg) f_2(y) f_1(x) dy dx\Bigg] \\ &= p(0, T_0)\Bigg[\sum_{k=1}^{n} D_{0,k} e^{-A_{0,k}} \int_{\bar{M}} \exp(-\tilde{B}^1_{0,k} x - \tilde{C}^{22}_{0,k} y^2) \\ &\quad \cdot \Bigg(\frac{e^{(\frac{1}{2}(\theta_k)^2 \bar{\beta}^3_{T_0} - \bar{\alpha}^3_{T_0} \theta_k)}}{\sqrt{1 + 2\bar{\beta}^3_{T_0} \tilde{C}^{33}_{0,k}}} \Big(\Phi(d^3_k(x, y)) - \Phi(d^1_k(x, y))\Big) \\ &\quad - e^{-\tilde{C}^{33}_{0,k} (\bar{z}^1)^2} \Big(\Phi(d^4_k(x, y)) - \Phi(d^2_k(x, y))\Big)\Bigg) f_2(y) f_1(x) dy dx\Bigg], \end{aligned} \tag{100}$$

where we have made use of (96) and (97), (98).

**(ii)** For $(x, y) \in \bar{M}^c$ we can conclude exactly as we did it for Case (1) and, by adding the two expressions in (i) and (ii), we obtain the statement also for Case (2).

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# Multi-curve Construction

## Definition, Calibration, Implementation and Application of Rate Curves

**Christian P. Fries**

**Abstract** In this chapter we discuss the definition, construction, interpolation and application of *curves*. We will discuss discount curves, a tool for the valuation of deterministic cash-flows and forward curves, a tool for the valuation of linear cash-flows of an index. A curve is mainly a tool to interpolate certain basic financial products (zero coupon bonds, FRAs) with respect to maturity date and fixing date, such that it can be used to value products, which can be represented as linear functions of possibly interpolated values of a discount or forward curve. For this, the chosen interpolation method and interpolation entity plays an important role. Distinguishing forward curves from discount curves (representing the collateralization of the forward) motivates an alternative interpolation method, namely interpolation of the forward value (the product of the forward and the discount factor). In addition, treating forward curves as native curves (instead of representing them by pseudo-discount curves) will avoid other problems, like that of overlapping instruments. Besides the interpolation, we discuss the calibration of the curves for which we give a generic object-oriented implementation in Fries (Curve calibration. Object-oriented reference implementation, 2010–2015, [11]). We give some numerical results, which have been obtained using this implementation and conclude with a remark on how to define term-structure models (analog to a LIBOR market model) based on the definition of the performance index of an accrual account associated with a discount curve.

**Keywords** Multi-curve construction · Interest rate curves · Interest rate curve interpolation · Cross-currency curves · Term structure models

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## 1 Introduction

Dynamic multi-curve term structure models, as the one discussed in this book, often use given interest rate curves as initial data. The classical (single curve) example is the HJM oder LMM model, where

$$\mathrm{d}f(t, T) = \mu(t, T)\mathrm{d}t + \Sigma(t, T)\mathrm{d}W(t), \qquad f(t_0, T) = f_0(T).$$

While research on multi-curve interest rates models was and is very active, see, e.g., [5, 6, 15, 20–22], references therein and the other chapters of in this book, the construction of the initial interest rate curve, here $f_0(T)$, naturally does not get a similar strong attention. However, a good curve construction is of high importance for practitioners, since it has a strong impact on the delta-hedge (that is, the first-order interest rate risk).

The market standard of (forward) curve construction is to calibrate an interpolated curve to given market instruments, often via an iterative procedure (bootstrapping). With respect to the interpolation of (interest rate) forward curves, a common approach is to represent a forward curve in terms of (pseudo-)discount factors (aka. synthetic discount factors) and apply an interpolation scheme on these discount factors. While this approach is in general not backed by an economic concept, it also introduces several (self-made) problems, e.g., the interpolation of (so-called) overlapping instruments, see Sect. 5.3.

In this paper we focus on the curve construction, provide an open source implementation and suggest appealing alternative interpolation schemes motivated from the multi-curve setup: direct interpolation of the forward curve or direct interpolation of the forward value curve, where the forward value is the product of a forward and the associated discount factor. While linear interpolation of the forward is a common scheme,[1] the interpolation of the forward value appears to be a new approach.

Nevertheless, the paper puts both methods on a solid foundation by deriving the schemes from the multi-curve definition of forward curves. Both interpolation schemes ease some of the complications associated with synthetic discount factors.

Once the curves and interpolations are defined, we are considering the problem of calibrating a set of curves to given market quotes. The value of an instrument is in general determined by a whole collection of curves, e.g., one or two discount curves and zero or more forward curves. To simplify the implementation, we define a generalized swap, which allows to represent most calibration instruments (FRAs, swaps, tenor basis swaps, cross-currency swaps, etc.) by a single class.

[1] Some trading systems, like Murex, do offer it as an option.

## 2 Foundations, Assumptions, Notation

Under well-known assumptions the valuation of a future cash flow can be written as an expectation,[2] that is the time $t_0$-value $V(t_0)$ is

$$V(t_0) \;=\; N(t_0) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(T)}{N(T)} \mid \mathscr{F}_{t_0}\right) \quad \text{for } t_0 \leq T, \tag{1}$$

where $V(T)$ is the time $T$ cash-flow, $N$ is the value process of a traded asset (or collateral account) which can serve as a *numéraire* and $\mathbb{Q}^N$ is the equivalent martingale measure associated with $N$. Equation (1) is the starting point for *curve construction* in the following sense: If the above valuation formula holds, then the value of a linear function of future cash-flows is the linear function of the values of the single cash flows. In other words: we can represent the valuation of so-called linear products by a *basis* consisting of the values of elementary products. This basis of elementary products is the set of curves, where "the curve" is formed by the parameter $T$.

Note that here and in the following, we consider the valuation for a fixed $t_0$. We are not concerned with the description of a dynamic model (describing $t \mapsto V(t)$ as a stochastic process).

**Definition 1** Let $I$ denote an index, that is, $I(T)$ is an $\mathscr{F}_T$-measurable random variable and $d > 0$ is some payment offset, then we define the (time $t_0$-)valuation curve with respect to $T$ as the map

$$T \mapsto C(T) \;:=\; N(t_0) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{I(T)}{N(T+d)} \mid \mathscr{F}_{t_0}\right). \tag{2}$$

For $I \equiv 1$ and $d = 0$ the curve in (2) represents the curve of (synthetical) zero-coupon bond prices $T \mapsto P(T; t_0)$, also known as *discount curve*.[3] For arbitrary indices $I$ (with fixed payment offset $d$[4]), the curve $T \mapsto C(T)/P(T; t_0)$ is known as the *forward curve*. Obviously both curves depend on $N$ and $t_0$.

Note that the specific stochastic behavior of $I$ and $N$ does not play a role when looking at $t_0$ only in the sense that we are only interested in the time $t_0$-expectation. That is, we could define $t \mapsto N(t)$ and $t \mapsto I(t)$ to be $\mathscr{F}_{t_0}$-measurable for all times $t$ and still generate any given discount curve and forward curve, respectively. There

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[2] Since we are only considering the linearity of the valuation at a fixed time $t_0$, we just require that some fundamental theorem of asset pricing holds, for example, assuming that the price processes are locally bounded semi-martingales and the *no free lunch with vanishing risk* condition holds, [7].

[3] We will use the notation $P(T; t)$ (instead of the more common $P(t, T)$) for a the time-$t$ value of zero-coupon bond maturing in $T$, since we consider $t = t_0$ as fixed. Sometimes we even drop the argument and just write $P(T)$. Similar for forward curves. The curves considered here are parametrized by $T$ for a fixed time $t$.

[4] In practice the payment offset may depend on $t_0$ and $T$ due to business day adjustments. Our implementation handles this, but to ease notation we drop the dependence here.

is no arbitrage constraint with respect to $t$ yet, since for different $t$ the index $I(t)$ represents different assets (underlyings).

Thus, with respect to the processes $1/N$ and $I/N$ we just require that they fulfill regularity assumptions such that (2) exists.[5]

On the other hand, the interpretation of the curve as a curve of valuations in the sense of (2) does play a role, when we consider the construction of the curve via interpolation of observed market prices. Here, the linearity of the expectation operator E allows to link market prices to different points of the curve.

Curves, like discount curves and forward curves solve, among others, two important problems:

- Valuation of linear instruments. This is performed by decomposing instruments into the value of single cash flows (zero coupon bonds and FRAs), which then allows to synthesize the valuation of linear functions of the individual cash flows (e.g., swaps).
- Valuation of a time $T$ cash-flow as interpolation of valuations of cash flows at discrete times $\{T_i\}_{i=0}^{n}$ (where $T_i \geq t_0$ for all $i$), e.g., swaps referencing cash flows on illiquid maturities.

Thus, curves are simply a methodology to interpolate on the cash flows with respect to their payment time.[6] Apart from this, the curves also represent the initial data for advanced term structure models (like the LIBOR market model). Hence, careful construction of curves is also key to (interest rate) derivatives valuation, when interpolated curves are the initial values of a dynamic model.

For details on the evolution of multi-curve construction see the recent book by Henrard, [18] (citing a preprint of the present paper). A very detailed description of multi-curve bootstrapping, which also details market conventions and convexity adjustments of the calibration instruments, can be found in [2]. For market conventions also see [17]. Here, we do not consider a possible convexity adjustment due to different market conventions (they should be part of the valuation formulas) and rather focus on the curves and their interpolation schemes. Also, we do not need to consider a bootstrapping, since we set up the calibration as a system of equations passed to a multi-dimensional optimization algorithm.

Usually (and here), the curves are used to interpolate at the fixed time $t_0$ only. If a curve interpolation should also be used for times $t > t_0$ within a dynamic multi-curve model, then this may impose additional constraints on the admissible interpolations schemes. For example, (2) implies that linear interpolation of time-$t$ zero-coupon bond prices for $t > t_0$ implies linear interpolation of the time-$t$ zero-coupon bond prices, which in turn implies a special interpolation of forward rates in a LIBOR market model, see Sect. 19.5 in [10]. In this case the linear interpolation of the discount curve and forward value curve would not introduce an arbitrage violation, given that

[5]For example, let $1/N$ and $I/N$ be Itô stochastic processes with integrable drift and bounded quadratic variation.

[6]This also applies to forward curve, see below, although in these cases there is also an associated fixing time of an index and it is maybe more consistent to parametrize the curve w.r.t. the fixing of the index.

the interpolation points are the same for all times. In practice term-structure models are often constructed with their own curve interpolations, such that the interpolation used for the initial data differs from the interpolation used for the simulated curves (while the model is still calibrated and arbitrage-free given that the drift is specified accordingly). In the following we focus on the interpolation of the initial data—that is, the time-$t_0$ curves, which is of greater importance for the deltas of interpolated products, where only the linear part matters.

In the above valuation formula (1) it is assumed that $V$ and $N$ are expressed in the same currency. If the two are in different currency, one of them has to be converted by an exchange rate, which we will denote by $FX$. Let $V$ be in currency $U_2$ and the numéraire $N$ in currency $U_1$, then the valuation formula is given by

$$V(t_0) \;=\; FX^{\frac{U_2}{U_1}}(t_0) \cdot N(t_0) \cdot \mathrm{E}^{\mathbb{Q}^N}\left( \frac{V(T)}{FX^{\frac{U_2}{U_1}}(T) \cdot N(T)} \mid \mathscr{F}_{t_0} \right),$$

where $FX^{\frac{U_2}{U_1}}(t)$ denotes the time $t$ exchange rate for one unit of currency $U_1$ into one unit of currency $U_2$. Furthermore, $FX^{\frac{U_1}{U_2}} \;=\; \left(FX^{\frac{U_2}{U_1}}\right)^{-1}$.

As discussed in [12], the valuation of a collateralized claim can be written as an expectation with respect to a specific numéraire, namely the collateral account $N = N^{\mathsf{C}}$.[7] We denote the currency of the collateral numéraire by $[\mathsf{C}]$. Let $U$ denote the currency of the cash flow $V(T)$. Assume that the cash flow $V(T)$ is collateralized by units of $N^{\mathsf{C}}$. In this case the Eq. (1) holds with the numéraire $N = N^{\mathsf{C}}$, $U_2 = U$, $U_1 = [C]$ (given that $V(t)$ is the collateral amount in the account $N^{\mathsf{C}}$).

*Remark 1* From the above we see that collateralization in a different currency can be interpreted twofold:

1. We may consider a payment converted to collateral currency and valued with respect to the collateral numéraire $N^{\mathsf{C}}$, or, alternatively,
2. we may consider a payment in the currency $U$ collateralized with respect to the collateral account $N^{U,\mathsf{C}} := FX^{\frac{U}{[\mathsf{C}]}} \cdot N^{\mathsf{C}}$.

We will adopt the latter interpretation, which will also make the valuation look more consistent[8]

$$V(t_0) \;=\; N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{U,N^{\mathsf{C}}}}\left( \frac{V(T)}{N^{U,\mathsf{C}}(T)} \mid \mathscr{F}_{t_0} \right). \qquad (3)$$

Note that this interpretation will then give rise to a new discount curve: the discount curve associated with $N^{U,\mathsf{C}}$, being the discount curve of a foreign currency ($U$) cash flow collateralized by a $C$.

[7] See also [8, 14].

[8] As has been noted in [12], the measures agree, i.e., $\mathbb{Q}^{U,N^{\mathsf{C}}} = \mathbb{Q}^{N^{\mathsf{C}}}$.

*Remark 2* For an uncollateralized product the role of the collateral account is taken by the funding account and the corresponding numéraire is the funding account. Since the valuation formulas are identical to the case of a "special" collateral account (agreeing with the funding account), we will consider an uncollateralized product as a product with a different collateralization.

In the following we use the notation $U$ for the currency unit of a cash flow, i.e., we may consider $U = 1€$ or $U = 1\$$. We will need this notation only when we consider cross-currency basis swaps. The symbols $V$ and $N$ (as well as $P$ defined below) will denote value processes including the corresponding currency unit, e.g., $V(t_0) = 0.25€$. The symbol $V$ refers to the value of the product under consideration, while $N$ denotes the numéraire, e.g., the OIS accrued collateral account. The symbol $X$ denotes a real number while $I$ denotes a real valued stochastic process, both can be considered as rates, i.e., unit-less indices, e.g., $X = 2.5\,\%$. For example $X$ will denote the fix rate in a swap, $I$ will denote the floating rate index in a swap, $U$ denotes the currency unit of the two legs, $N$ will be used to define the discount factor and the value of the swap. The value of the swap is then denoted by $V$.

## 3 Discount Curves

Consider a fixed constant cash flow $X$, paid in currency $U$ in time $T$, collateralized by an account $\mathsf{C}$. Since $X$ is a constant and the expectation operator is linear, we can express the time-$t_0$ value $V(t_0)$ of this cash flow as

$$V(t_0) \ = \ X \cdot P^{U,\mathsf{C}}(T; t_0), \tag{4}$$

where

$$P^{U,\mathsf{C}}(T; t_0) \ := \ N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}} \left( \frac{1 \cdot U}{N^{U,\mathsf{C}}(T)} \mid \mathscr{F}_{t_0} \right) \tag{5}$$

defines the value of a theoretical zero coupon bond. Note that Eq. (4) can be used in two ways. First, for given market prices we may determine $P^{U,\mathsf{C}}(T; t_0)$—that is we calibrate the curve $T \mapsto P^{U,\mathsf{C}}(T; t_0)$. Second, for given $P^{U,\mathsf{C}}(T; t_0)$ we may value a constant cash flow.

This defines the discount curve:

**Definition 2** Let $P^{U,\mathsf{C}}(T; t_0)$ denote the time $t_0$ value expressed in currency unit $U$ of a unit cash-flow of 1 unit of the currency $U$ in $T$, collateralized by a collateral account $\mathsf{C}$. In this case we call $T \mapsto P^{U,\mathsf{C}}(T; t_0)$ given by (5) the *discount curve* for cash flows in currency $U$ collateralized by the account $C$.

*Remark 3* By assumption (of a frictionless no-arbitrage market, (1)) the value of a fixed constant future cash-flow $X$ is a linear function of its amount. Hence, we have that the time $t_0$ value of a cash flow $X$ in $T$ and currency $U$, collateralized with an account $C$ is

$$X \cdot P^{U,\mathsf{C}}(T; t_0).$$

In other words, the discount curve allows us to valuate all fixed (deterministic) cash flows in a given currency, collateralized by a given account.

The discount factor $P^{U,\mathsf{C}}(T; t_0)$ represents the price of an (idealized) zero-coupon bond. Although a zero-coupon bond is usually not a market-traded asset, we may represent market-traded coupon bonds as a linear combination of zero-coupon bonds, and vice versa. If $\mathsf{C}$ denotes some cash-collateral account, there is no such thing as a collateralized bond, but in that case $P^{U,\mathsf{C}}(T; t_0)$ has the natural interpretation of representing the time-$t$ value of a collateralized unit currency time-$T$ cash flow. In any case, $P^{U,\mathsf{C}}(T; t_0)$ can be considered a linear function of traded asset (within its collateralization scheme).

## 4 Forward Curves

The same approach can now be applied to a payoff of a cash flow $X \cdot I(T_1)$, paid in currency $U$ in time $T_2$ ($T_1 \leq T_2$), collateralized by account $\mathsf{C}$, where $X$ is a constant and $I$ is an adapted process representing index.[9] Its value is

$$V(t_0) \;=\; N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}} \left( \frac{X \cdot I(T_1) \cdot U}{N^{U,\mathsf{C}}(T_2)} \mid \mathscr{F}_{t_0} \right).$$

We can express the value as $V(t_0) \;=\; X \cdot F_I^{U,\mathsf{C}}(T_1, T_2; t_0) \cdot P^{U,\mathsf{C}}(T_2; t_0)$, where

$$F_I^{U,\mathsf{C}}(T_1, T_2; t_0) \;=\; N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}} \left( \frac{I(T_1) \cdot U}{N^{U,\mathsf{C}}(T_2)} \mid \mathscr{F}_{t_0} \right) / P^{U,\mathsf{C}}(T_2; t_0). \qquad (6)$$

This definition allows us to derive $F_I^{U,\mathsf{C}}(T_1, T_2; t_0)$ from given market prices. Conversely, given $P^{U,\mathsf{C}}(T_2; t_0)$ and $F_I^{U,\mathsf{C}}(T_1, T_2; t_0)$ we may value all linear payoff functions of $I(T_1)$ paid in $T_2$.

In (6) the forward depends on the fixing time $T_1$ and the payment time $T_2$. However, the offset of the payment time from the fixing time $d = T_2 - T_1$ can be viewed as a property of the index (a constant) and hence, the forward represents a curve $T \mapsto F_I^{U,\mathsf{C}}(T, T + d; t_0)$.

**Definition 3** Let $t \mapsto I(t)$ denote an index, that is $I$ is an adapted stochastic real valued process. Let

$$V_I^{U,\mathsf{C}}(T, T + d; t_0) := N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}} \left( \frac{I(T) \cdot U}{N^{U,\mathsf{C}}(T + d)} \mid \mathscr{F}_{t_0} \right)$$

[9] Examples for $I$ are LIBOR rates or the performance of an EONIA accrual account.

denote the time $t_0$-value of a payment of $I(T)$ paid in $T+d$ in currency $U$, collateralized by an account $\mathsf{C}$ (where $d \geq 0$). We assume that $I$ and $N$ is such that the expectation exists for all $T$. Then we define the *forward of a payment of* $I(T)$ *paid in* $T+d$ *in currency* $U$, *collateralized by an account* $C$ as

$$F_I^{U,\mathsf{C}}(T;t_0) \ := \ \frac{V_I^{U,\mathsf{C}}(T,T+d;t_0)}{P^{U,\mathsf{C}}(T+d;t_0)}.$$

*Remark 4* The forward curve allows us to value a future payment of the index $I$ by

$$V_I^{U,\mathsf{C}}(T,T+d;t_0) \ = \ F_I^{U,\mathsf{C}}(T;t_0)\cdot P^{U,\mathsf{C}}(T+d;t_0)$$

and by assumption (of a frictionless no-arbitrage market, (1)), the forward curve allows us to evaluate all linear cash flows $X \cdot I$ (in currency $U$, collateralized by an account $\mathsf{C}$) by $X \cdot F_I^{U,\mathsf{C}}(T;t_0)\cdot P^{U,\mathsf{C}}(T+d;t_0)$.

Note that $F_I^{U,\mathsf{C}}$ is not a classical single curve forward rate, related to some discount curve. Due to our definition of the forward curve, the curve includes all valuation effects related to the index, in particular a possible convexity adjustment. For example: if we would consider an in-arrears index and an in-advance index we would obtain two different forward curves which differ by the in-arrears convexity adjustment!

### 4.1 *Performance Index of a Discount Curve (or "Self-Discounting")*

The OIS swap pays the performance of an account, accruing with the overnight rate, that is:

**Definition 4** (*Overnight Index Swap*) Let $N^{\mathsf{C}}(t)$ denote the account accruing at the overnight rate $r(t)$, $N^{\mathsf{C}}(t_0) = 1U$, i.e. on a given time discretization (accrual periods) $\{t_i\}_{i=0}^n$

$$N^{\mathsf{C}}(t_k) \ := \ \prod_{i=0}^{k}(1+r(t_i)\Delta t_i) \ \approx \ \exp\left(\int_{t_0}^{t_k} r(s)\mathrm{d}s\right).$$

The overnight index swap pays a fix coupon and receives the performance $I_i^{\mathsf{C}}$ of the accrual account, that is

$$I_i^{\mathsf{C}}(T_i,T_{i+1}) \ := \ \frac{N^{\mathsf{C}}(T_{i+1})}{N^{\mathsf{C}}(T_i)} - 1.$$

in $T_{i+1}$ with a quarterly tenor $T_0, T_1, \ldots$.

The time-$t_0$ linear forward of the index above is $\frac{P^{U,\mathsf{C}}(T_i;t_0)-P^{U,\mathsf{C}}(T_{i+1};t_0)}{P^{U,\mathsf{C}}(T_{i+1};t_0)}$ (and dividing by $T_{i+1} - T_i$ this gives the linear forward rate). Hence, this is the same situation as for single curve interest rate theory swaps.

The OIS swap is collateralized with respect to the account $N^{\mathsf{C}}$. Due to this, it is sometimes called "self-discounted". However, we may give an appealing alternative view, defining the forward curve from the discount curve (and not the other way around):

Let us consider a discount factor curve $P^{U,\mathsf{C}}(T;t)$ as seen in time $t$. The curve allows the definition of a special index, namely the performance rate of the collateral account $\mathsf{C}$ in currency $U$ over a period of period length $d$:
Let $I^{\mathsf{C}}(T_i) := \frac{1-P^{U,\mathsf{C}}(T_i+d;T_i)}{P^{U,\mathsf{C}}(T_i+d;T_i)}$, where $P^{U,\mathsf{C}}(T_i+d;T_i)$ is the discount factor for the maturity $T_i + d$ as seen in time $T_i$. The index $I^{\mathsf{C}}(T_i)$ is the payment we have to receive in $T_i + d$ collateralized with respect to the collateral account $\mathsf{C}$, such that $1 + I^{\mathsf{C}}(T_i)$ in $T_{i+1}$ has the same value as 1 in $T_i$. This index has a special property, namely that its forward can be expressed in terms of the discount factor curve $P^{U,\mathsf{C}}$ too: The time $t_0$ forward of $I^{\mathsf{C}}(T_i)$ is $F^{U,\mathsf{C}}(T_i;t_0)$ where

$$
\begin{aligned}
F^{U,\mathsf{C}}(T_i;t_0) \cdot P^{U,\mathsf{C}}(T_i+d;t_0) &= N^{U,\mathsf{C}}(t_0) \cdot \mathrm{E}^{\mathbb{Q}^{N^{U,\mathsf{C}}}}\left(\frac{I^{\mathsf{C}}(T_i) \cdot U}{N^{U,\mathsf{C}}(T_i+d)} \mid \mathscr{F}_{t_0}\right) \\
&= P^{U,\mathsf{C}}(T_i;t_0) - P^{U,\mathsf{C}}(T_i+d;t_0).
\end{aligned}
$$

Consequently this index has the special property that its forward can be expressed by the associated discount factors evaluated at different maturities.

**Definition 5** (*Forward associated with a Discount Curve*) Let $P^{U,\mathsf{C}}(T_i+d;t_0)$ denote a discount curve. For a given period length $d$ we define the forward $F^{d,U,\mathsf{C}}(T_i;t_0)$ as

$$
F^{d,U,\mathsf{C}}(T_i;t_0) := \frac{P^{U,\mathsf{C}}(T_i;t_0) - P^{U,\mathsf{C}}(T_i+d;t_0)}{P^{U,\mathsf{C}}(T_i+d;t_0) \cdot d}. \tag{7}
$$

$F^{d,U,\mathsf{C}}(T_i;t_0)$ is the forward associated with the performance index of $P^{U,\mathsf{C}}$ over a period of length $d$.

*Remark 5* The above definition relates a forward curve and discount factor curve. Note however, that we define a forward from a discount factor curve and that this definition is backed by a clear interpretation of the underlying index. Conversely, we may define a discount curve from a forward curve "implicitly" such that the relation (7) holds. Note however, that a generalization of this relation should be considered with care, since the associated product may not exist.

The definition above is an idealization in the sense that we assume that interval points over which the performance is measured correspond to the payment dates. In practice (EONIA is an example) there might be some small deviations from this assumption (e.g. payment offsets of a few days). In this case (7) does not hold (but may be still considered an approximation).

Products like the OIS swaps are sometimes called "self-discounting" since the discounting is performed on a curve corresponding to the index they fix. From the above, we find an alternative (and maybe more natural) interpretation, namely that the swap pays the performance index of its collateral account, i.e., it pays the index associated with the discount curve.

## 5 Interpolation of Curves

In this section we consider a discount curve $P^{U,\mathsf{C}}$ and an associated forward curve $F^{U,\mathsf{C}}$. To simplify notation we set $D(T) := P^{U,\mathsf{C}}(T; t_0)$ and $F(T) := F^{U,\mathsf{C}}(T; t_0)$.

Forwards and discount factors are linked together by Definition 3, which says that the time-$t_0$ value of a forward contract $V(t_0, T)$ with fixing in $T$ is the product of the forward $F(T)$ and the associated discount factor $D(T + d)$, i.e., $V(t_0, T) = F(T) \cdot D(T + d)$. Note that $T \mapsto V(t_0, T)$ and $T \mapsto D(T)$ are *value curves*, i.e., for a fixed $T$ the quantities $V(t_0, T)$ and $D(T)$ are values of financial products. However, $F(T)$ is a derived quantity, the forward.

Since $V$ and $D$ represent values of financial products, there is a natural interpretation for a linear interpolation of different values $V(T_i)$ and of different values of $D(T_i)$, since this would correspond to a portfolio of such products. Note that defining an interpolation method for $V$ and $D$ implies a (possible more complex) interpolation method of $F$.

On the other hand, it is common practice to define an interpolation method for a rate curve (both forward curve and discount factor curve) via zero rates, sometimes even regardless of the nature of the curve, which then implies the interpolation of the value curves $D$ and $V$. Some of these interpolations will result in natural interpolations on the value process $V$, others not. Other examples for interpolations of $F$ and $D$ are:

- log-linear interpolation of the forward, log-linear interpolation of the discount factor: the case is equivalent to log-linear interpolation of the value.
- linear interpolation of the forward, log-linear interpolation of the discount factor: the case is equivalent with a linear interpolation of the value, with an interpolation weight being a function of the discount factor ratio.

In [16] interpolations on the discount factors, on the logarithm of discount factors, on the yield and directly on the forwards were discussed. Highlighting some disadvantages of cubic splines, they introduced two new interpolation methods (monotone convex spline and minimal cubic spline) which overcome most of the shortfalls of the other interpolations. In [19] some issues of these methods were pointed out, favoring a harmonic spline interpolation. In [1] a modified Bessel spline on the logarithm of the discount factors was proposed.

Based on the formal setup presented in the present paper, the stability of cumulated error of a dynamic hedge was considered as a criterion for the interpolation methods and compared for a large collection of methods in [13].

At this point, we would like to stress the importance of the *interpolation entity*, that is, whether we interpolate on a forward or on a synthetic discount factor (in the sense of Definition 5). While the interpolation method (e.g., *linear* compared to *spline*) is often in the focus (discussing locality versus smoothness, [16]), the choice of the interpolation entity has a strong impact on the delta hedge, see Table 1.

Depending on the application, it is popular to represent a curve by a parametric curve. This is done especially for discount curves. Examples are the Nelson–Siegel (NS) and the Nelson–Siegel–Svensson (NSS) parametrization. Our benchmark implementation in [11] allows to use NS or NSS in the calibration.[10]

### 5.1 Implementing the Interpolation of a Curve: Interpolation Method and Interpolation Entities

In this paper we focus on interpolation schemes based on given interpolation points. Implementing the interpolation of a curve that way, it is convenient to distinguish the *interpolation method*, e.g., linear interpolation of interpolation points $\{(T_i, x_i)\}$, and the *interpolation entity*, that is, a (bijective) transformation from $(T, x)$ to the actual curve. For example, for discount curves one might consider a linear interpolation of the zero rate. In this case the interpolation method is *linear interpolation* and the interpolation entity is $(T, x(T)) = (T, \frac{\log(D(T))}{T})$ for $T > 0$, where $D$ denotes the discount curve. Given $0 < T_i \leq T \leq T_{i+1}$ and discount factors $D(T_j)$, a linear interpolation of the zero rates would then imply the interpolation

$$D(T) := \exp\left(\left(\frac{T - T_i}{T_{i+1} - T_i} \frac{\log(D(T_{i+1}))}{T_{i+1}} + \frac{T_{i+1} - T}{T_{i+1} - T_i} \frac{\log(D(T_i))}{T_i}\right) \cdot T\right).$$

In our benchmark implementation [11], this functionality is provided for a large number of interpolation methods (constant, linear, Akima, spline, etc.) and interpolation entities (value, log-value, log-value-per-time) by the class `net.finmath.marketdata.model.curves.Curve`.[11] For forward curves we provide two additional interpolation entities: forward and synthetic discount factor (see below).

### 5.2 Interpolation Time

For both, parametric curves (like NSS) and non-parametric interpolation schemes, it is important to specify the convention used to transform product maturities (dates) to real numbers (time $T$). For example, we might use a daycount convention (like

[10] See http://finmath.net/finmath-lib/apidocs/net/finmath/marketdata/model/curves/DiscountCurveNelsonSiegelSvensson.html.

[11] See http://finmath.net/finmath-lib/apidocs/net/finmath/marketdata/model/curves/Curve.html.

ACT/365) and measure $T$ as a daycount fraction between evaluation date and maturity date, that is $T := \mathrm{dcf}(\text{evaluation date, maturity date})$. Clearly, a change in the time parametrization will change the interpretation of the curve parameters (for a parametric curve). Also, some daycount convention actually introduces non-linear time transformations.

## 5.3 Interpolation of Forward Curves

### 5.3.1 The Classical Approach

For forward curves, a common approach is to consider an interpolation of the forward as an independent entity (like for the discount curve). For interest rate forwards, a popular interpolation scheme (coming from the single curve interpretation of interest rates forwards) is to represent the forward in terms of synthetic discount factors. That is, if $d$ denotes a period length associated with the forward and if $F(T_i)$ is given for $T_i = i \cdot d$, then one might consider interpolation of (pseudo-)discount factor $D^F(T_i) := \prod_{k=0}^{i-1}(1 + F(T_k) \cdot d)^{-1}$, possibly considering another transformation on $D^F(T)$ to define the actual interpolation entity. See [3] for a corresponding multi-curves bootstrap algorithm.

It is obvious that this definition of the interpolation entity for forward curve is complex, results in problems for non-equidistant interpolation points and is—without further assumptions—not backed by a meaningful interpretation. First, in a multi-curve setup this approach lacks an economic justification. Second, it may introduce problems:

- The common approach of a linear interpolation of the logarithm of the synthetic discount factor representing the forward curve results in an almost piecewise constant interpolation of the forward, see [13]. This may result into "jumps" when products are aging.
- The use of synthetic discount factors defines a forward with fixing time $T$ in terms of (interpolated) discount factors at times $T$ and $T + d$ (where $d$ is the period length). The method is a common practice (also considered in [1]). However, considering forwards for overlapping periods, this may introduce oscillations and result in implausible delta-hedges (see Table 1).

### 5.3.2 Alternative Interpolation Schemes for Forward Curves

The definition of the forward curve in the multi-curve setup suggests an appealing alternative for the creation of an interpolated forward: Like a discount factor curve, the curve $V(T) = F(T) \cdot D(T + d)$ represents the value of a financial product. Hence, we may consider the interpolation of $V$ like we did for the curve $D$. For example, if we consider linear interpolation of the value curve $V$, we interpolate the forward curve $F$ by considering the interpolation entity $F(T) \cdot D(T + d)$ with a given discount curve $D$, i.e., we have

$$F(T) := \frac{1}{D(T+d)} \left( \frac{T - T_i}{T_{i+1} - T_i} F(T_{i+1}) D(T_{i+1} + d) + \frac{T_{i+1} - T}{T_{i+1} - T_i} F(T_i) D(T_i + d) \right)$$

for $T_i \leq T \leq T_{i+1}$ and given points $F(T_j)$.

Given that log-linear interpolation is a popular interpolation scheme for discount curves one may consider log-linear interpolation of $V$. This interpolation scheme has the restriction that the forward is required to be positive. Since negative interest rates are possible, this interpolation scheme is not appropriate for interest rate curves.

### *5.4 Assessment of the Interpolation Method*

The assessment of the quality of performance of an interpolation method is difficult. Some basic criteria (like continuity, locality, etc.) have been reviewed in [16]. Locality, i.e., how does a local change in input data affect the curve, is a desired property from a hedging perspective. In [13] a long-term dynamic hedging is used to asses the performance of an interpolation scheme. The results in [13] suggest that among the local methods, linear interpolation of the forward curve and log-linear interpolation of the discount curve were the best performing schemes when using the cumulated dynamic hedge error as a primary criterion.

## 6 Implementation of the Calibration of Curves

A curve (discount curve or forward curve) is used to encode values of market instruments. A forward curve together with its associated discount curve, allows to value all linear products (linear payoffs) in the corresponding currency under the corresponding collateralization.

The standard way to calibrate a curve is, hence, to obtain given market values of (linear) instruments (e.g., swaps). For each market value a single "point" in a single curve is calibrated. Hence the total number of calibrated curve interpolation points (aggregated across all curves) equals the number of market instruments.

By "sorting" and combining the calibration instruments, the corresponding equations can be brought into the form of a system of equations with a triangular structure, i.e., the value of the $n$th calibration instrument only depends on the first $n$ curve points. This allows for an iterative construction of the curve.

However, here (and in the associated reference implementation [11]) we propose the calibration of the curves using a multi-variate optimization algorithm, like the Levenberg–Marquardt algorithm or a Differential Evolution algorithm. This approach brings several advantages, e.g., the freedom to specify the calibration instruments and the ability to extend the approach to over-determined systems of equations.

In addition, we can handle the case of curve-interdependence, for example to calibrate certain discount curves from cross-currency swaps. This comes at the cost of slower performance in terms of required calculation time.

What remains is to specify the valuation equations for the calibration instruments. To simplify implementation, we may generalize the definition of a "swap" comprising plain swaps, tenor basis swaps and cross-currency swaps.

## 6.1 Generalized Definition of a Swap

Many of the following calibration instruments (from OIS swaps to cross-currency basis-swaps) fit under a generalized definition of a swap. The swap consists of two legs. Each leg consists of several periods $[T_i, T_{i+1}]$. To ease notation, we do not distinguish between period start time, period end time, fixing time of the index and payment time. We assume that for the period $[T_i, T_{i+1}]$ index fixing is in $T_i$ and payment is in $T_{i+1}$. This is done purely to ease notation, the generalization to distinguished times is straightforward.

**Definition 6** (*Swap Leg*) A swap leg pays a multiple $\alpha$ of the index $I$ fixed in $T_i$ plus some fixed payment $X$, both in currency unit $U$ collateralized by the collateral account $\mathsf{C}$ and paid in time $T_{i+1}$. Here $\alpha$ and $X$ are constants (possibly zero). The value of the swap leg can be expressed in terms of forwards and discount factors as

$$V^{U,\mathsf{C}}_{SwapLeg}(\alpha I, X, \{T_i\}_{i=0}^{n}) \;=\; \sum_{i=0}^{n-1} \big(\alpha F^{U,\mathsf{C}}(T_i) + X\big) \cdot P^{U,\mathsf{C}}(T_{i+1}),$$

where $F^{U,\mathsf{C}}$ denotes the forward curve of the index $I$ paid in currency $U$ collateralized with respect to $\mathsf{C}$ and $P^{U,\mathsf{C}}$ denotes the corresponding discount curve.

A swap leg with notional exchange has the payments as in Definition 6 together with an additional payment of $-1$ in $T_i$ and $+1$ in $T_{i+1}$. The value of the swap leg with notional exchange can be expressed in terms of forwards and discount factors as

$$\begin{aligned} V^{U,\mathsf{C}}_{SwapLeg}(\alpha I, X, \{T_i\}_{i=0}^{n}) = \sum_{i=0}^{n-1} \Big(\big(\alpha F^{U,\mathsf{C}}(T_i) + X\big) \cdot P^{U,\mathsf{C}}(T_{i+1}) \\ + P^{U,\mathsf{C}}(T_{i+1}) - P^{U,\mathsf{C}}(T_i)\Big)\,, \end{aligned}$$

where $F^{U,\mathsf{C}}$ denotes the forward curve of the index $I$ paid in currency $U$ collateralized with respect to $\mathsf{C}$ and $P^{U,\mathsf{C}}$ denotes the corresponding discount curve.

**Definition 7** (*Swap*) A swap exchanges the payments of two swap legs, the receiver leg and the payer leg. We allow that the legs have different indices, different fixed payments, different payment times, different currency units, but are collateralized with respect to the same account $\mathsf{C}$. The swaps receive a swap leg with value $V^{U_1,\mathsf{C}}_{SwapLeg}(\alpha_1 I_1, X_1, \{T^1_i\}^{n_1}_{i=0})$ and pay a leg with value $V^{U_2,\mathsf{C}}_{SwapLeg}(\alpha_2 I_2, X_2, \{T^2_i\})$. Since the currency unit of the two legs may be different, the value of the swap in currency $U_1$ is

$$V_{Swap} = V^{U_1,\mathsf{C}}_{SwapLeg}(\alpha_1 I_1, X_1, \{T^1_i\}^{n_1}_{i=0}) - V^{U_2,\mathsf{C}}_{SwapLeg}(\alpha_2 I_2, X_2, \{T^2_i\}^{n_2}_{i=0}) \cdot FX^{\frac{U_1}{U_2}}$$

Many instruments can be represented (and hence valued) in this form. We will now list a few of them.

## 6.2 Calibration of Discount Curve to Swap Paying the Collateral Rate (aka. Self-Discounted Swaps)

Discount curves can be calibrated to swaps paying the performance index of their collateral account. For example, a swap as in Definition 7 where both legs pay in the same currency $U = U_1 = U_2$. In a receiver swap the receiver leg pays a fixed rate $C$, and the payer leg pays an index $I$. Thus the value of the swap can be expressed in terms of the discount factors $P^{U,\mathsf{C}}(T_{i+1}; t)$ only, which allows to calibrate this curve using these swaps. Overnight index swaps are an example.

For the swap paying the performance of the collateral account we have

$$\begin{aligned} X_1 &= C = \text{const.} = \text{given}, \; X_2 = 0, \\ F^{U_1,\mathsf{C}}_1(T^1_i; t_0) &= 0, \qquad F^{U_2,\mathsf{C}}_2(T^2_i; t_0) = \frac{P^{U,\mathsf{C}}(T^2_i; t_0) - P^{U,\mathsf{C}}(T^2_{i+1}; t_0)}{P^{U,\mathsf{C}}(T^2_{i+1}; t_0)(T^2_{i+1} - T^2_i)}, \\ P^{U_1,\mathsf{C}}_1 &= P^{U,\mathsf{C}} = \text{calibrated}, \; P^{U_2,\mathsf{C}}_2 = P^{U,\mathsf{C}} = \text{calibrated}. \end{aligned}$$

In a situation where the number of interpolation points matches the number of swaps (e.g., a bootstrapping), we calibrate the time $T$ discount factor $P^{U,\mathsf{C}}(T; t_0)$ with $T = \max(T^1_n, T^2_n)$ being the last payment time from a given swap.

## 6.3 Calibration of Forward Curves

Given a calibrated discount curve $P^{U,\mathsf{C}}$ we consider a swap with payments in currency $U$ collateralized with respect to the account $\mathsf{C}$, paying some index $I$ and receiving some fixed cash flow $C$. An example is swaps paying the 3M LIBOR rate. For such a swap we have

$$\begin{aligned} &X_1 = C = \text{const.} = \text{given}, && X_2 = 0, \\ &F_1^{U_1,\mathsf{C}}(T_i^1) = 0, && F_2^{U_2,\mathsf{C}}(T_i^2) = F^{U,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ &P_1^{U_1,\mathsf{C}} = P^{U,\mathsf{C}} = \text{given}, && P_2^{U_2,\mathsf{C}} = P^{U,\mathsf{C}} = \text{given}. \end{aligned}$$

From one such swap we calibrate the time $T$ forward $F^{U,\mathsf{C}}(T)$ of $I(T)$ with $T = T_{n-1}^2$ (the last fixing time).

Given a calibrated discount curve $P^{U,\mathsf{C}}$ and a calibrate forward curve $F_1^{U,\mathsf{C}}$ belonging to the index $I_1$, both in currency $U$ and collateralized with respect to the account $\mathsf{C}$, we consider a swap collateralized with respect to the account $\mathsf{C}$, paying some index $I_2 = I$ in currency $U$, receiving the index $I_1$ in currency $U$. An example is tenor basis swaps paying the 6M LIBOR rate, receiving the 3M LIBOR rate. For such a swap we have

$$\begin{aligned} &X_1 = C_1 = \text{const.} = \text{given}, && X_2 = C_2 = \text{const.} = \text{given}, \\ &F_1^{U_1,\mathsf{C}}(T_i^1) = F_1^{U,\mathsf{C}}(T_i^1) = \text{given}, && F_2^{U_2,\mathsf{C}}(T_i^2) = F_2^{U,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ &P_1^{U_1,\mathsf{C}} = P^{U,\mathsf{C}} = \text{given}, && P_2^{U_2,\mathsf{C}} = P^{U,\mathsf{C}} = \text{given}. \end{aligned}$$

From one such swap we calibrate the time $T$ forward $F_2^{U,\mathsf{C}}(T)$ of $I(T)$ with $T = T_{n-1}^2$ (the last fixing time of index $I_2$).

## *6.4 Calibration of Discount Curves When Payment and Collateral Currency Differ*

### 6.4.1 Fixed Payment in Other Currency

Given a calibrated discount curve $P^{U_1,\mathsf{C}}$ we consider a swap collateralized with respect to the account $\mathsf{C}$, paying some index $I_1$ in currency $U_1$, and receiving some fixed cash flow $C_2$ in currency $U_2$. An example for such a swap is a cross-currency swap paying floating index $I$ in collateral currency and receiving fixed $C_2$ in a different currency.[12] For such a swap we have

$$\begin{aligned} &X_1 = C_1 = \text{const.} = \text{given}, && X_2 = C_2 = \text{const.} = \text{given}, \\ &F_1^{U_1,\mathsf{C}}(T_i^1) = F_1^{U_1,\mathsf{C}}(T_i^1) = \text{given}, && F_2^{U_2,\mathsf{C}}(T_i^2) = 0, \\ &P_1^{U_1,\mathsf{C}} = P^{U_1,\mathsf{C}} = \text{given}, && P_2^{U_2,\mathsf{C}} = P^{U_2,\mathsf{C}} = \text{calibrated}. \end{aligned}$$

We calibrate the discount factor $P^{U_2,\mathsf{C}}(T; t_0)$ with $T = T_n^2$ (last payment time in currency $U_2$).

[12] Usually cross-currency swaps exchange two floating indices, we will consider this case below.

### 6.4.2 Float Payment in Other Currency

If instead of a fixed payment we have that an index $I_2$ is paid in an other currency $U_2$ we may encounter the problem that the swap has two unknowns, namely the discount curve $P^{U_2,\mathsf{C}}$ for payments in currency $U_2$ collateralized with respect to $\mathsf{C}$ and the forward curve $F_2^{U_2,\mathsf{C}}$ of the index $I_2$ paid in currency $U_2$ collateralized with respect to $\mathsf{C}$. The two curves can be obtained jointly from two different swaps: first a fix-versus-float swaps in currency $U_2$ collateralized by $\mathsf{C}$, and second, a cross-currency swap exchanging the index $I_2$ with an index $I_1$ in currency $U_1$ for which the forward $F_1^{U_1,\mathsf{C}}$ is known. For the first instrument we denote the fixed payment by $C_1$, $C_2$. For the second instrument we denote the fixed payment by $s_1$, $s_2$ (usually a spread). For the first instrument we have

$$\begin{aligned} X_1 &= C_1 = \text{const.} = \text{given}, & X_2 &= C_2 = \text{const.} = \text{given}, \\ F_1^{U_1,\mathsf{C}}(T_i^1) &= 0, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F_2^{U_2,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ P_1^{U_1,\mathsf{C}} &= P_2^{U_2,\mathsf{C}} = \text{calibrated}, & P_2^{U_2,\mathsf{C}} &= \text{calibrated}. \end{aligned}$$

For the second swap we have

$$\begin{aligned} X_1 &= s_1 = \text{const.} = \text{given}, & X_2 &= s_2 = \text{const.} = \text{given}, \\ F_1^{U_1,\mathsf{C}}(T_i^1) &= F_1^{U_1,\mathsf{C}}(T_i^1) = \text{given}, & F_2^{U_2,\mathsf{C}}(T_i^2) &= F_2^{U_2,\mathsf{C}}(T_i^2) = \text{calibrated}, \\ P_1^{U_1,\mathsf{C}} &= \text{given}, & P_2^{U_2,\mathsf{C}} &= \text{calibrated}. \end{aligned}$$

We calibrate the discount factor $P^{U_2,\mathsf{C}}(T;t_0)$ with $T = T_n^2$ and the forward $F_2^{U_2,\mathsf{C}}(\mathrm{T})$ with $T = T_{n-1}^2$.

Often market data are not available to calibrate the forward $F_2^{U_2,\mathsf{C}}$, but the forward $F_2^{U_2,\mathsf{C}_2}$ collateralized with respect to a different account $\mathsf{C}_2$ is available. The two forwards differ by a possible convexity adjustment. One possible approximation (which would follow from the assumption that forwards are independent of their collateralization) is to use $F_2^{U_2,\mathsf{C}} \approx F_2^{U_2,\mathsf{C}_2}$.

The joint calibration of the two curves can be decomposed into two independent calibration steps, which would then allow to re-use a traditional bootstrap algorithm, see, e.g., [4].

#### *Calibration of Discount Curves as Spread Curves*

We consider a swap leg with notional exchange and tenor $\{T_i\}_{i=0}^n$, paying an index $I$ plus some constant $X = s(T_n) = const.$ Here $s(T_n)$ has the interpretation of a maturity-dependent spread. If this leg is in currency $U$ and with respect to a collateral account (here funding account) $\mathsf{D}$, then its value is

$$\begin{aligned} V_{SwapLeg}^{U,\mathsf{D}}(\alpha I, X, \{T_i\}_{i=0}^n) = \sum_{i=0}^{n-1} \Big( \big(\alpha F^{U,\mathsf{D}}(T_i) + X\big) \cdot P^{U,\mathsf{D}}(T_{i+1}) \\ + P^{U,\mathsf{D}}(T_{i+1}) - P^{U,\mathsf{D}}(T_i) \Big) . \end{aligned}$$

An example of such an instrument is an (uncollateralized) floating rate bond, paying a 3M rate plus some spread. If we assume that the forward $F^{U,\mathsf{D}}(T_i)$ is known, this instrument can be used to calibrate the discount curve $P^{U,\mathsf{D}}$. In fact $I + X$ represents the performance of the funding account associated with $P^{U,\mathsf{D}}$.

If the forward $F^{U,\mathsf{D}}(T_i)$ is not known, we encounter the same problem as for cross-currency swaps, namely that the forward curve $F^{U,\mathsf{D}}(T_i)$ and the discount curve $P^{U,\mathsf{D}}$ need to be calibrated jointly to two instruments. The first one is a swap which is collateralized with respect to the funding account $\mathsf{D}$, i.e., it is an uncollateralized swap. The second is the funding floater.

For the first instrument, the uncollateralized swap, we have

$$\begin{aligned} X_1 &= C_1 = \text{const.} = \text{given}, & X_2 &= C_2 = \text{const.} = \text{given}, \\ F_1^{U,\mathsf{D}}(T_i^1) &= 0 = \text{given}, & F_2^{U,\mathsf{D}}(T_i^2) &= F^{U,\mathsf{D}}(T_i^2) = \text{calibrated}, \\ P_1^{U,\mathsf{D}} &= P^{U,\mathsf{D}} = \text{calibrated}, & P_2^{U,\mathsf{D}} &= P^{U,\mathsf{D}}. \end{aligned}$$

For the second instrument, the funding floating rate bond (uncollateralized swap leg with notional exchange) we have

$$\begin{aligned} X_1 &= S = \text{const.} = \text{given}, \\ F_1^{U,\mathsf{D}}(T_i^1) &= F^{U,\mathsf{D}}(T_i^1) = \text{calibrated}, \\ P_1^{U,\mathsf{D}} &= P^{U,\mathsf{D}} = \text{calibrated}. \end{aligned}$$

*Remark 6* The calibration of the funding curve $P^{U,\mathsf{D}}$ is analog to the calibration of the cross-currency discount curve $P^{U_2,\mathsf{C}}$.

In the above, we consider the *funding floater* as a floating rate bond. Note however, that bonds (in contrast to swaps) do not permit negative coupons, hence they have an implicit floor. There are ways to solve this problem: either one has to incorporate an option premium in the calibration procedure (which does require a model for the volatility) or one considers only market data of fixed bonds together with uncollateralized swaps (which likely requires some assumption since usually this calibration instrument is not observed). See the following section.

## 6.5 Lack of Calibration Instruments (for Difference in Collateralization)

The calibration of cross-currency curves (forward curve and discount curves for currency $U_2$ with collateralization in currency $U_1$, see Sect. 6.4) and the calibration of un-collateralized curves (forward curves and discount curves for uncollateralized products, see section "Calibration of Discount Curves as Spread Curves") may require market data which are not available, e.g., the forward of an index $I$ paid in currency $U_2$ collateralized in a different currency or by a different account. This issue has been pointed out by [14].

In this case the curve can be obtained by adding additional assumptions. Two simple examples are:

- the market rates are assumed to be independent of the type of collateralization, or
- the forward rates are assumed to be independent of the type of collateralization.

The two assumptions lead to different results, since they imply different correlations which will lead to different (convexity) adjustments. For details on the example see [11], where a sample calculation with assuming identical market rates for 3M swaps collateralized in USD-OIS or EUR-OIS results in a difference of around 1 or 2 basis points (0.01 %) for the forward curves.

### *6.6 Implementation*

The definition of the various calibration instruments indicated that an iterative bootstrapping algorithm (there the curve is built in a step-by-step process solving only one dimensional problems in one variable) is no longer straightforward. This is due to the interdependence of discount and forward curves. While this problem may be solved in some cases via a pre-processing (see [4]), we suggest a different route: we propose to solve the calibration problem via a single optimization run on the full multi-dimensional problem. This also allows to calibrate curve in the sense of a best fit in cases where we use more calibration instruments than curve points, resulting in an overdetermined system.

We provide an object-oriented implementation at [11] implementing the Java classes for `Curves`, `DiscountCurves`, `ForwardCurves`, `Solver`, `SwapLeg` and `Swap`.

A detailed discussion of the implementation can be found in the associated JavaDocs and is left out here to shorten the presentation.

## 7 Redefining Forward Rate Market Models

Having discussed the setup of curves, we would like to conclude with a remark on how the curves are integrated into term-structure models, specifically, how the multi-curve setup harmonizes with a classical single curve standard LIBOR market model, which can then be extended to a fully multi-curve model.

If $N^{\mathsf{C}}$ denotes an accrual account, i.e., $N^{\mathsf{C}}$ is a process with $N^{\mathsf{C}}(t_0) = 1U$ (e.g., a collateral account), then $N^{\mathsf{C}}$ defines a discount curve, namely the discount curve $T \mapsto P^{U,\mathsf{C}}(T;t_0) =: P^{\mathsf{C}}(T;t_0)$ of fixed payments made in $T$, valued in $t$ and collateralized by units of $N^{\mathsf{C}}$.

Now let $\{T_i\}$ denote a given tenor discretization. As shown in Sect. 4.1 the period-$[T_i, T_{i+1}]$ performance index $I^{\mathsf{C}}(T_i, T_{i+1})$ of the an accrual account, i.e., $I^{\mathsf{C}}(T_i, T_{i+1}; T_i) := \frac{N^{\mathsf{C}}(T_{i+1})}{N^{\mathsf{C}}(T_i)} - 1$ has the property that its time $t$ *forward* (of a payment of $I^{\mathsf{C}}(T_i, T_{i+1})$ made in $T_{i+1}$, collateralized in units of $N^{\mathsf{C}}$) (following the definition of a forward from Definition 3) is given as $F^{U,\mathsf{C}}(T_i, T_{i+1}; t) := \frac{P^{\mathsf{C}}(T_i;t) - P^{\mathsf{C}}(T_{i+1};t)}{P^{\mathsf{C}}(T_{i+1};t_0)}$.

This relation allows us to create a term-structure model for the curve $P^{\mathsf{C}}$ which has the same structural properties as a standard single curve (LIBOR) market model. This model is given by a joint modeling of the processes $L_i(t) := \frac{F^{U,\mathsf{C}}(T_i,T_{i+1};t)}{T_{i+1}-T_i}$, e.g., as log-normal processes under the measure $\mathbb{Q}^{N^{\mathsf{C}}}$ and the additional assumption that the process $P^{\mathsf{C}}(T_i; t)$ is deterministic on its short period $t \in (T_{i-1}, T_i]$.

From these two assumptions it follows that the processes $L_i$ have the structure of a standard LIBOR market model and $\mathbb{Q}^{N^{\mathsf{C}}}$ corresponds to the spot measure. Indeed we have $\prod_{j=0}^{i-1} 1 + L_j(T_j) \cdot (T_{j+1} - T_j) = N^{\mathsf{C}}(T_i)$.

What we have described is how to use the standard LIBOR market model as a term structure model for the collateral account $N^{\mathsf{C}}$ (e.g., the OIS curve). Now, modeling all other rates (including LIBOR) can be performed by modeling (possibly stochastic) spreads over this curve. This is analog to a defaultable market model.

An alternative is to start with a stochastic model for the forward rates, where now the forward curve defines the initial value of the model SDEs, and then define the discount curve (numéraire) via deterministic or stochastic spreads. This approach has a practical advantage, since for LIBOR rates implied volatilities are more liquid than for OIS rates. See, e.g., [20] and references therein. An implementation of the standard LMM with a deterministic adjustment for the discount curve is provided by the author at [9].

## 8 Some Numerical Results

### 8.1 *Impact of the Interpolation Entity of a Forward Curve on the Delta Hedge*

Using our reference implementation [11], we investigate the interpolation of forward curves using different interpolation methods and interpolation entities. While interpolation of (synthetic) discount factors is—motivated from its single curve origin—a very popular interpolation method, it may result in very implausible deltas, if the curve is constructed from overlapping instruments. Table 1 shows the delta of an 8x11 FRA calculated on a curve constructed from 0x3, 1x4, 2x5, 3x6, 4x7, 5x8, 6x9, 7x10, 9x12 FRA (note that the 8x11 is missing in the curve construction). The plausible hedge would be to use the adjacent 7x10 and 9x12 FRAs. Using the interpolation entity `DISCOUNTFACTOR` we find non-zero deltas for instruments prior to the 7x10 FRA, summing up to zero. This effect stems from the error propagation inherent in the definition of the interpolation entity. The interpolation entity `FORWARD` does not show this effect.

**Table 1** The delta of an 7Mx10M FRA with respect to different calibration instruments, where the 7Mx10M FRA is not part of the calibration instruments, hence interpolates

| Risk Factor | Delta of an 7Mx10M FRA using the interpolation entity | |
|---|---|---|
| | `DISCOUNTFACTOR` (%) | `FORWARD` (%) |
| 0Dx3M | 44.5 | 0.0 |
| 1Mx4M | −95.9 | 0.0 |
| 2Mx5M | 52.4 | 0.0 |
| 3Mx6M | 44.0 | 0.0 |
| 4Mx7M | −97.0 | 0.0 |
| 5Mx8M | 52.4 | 0.0 |
| 6Mx9M | 47.6 | 48.4 |
| 7Mx10M | 0.0 | 0.0 |
| 8Mx11M | 51.9 | 51.6 |
| 9Mx12M | 0.0 | 0.0 |

Different interpolation entities result in very different delta hedges. The popular interpolation entity of a synthetic discount factor results in counterintuitive hedges. The interpolation method is `LINEAR` in both cases. It is the choice of the interpolation entity which introduces the effect

### *8.2 Impact of the Lack of Calibration Instruments for the Case of a Foreign Swap Collateralized in Domestic Currency*

Based on the curve framework and the calibration instruments defined in this paper and implemented at [11] we have investigated the impact of the assumptions, which had to be made due to the lack of calibration instruments for foreign currency swaps. Since a foreign currency swap collateralized in domestic currency is (currently) not a liquid instrument, the *foreign forward with respect to domestic collateralization* cannot be calibrated. Hence, a model assumption is required. Two possible assumptions are: (1) the forward rate is independent from its collateralization—that is, use the foreign forward curve derived from instruments collateralized in *foreign* currency, or, (2) the market (swap) rates are independent from its collateralization—that is, use the foreign market (par-)swap rates form foreign currency swaps collateralized in foreign currency together with a domestic currency discount curve to calibrate a foreign currency forward rate curve with respect to domestic collateralization. Both approaches result in different forward curves. The impact can be assessed using the spreadsheet available at [11]. For 2012 market data the difference for an USD forward curve collateralized in EUR can be found to be around two basis points. While the first assumption (re-using the forward curve) is likely the more natural one, and maybe a market standard, the calculation shows that the assumption has a considerable impact on the resulting curve, see Fig. 1.

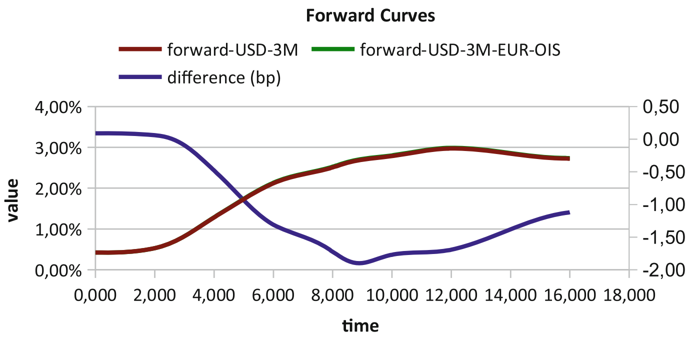


**Fig. 1** Forward curve (USD-3M) calibrated from swaps with different collateralization (USD-OIS and EUR-OIS) assuming independence of the market rates of from the type of collateralization

### *8.3 Impact of the Interpolation Scheme on the Hedge Efficiency*

Also based on the framework presented here, the impact of the different interpolation schemes has been investigated in [13], where indication was found that among the local interpolation schemes, it is indeed better to use a different interpolation scheme for forward curves than for discount curves. For details we refer to [13].

## 9 Conclusion

We have presented the re-definition of discount curves and forward curves, which clearly distinguishes the two as different objects (with some relation for the special case of OIS curves). This re-definition results in curves, representing values with well-defined economic interpretations. We then discussed some interpolation schemes for these curves, where our re-definition suggests to apply different interpolation schemes for discount and forward curves. This stands in contrast to the classical approach where a forward curve had been represented via synthetic discount factors, using the same interpolation schemes for both types of curves.

We have presented the calibration, defining the calibration instruments. Based on this, we provide an open source, object-oriented implementation at [11].[13]

Based on this benchmark implementation it was possible to assess the impact of assumptions, which had to be made due to the lack of calibration instruments, e.g., for the case of cross-currency swaps, and the impact of the different interpolation schemes. Indication was found that it is better to use a different interpolation scheme for forward curves than for discount curves. With respect to delta hedges one should

[13] A complete description of the implementation is given at http://www.finmath.net/finmath-lib, including source code and numerical examples. They are left out in this paper.

favor forward interpolation over synthetic discount factor interpolation. Among forward interpolation, linear interpolation performed well with respect to the hedge performance.

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# Impact of Multiple-Curve Dynamics in Credit Valuation Adjustments

**Giacomo Bormetti, Damiano Brigo, Marco Francischello and Andrea Pallavicini**

**Abstract** We present a detailed analysis of interest rate derivatives valuation under credit risk and collateral modeling. We show how the credit and collateral extended valuation framework in Pallavicini et al. (2011) can be helpful in defining the key market rates underlying the multiple interest rate curves that characterize current interest rate markets. We introduce the collateralized valuation measures and formulate a consistent realistic dynamics for the rates emerging from our analysis. We point out limitations of multiple curve models with deterministic basis considering valuation of particularly sensitive products such as basis swaps.

**Keywords** Multiple curves · Evaluation adjustments · Basis swaps · Collateral · HJM model

## 1 Introduction

After the onset of the crisis in 2007, all market instruments are quoted by taking into account, more or less implicitly, credit- and collateral-related adjustments. As a consequence, when approaching modeling problems one has to carefully check standard theoretical assumptions which often ignore credit and liquidity issues. One has to go back to market processes and fundamental instruments by limiting oneself to use models based on products and quantities that are available on the market.

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Referring to market observables and processes is the only means we have to validate our theoretical assumptions, so as to drop them if in contrast with observations. This general recipe is what is guiding us in this paper, where we try to adapt interest rate models for valuation to the current landscape.

A detailed analysis of the updated valuation problem one faces when including credit risk and collateral modeling (and further funding costs) has been presented elsewhere in this volume, see for example [6, 7]. We refer to those papers and references therein for a detailed discussion. Here we focus our updated valuation framework to consider the following key points: (i) focus on interest rate derivatives; (ii) understand how the updated valuation framework can be helpful in defining the key market rates underlying the multiple interest rate curves that characterize current interest rate markets; (iii) define collateralized valuation measures; (iv) formulate a consistent realistic dynamics for the rates emerging from the above analysis; (v) show how the framework can be applied to valuation of particularly sensitive products such as basis swaps under credit risk and collateral posting;(vi) point out limitations in some current market practices such as explaining the multiple curves through deterministic fudge factors or shifts where the option embedded in the credit valuation adjustment (CVA) calculation would be priced without any volatility. For an extended version of this paper we remand to [3]. This paper is an extended and refined version of ideas originally appeared in [24].

## 2 Valuation Equation with Credit and Collateral

Classical interest-rate models were formulated to satisfy no-arbitrage relationships by construction, which allowed one to price and hedge forward-rate agreements in terms of risk-free zero-coupon bonds. Starting from summer 2007, with the spreading of the credit crunch, market quotes of forward rates and zero-coupon bonds began to violate usual no-arbitrage relationships. The main driver of such behavior was the liquidity crisis reducing the credit lines along with the fear of an imminent systemic break-down. As a result the impact of counterparty risk on market prices could not be considered negligible any more.

This is the first of many examples of relationships that broke down with the crisis. Assumptions and approximations stemming from valuation theory should be replaced by strategies implemented with market instruments. For instance, inclusion of CVA for interest-rate instruments, such as those analyzed in [8], breaks the relationship between risk-free zero-coupon bonds and LIBOR forward rates. Also, funding in domestic currency on different time horizons must include counterparty risk adjustments and liquidity issues, see [15], breaking again this relationship. We thus have, against the earlier standard theory,

$$L(T_0, T_1) \neq \frac{1}{T_1 - T_0}\left(\frac{1}{P_{T_0}(T_1)} - 1\right), \quad F_t(T_0, T_1) \neq \frac{1}{T_1 - T_0}\left(\frac{P_t(T_0)}{P_t(T_1)} - 1\right), \tag{1}$$

where $P_t(T)$ is a zero-coupon bond price at time $t$ for maturity $T$, $L$ is the LIBOR rate and $F$ is the related LIBOR forward rate. A direct consequence is the impossibility to describe all LIBOR rates in terms of a unique zero-coupon yield curve. Indeed, since 2009 and even earlier, we had evidence that the money market for the Euro area was moving to a multi-curve setting. See [1, 19, 20, 27].

## 2.1 Valuation Framework

In order to value a financial product (for example a derivative contract), we have to discount all the cash flows occurring after the trading position is entered. We follow the approach of [25, 26] and we specialize it to the case of interest-rate derivatives, where collateralization usually happens on a daily basis, and where gap risk is not large. Hence we prefer to present such results when cash flows are modeled as happening in a continuous time-grid, since this simplifies notation and calculations. We refer to the two names involved in the financial contract and subject to default risk as investor (also called name "I") and counterparty (also called name "C"). We denote by $\tau_I$, and $\tau_C$, respectively, the default times of the investor and counterparty. We fix the portfolio time horizon $T > 0$, and fix the risk-neutral valuation model $(\Omega, \mathscr{G}, \mathbb{Q})$, with a filtration $(\mathscr{G}_t)_{t\in[0,T]}$ such that $\tau_C$, $\tau_I$ are $(\mathscr{G}_t)_{t\in[0,T]}$-stopping times. We denote by $\mathbb{E}_t[\,\cdot\,]$ the conditional expectation under $\mathbb{Q}$ given $\mathscr{G}_t$, and by $\mathbb{E}_{\tau_i}[\,\cdot\,]$ the conditional expectation under $\mathbb{Q}$ given the stopped filtration $\mathscr{G}_{\tau_i}$. We exclude the possibility of simultaneous defaults, and define the first default event between the two parties as the stopping time $\tau := \tau_C \wedge \tau_I$.

We will also consider the market sub-filtration $(\mathscr{F}_t)_{t\geq 0}$ that one obtains implicitly by assuming a separable structure for the complete market filtration $(\mathscr{G}_t)_{t\geq 0}$. $\mathscr{G}_t$ is then generated by the pure default-free market filtration $\mathscr{F}_t$ and by the filtration generated by all the relevant default times monitored up to $t$ (see for example [2]).

We introduce a risk-free rate $r$ associated with the risk-neutral measure. We therefore need to define the related stochastic discount factor $D(t, u, r)$ that in general will denote the risk-neutral default-free discount factor, given by the ratio

$$D(t, u, r) = B_t/B_u \ , \quad dB_t = r_t B_t dt,$$

where $B$ is the bank account numeraire, driven by the risk-free instantaneous interest rate $r_t$ and associated to the risk-neutral measure $\mathbb{Q}$. This rate $r_t$ is assumed to be $(\mathscr{F}_t)_{t\in[0,T]}$ adapted and is the key variable in all pre-crisis term structure modeling.

We now want to price a collateralized derivative contract, and in particular we assume that collateral re-hypothecation is allowed, as done in practice (see [4] for a discussion on re-hypothecation). We thus write directly the adjustment payout terms as carry costs cash flows, each accruing at the relevant rate, namely the price $V_t$ of a derivative contract, inclusive of collateralized credit and debit risk, margining costs, can be derived by following [25, 26], and is given by:

$$V_t = \mathbb{E}\left[ \int_t^T D(t,u;r) \left( 1_{\{u<\tau\}} d\pi_u + 1_{\{\tau \in du\}} \theta_u + (r_u - c_u) C_u du \right) \mid \mathscr{G}_t \right] \quad (2)$$

where

- $\pi_u$ is the coupon process of the product, without credit or debit risk and without collateral cash flows;
- $C_u$ is the collateral process, and we use the convention that $C_u > 0$ while $I$ is the collateral receiver and $C_u < 0$ when $I$ is the collateral poster. $(r_u - c_u)C_u$ are the collateral margining costs and the collateral rate is defined as $c_t := c_t^+ 1_{\{C_t>0\}} + c_t^- 1_{\{C_t<0\}}$ with $c^\pm$ defined in the CSA contract. In general we may assume the processes $c^+, c^-$ to be adapted to the default-free filtration $\mathscr{F}_t$.
- $\theta_u = \theta_u(C, \varepsilon)$ is the on-default cash flow process that depends on the collateral process $C_u$ and the close-out value $\varepsilon_u$.[1] It is primarily this term that originates the credit and debit valuation adjustments (CVA/DVA) terms, that may also embed collateral and gap risk due to the jump at default of the value of the considered deal (e.g. in a credit derivative), see for example [5].

Notice that the above valuation equation (2) is not suited for explicit numerical evaluations, since the right-hand side is still depending on the derivative price via the indicators within the collateral rates and possibly via the close-out term, leading to recursive/nonlinear features. We could resort to numerical solutions, as in [11], but, since our goal is valuing interest-rate derivatives, we prefer to further specialize the valuation equation for such deals.

## 2.2 *The Master Equation Under Change of Filtration*

In this first work we develop our analysis without considering a dependence between the default times if not through their spreads, or more precisely by assuming that the default times are $\mathscr{F}$-conditionally independent. Moreover, we assume that the collateral account and the close-out processes are $\mathscr{F}$-adapted. Thus, we can simplify the valuation equation given by (2) by switching to the default-free market filtration. By following the filtration switching formula in [2], we introduce for any $\mathscr{G}_t$-adapted process $X_t$ a unique $\mathscr{F}_t$-adapted process $\widetilde{X}_t$, defined such that $1_{\{\tau>t\}} X_t = 1_{\{\tau>t\}} \widetilde{X}_t$. Hence, we can write the pre-default price process as given by $1_{\{\tau>t\}} \widetilde{V}_t = V_t$ where the right-hand side is given in Eq. (2) and where $\tilde{V}_t$ is $\mathscr{F}_t$-adapted. Before changing filtration, we have to specify the form of the close-out payoff:

$$\theta_\tau = \varepsilon_\tau(\tau, T) - 1_{\{\tau_C < \tau_I\}} LGD_C (\varepsilon_\tau(\tau, T) - C_\tau)^+ - 1_{\{\tau_I < \tau_C\}} LGD_I (\varepsilon_\tau(\tau, T) - C_\tau)^-$$

[1] The closeout value is the residual value of the contract at default time and the CSA specifies the way it should be computed.

where $LGD \leq 1$ is the loss given default, $(x)^+$ indicates the positive part of $x$ and $(x)^- = -(-x)^+$. For an extended discussion of the term $\theta_\tau$ we refer to [3]. Moreover, to derive an explicit valuation formula we assume that gap risk is not present, namely $\tilde{V}_{\tau-} = \tilde{V}_\tau$, and we consider a particular form for collateral and close-out prices, namely we model the close-out value as

$$\varepsilon_s(t,T) = \mathbb{E}\left[\int_t^T D(t,u,r)d\pi_u \mid \mathscr{G}_s\right], \quad C_t \doteq \alpha_t \varepsilon_t(t,T)$$

with $0 \leq \alpha_t \leq 1$ and where $\alpha_t$ is $\mathscr{F}_t$-adapted. This means that the close-out is the risk-free mark to market at first default time and the collateral is a fraction $\alpha_t$ of the close-out value. An alternative approximation that does not impose a proportionality between the account value processes can be found in [9]. We obtain, by switching to the default-free market filtration $\mathscr{F}$ the following.[2]

**Proposition 1** (Master equation under $\mathscr{F}$-conditionally independent default times, no gap risk and $\mathscr{F}_t$ measurable payout $\pi_t$) *Under the above assumption, Valuation Equation (2) is further specified as* $V_t = 1_{\{\tau>t\}}\widetilde{V}_t$

$$\begin{aligned}\widetilde{V}_t =& \varepsilon_t(t,T) + \mathbb{E}\left[\int_t^T D(t,u;r+\lambda)(r_u - c_u)\alpha_u\varepsilon_u(u,T)du \mid \mathscr{F}_t\right] \\ & - \mathbb{E}\left[\int_t^T D(t,u;r+\lambda)\lambda_u^C(1-\alpha_u)LGD_C(\varepsilon_u(u,T))^+du \mid \mathscr{F}_t\right] \\ & - \mathbb{E}\left[\int_t^T D(t,u;r+\lambda)\lambda_u^I(1-\alpha_u)LGD_I(\varepsilon_u(u,T))^-du \mid \mathscr{F}_t\right]\end{aligned}$$

*where we introduced the pre-default intensity* $\lambda_t^I$ *of the investor and the pre-default intensity* $\lambda_t^C$ *of the counterparty as*

$$1_{\{\tau_I>t\}}\lambda_t^I\,dt := \mathbb{Q}\{\tau_I \in dt \mid \tau_I > t, \mathscr{F}_t\},\; 1_{\{\tau_C>t\}}\lambda_t^C\,dt := \mathbb{Q}\{\tau_C \in dt \mid \tau_C > t, \mathscr{F}_t\}$$

*along with their sum* $\lambda_t$ *and the discount factor for any rate* $x_u$, *namely* $D(t,T,x) := \exp\{-\int_t^T x_u du\}$.

## 3 Valuing Collateralized Interest-Rate Derivatives

As we mentioned in the introduction, we will base our analysis on real market processes. All liquid market quotes on the money market (MM) correspond to instruments with daily collateralization at overnight rate ($e_t$), both for the investor and the counterparty, namely $c_t \doteq e_t$ .

[2] We refer to [3] and [6] for a precise derivation of the proposition.

Notice that the collateral accrual rate is symmetric, so that we no longer have a dependency of the accrual rates on the collateral price, as opposed to the general master equation case. Moreover, we further assume $r_t \doteq e_t$ .

This makes sense because $e_t$ being an overnight rate, it embeds a low counterparty risk and can be considered a good proxy for the risk-free rate $r_t$. We will describe some of these MM instruments, such as OIS and Interest Rate Swaps (IRS), along with their underlying market rates, in the following sections. For the remaining of this section we adopt the perfect collateralization approximation of Eq. (1) to derive the valuation equations for OIS and IRS products, hence assuming no gap-risk, while in the numeric experiments of Sect. 4 we will consider also uncollateralized deals. Furthermore, we assume that daily collateralization can be considered as a continuous-dividend perfect collateralization. See [4] for a discussion on the impact of discrete-time collateralization on interest-rate derivatives.

### *3.1 Overnight Rates and OIS*

Among other instruments, the MM usually quotes the prices of overnight indexed swaps (OIS). Such contracts exchange a fix-payment leg with a floating leg paying a discretely compounded rate based on the same overnight rate used for their collateralization. Since we are going to price OIS under the assumption of perfect collateralization, namely we are assuming that daily collateralization may be viewed as done on a continuous basis, we approximate also daily compounding in OIS floating leg with continuous compounding, which is reasonable when there is no gap risk. Hence the discounted payoff of a one-period OIS with tenor $x$ and maturity $T$ is given by

$$D(t, T, e)\left(1 + xK - \exp\left\{\int_{T-x}^{T} e_u du\right\}\right)$$

where $K$ is the fixed rate payed by the OIS. Furthermore, we can introduce the (par) fix rates $K = E_t(T, x; e)$ that make the one-period OIS contract fair, namely priced 0 at time $t$. They are implicitly defined via

$$\widetilde{V}_t^{\mathrm{OIS}}(K) := \mathbb{E}\left[\left(1 + xK - \exp\left\{\int_{T-x}^{T} e_u du\right\}\right) D(t, T; e) \mid \mathscr{F}_t\right]$$

with $\widetilde{V}_t^{\mathrm{OIS}}(E_t(T, x; e)) = 0$ leading to

$$E_t(T, x; e) := \frac{1}{x}\left(\frac{P_t(T-x; e)}{P_t(T; e)} - 1\right) \tag{3}$$

where we define collateralized zero-coupon bonds[3] as

$$P_t(T; e) := \mathbb{E}\left[ D(t, T; e) \mid \mathscr{F}_t \right]. \tag{4}$$

One-period OIS rates $E_t(T, x; e)$, along with multi-period ones, are actively traded on the market. Notice that we can bootstrap collateralized zero-coupon bond prices from OIS quotes.

## 3.2 LIBOR Rates, IRS and Basis Swaps

LIBOR rates ($L_t(T)$) used to be linked to the term structure of default-free interlink interest rates in a fundamental way. In the classical term structure theory, LIBOR rates would satisfy fundamental no-arbitrage conditions with respect to zero-coupon bonds that we no longer consider to hold, as we pointed out earlier in (1). We now deal with a new definition of forward LIBOR rates that may take into account collateralization. LIBOR rates are still the indices used as reference rate for many collateralized interest-rate derivatives (IRS, basis swaps, …). IRS contracts swap a fix-payment leg with a floating leg paying simply compounded LIBOR rates. IRS contracts are collateralized at overnight rate $e_t$. Thus, a discounted one-period IRS payoff with maturity $T$ and tenor $x$ is given by

$$D(t, T, e)x(K - L_{T-x}(T))$$

where $K$ is the fix rate payed by the IRS. Furthermore, we can introduce the (par) fix rates $K = F_t(T, x; e)$ that render the one-period IRS contract fair, i.e. priced at zero. They are implicitly defined via

$$\widetilde{V}_t^{\text{IRS}}(K) := \mathbb{E}\left[ (xK - xL_{T-x}(T))\, D(t, T; e) \mid \mathscr{F}_t \right]$$

with $\widetilde{V}_t^{\text{IRS}}(F_t(T, x; e)) = 0$, leading to the following definition of forward LIBOR rate

$$F_t(T, x; e) := \frac{\mathbb{E}\left[ L_{T-x}(T) D(t, T; e) \mid \mathscr{F}_t \right]}{\mathbb{E}\left[ D(t, T; e) \mid \mathscr{F}_t \right]} = \frac{\mathbb{E}\left[ L_{T-x}(T) D(t, T; e) \mid \mathscr{F}_t \right]}{P_t(T; e)}$$

The above definition may be simplified by a suitable choice of the measure under which we take the expectation. In particular, we can consider the following Radon–Nikodym derivative, defining the collateralized $T$-forward measure $\mathbb{Q}^{T;e}$,

[3]Notice that we are only defining a price process for hypothetical collateralized zero-coupon bond. We are not assuming that collateralized bonds are assets traded on the market.

$$Z_t(T;e) := \left.\frac{d\mathbb{Q}^{T;e}}{d\mathbb{Q}}\right|_{\mathscr{F}_t} := \frac{\mathbb{E}\left[D(0,T;e) \mid \mathscr{F}_t\right]}{P_0(T;e)} = \frac{D(0,t;e)P_t(T;e)}{P_0(T;e)}$$

which is a positive $\mathbb{Q}$-martingale, normalized so that $Z_0(T;e) = 1$.

Thus, for any payoff $\phi_T$, perfectly collateralized at overnight rate $e_t$, we can express prices as expectations under the collateralized $T$-forward measure and in particular, we can write LIBOR forward rates as

$$F_t(T,x;e) := \frac{\mathbb{E}\left[L_{T-x}(T)D(t,T;e) \mid \mathscr{F}_t\right]}{\mathbb{E}\left[D(t,T;e) \mid \mathscr{F}_t\right]} = \mathbb{E}^{T;e}\left[L_{T-x}(T) \mid \mathscr{F}_t\right]. \tag{5}$$

One-period forward rates $F_t(T,x;e)$, along with multi-period ones (swap rates), are actively traded on the market. Once collateralized zero-coupon bonds are derived, we can bootstrap forward rate curves from such quotes. See, for instance, [1] or [27] for a discussion on bootstrapping algorithms.

Basis swaps are an interesting product that became more popular after the market switched to a multi-curve structure. In fact, in a basis swap there are two floating legs, one pays a LIBOR rate with a certain tenor and the other pays the LIBOR rate with a shorter tenor plus a spread that makes the contract fair at inception. More precisely, the payoff of a basis swap whose legs pay respectively a LIBOR rate with tenors $x < y$ with maturity $T = nx = my$ is given by

$$\sum_{i=1}^{n} D(t, T-(n-i)x, e)x(L_{T-(n-i-1)x}(T-(n-i)x) + K)$$
$$- \sum_{j=1}^{m} D(t, T-(m-j)y, e)yL_{T-(m-j-1)y}(T-(m-j)y).$$

It is clear that apart from being traded per se, this instrument is naturally present in the banks portfolios as result of the netting of opposite swap positions with different tenors.

### 3.3 Modeling Constraints

Our aim is to set up a multiple-curve dynamical model starting from collateralized zero-coupon bonds $P_t(T;e)$, and LIBOR forward rates $F_t(T,x;e)$. As we have seen we can bootstrap the initial curves for such quantities from directly observed quotes in the market. Now, we wish to propose a dynamics that preserves the martingale properties satisfied by such quantities. Thus, without loss of generality, we can define collateralized zero-coupon bonds under the $\mathbb{Q}$ measure as

$$dP_t(T;e) = P_t(T;e)\left(e_t\,dt - \sigma_t^P(T;e)^*\,dW_t^e\right)$$

and LIBOR forward rates under the $\mathbb{Q}^{T;e}$ measure as

$$dF_t(T,x;e) = \sigma_t^F(T,x;e)^* \, dZ_t^{T;e}$$

where $W^e$ and $Z^{T;e}$ are correlated standard (column) vector[4] Brownian motions with correlation matrix $\rho$, and the volatility vector processes $\sigma^P$ and $\sigma^F$ may depend on bonds and forward LIBOR rates themselves.

The following definition of $f_t(T,e)$ is not strictly necessary, and we could keep working with bonds $P_t(T;e)$, using their dynamics. However, as it is customary in interest rate theory to model rates rather than bonds, we may try to formulate quantities that are closer to the standard HJM framework. In this sense we can define instantaneous forward rates $f_t(T;e)$, by starting from (collateralized) zero-coupon bonds, as given by

$$f_t(T;e) := -\partial_T \log P_t(T;e)$$

We can derive instantaneous forward-rate dynamics by Itô lemma, and we obtain the following dynamics under the $\mathbb{Q}^{T;e}$ measure

$$df_t(T;e) = \sigma_t(T;e)\, dW_t^{T;e} \ , \quad \sigma_t(T;e) := \partial_T \, \sigma_t^P(T;e)$$

where the $W^{T;e}$s are Brownian motions and partial differentiation is meant to be applied component-wise.

Hence, we can summarize our modeling assumptions in the following way. Since linear products (OIS, IRS, basis swaps…) can be expressed in terms of simpler quantities, namely collateralized zero-coupon bonds $P_t(T;e)$ and LIBOR forward rates $F_t(T,x;e)$, we focus on their modeling. The initial term structures for collateralized products may be bootstrapped from market data, and for volatility and dynamics, we can write rates dynamics by enforcing suitable no-arbitrage martingale properties, namely

$$df_t(T;e) = \sigma_t(T;e) \cdot dW_t^{T;e}, \ \ dF_t(T,x;e) = \sigma_t^F(T,x;e) \cdot dZ_t^{T;e}. \tag{6}$$

As we explained in the introduction, this is where the multiple curve picture finally shows up: we have a curve with LIBOR-based forward rates $F_t(T,x;e)$, that are collateral adjusted expectation of LIBOR market rates $L_{T-x}(T)$ we take as primitive rates from the market, and we have instantaneous forward rates $f_t(T;e)$ that are OIS-based rates. OIS rates $f_t(T;e)$ are driven by collateral fees, whereas LIBOR forward rates $F_t(T,x;e)$ are driven both by collateral rates and by the primitive LIBOR market rates.

[4]In the following we will consider $N$-dimensional vectors as $N \times 1$ matrices. Moreover, given a matrix $A$, we will indicate $A^*$ its transpose, and if $B$ is another conformable matrix we indicate $AB$ the usual matrix product.

## 4 Interest-Rate Modeling

We can now specialize our modeling assumptions to define a model for interest-rate derivatives which is on one hand flexible enough to calibrate the quotes of the MM, and on the other hand robust. Our aim is to use an HJM framework using a single family of Markov processes to describe all the term structures and interest rate curves we are interested in.

In the literature many authors proposed generalizations of the HJM framework to include multiple yield curves. In particular, we cite the works of [12–14, 16, 20–23]. A survey of the literature can be found in [17].

In such works the problem is faced in a pragmatic way by considering each forward rate as a single asset without investigating the microscopical dynamics implied by liquidity and credit risks. However, the hypothesis of introducing different underlying assets may lead to over-parametrization issues that affect the calibration procedure. Indeed, the presence of swap and basis-swap quotes on many different yield curves is not sufficient, as the market quotes swaption premia only on few yield curves. For instance, even if the Euro market quotes one-, three-, six- and twelve-month swap contracts, liquidly traded swaptions are only those indexed to the three-month (maturity one-year) and the six-month (maturities from two to thirty years) Euribor rates. Swaptions referring to other Euribor tenors or to overnight rates are not actively quoted.

In order to solve such problem [23] introduces a parsimonious model to describe a multi-curve setting by starting from a limited number of (Markov) processes, so as to extend the logic of the HJM framework to describe with a unique family of Markov processes all the curves we are interested in.

### *4.1 Multiple-Curve Collateralized HJM Framework*

We follow [22, 23] by reformulating their theory under the $\mathbb{Q}^{T;e}$ measure. We model only observed rates as in market model approaches and we consider a common family of processes for all the yield curves of a given currency, so that we are able to build parsimonious yet flexible models. Hence let us summarize the basic requirements the model must fulfill:

(i) existence of OIS rates, which we can describe in terms of instantaneous forward rates $f_t(T; e)$;
(ii) existence of LIBOR rates assigned by the market, typical underlyings of traded derivatives, with associated forwards $F_t(T, x; e)$;
(iii) no arbitrage dynamics of the $f_t(T; e)$ and the $F_t(T, x; e)$ (both being $(T, e)$-forward measure martingales);
(iv) possibility of writing both $f_t(T; e)$ and $F_t(T, x; e)$ as functions of a common family of Markov processes, so that we are able to build parsimonious yet flexible models.

While the first two points are related to the set of financial quantities we are about to model, the last two are conditions we impose on their dynamics, and will be granted by the right choice of model volatilities. Hence, we choose under $\mathbb{Q}^{T;e}$ measure, the following dynamics:

$$df_t(T;e) = \sigma_t(T)^* dW_t^{T;e} \qquad (7)$$
$$dF_t(T,x;e) = (k(T,x) + F_t(T,x;e))\, \Sigma_t(T,x)^* dW_t^{T;e}$$

where we introduce the families of (stochastic $N$-dimensional) volatility processes $\sigma_t(T)$ and $\Sigma_t(T,x)$, the vector of $N$ independent $\mathbb{Q}^{T;e}$-Brownian motions $W_t^{T;e}$, and the set of deterministic shifts $k(T,x)$, such that $\lim_{x\to 0} xk(T,x) = 1$. This limit condition ensures that the model approaches a standard default- and liquidity-free HJM model when the tenor goes to zero. We bootstrap $f_0(T;e)$ and $F_0(T,x;e)$ from market quotes.

In order to get a model with a reduced number of common driving factors in the spirit of HJM approaches, it is sufficient to conveniently tie together the volatility processes $\sigma_t(T)$ and $\Sigma_t(T,x)$ through a third volatility process $\sigma_t(u,T,x)$.

$$\sigma_t(T) := \sigma_t(T;T,0)\ , \quad \Sigma_t(T,x) := \int_{T-x}^{T} \sigma_t(u;T,x)\, du. \qquad (8)$$

Under this parametrization the OIS curve dynamics is the very same as the risk-free curve in an ordinary HJM framework. Indeed, we have for linearly compounding forward rates

$$dE_t(T,x;e) = (1/x + E_t(T,x;e)) \int_{T-x}^{T} \sigma_t(u)^* du\ dW_t^{T;e}.$$

In the generalized version of the HJM framework proposed by [23] we have an explicit expression for both the collateralized zero-coupon bonds $P_t(T;e)$ and the LIBOR forward rates $F_t(T,x;e)$. The first result is a direct consequence of modeling the OIS curve as the risk-free curve in a standard HJM framework, while the second result can be achieved only if a particular form of the volatilities is selected. We obtain this if we generalize the approach of [28] by introducing the following separability constraint

$$\sigma_t(u,T,x) := h(t)q(u,T,x)g(t,u),$$
$$g(t,u) := \exp\left\{-\int_t^u a(s)ds\right\}\ , \quad q(u;u,0) := Id, \qquad (9)$$

where $h_t$ is an $N \times N$ matrix process, $q(u,T,x)$ is a deterministic $N \times N$ diagonal matrix function, and $a(s)$ is a deterministic $N$-dimensional vector function. The condition on $q(u;T,x)$ being the identity matrix, when $T = u$ ensures that a standard HJM framework holds for collateralized zero-coupon bonds.

We can work out an explicit expression for the LIBOR forward rates, by plugging the expression of the volatilities into Eq. (7). We obtain

$$\log\left(\frac{k(T,x)+F_t(T,x;e)}{k(T,x)+F_0(T,x;e)}\right) = G(t,T-x,T;T,x)^*\left(X_t + Y_t\left(G_0(t,t,T) - \frac{1}{2}G(t,T-x,T;T,x)\right)\right), \tag{10}$$

where the stochastic vector process $X_t$ and the auxiliary matrix process $Y_t$ are defined under the $\mathbb{Q}$ measure as in the ordinary HJM framework

$$X_t^i = \sum_{k=1}^{N}\int_0^t g_i(s,t)\left(h_{ik,s}dW_{k,s} + (h_s^* h_s)_{ik}\int_s^t dy g_k(s,y)\, ds\right) \quad, i = 1\dots N$$
$$Y_t^{ik} = \int_0^t g_i(s,t)(h_s^* h_s)_{ik} g_k(s,t)ds \quad i,k = 1\dots N$$

and

$$G_0(t,T_0,T_1) = \int_{T_0}^{T_1} g(t,s)ds, \quad G(t,T_0,T_1,T,x) = \int_{T_0}^{T_1} q(s,T,x)g(t,s)ds.$$

It is worth noting that the integral representation of forward LIBOR volatilities given by Eq. (8), together with the common separability constraint given in Eq. (9) are sufficient conditions to ensure the existence of a reconstruction formula for all OIS and LIBOR forward rates based on the very same family of Markov processes (see [3]).

We are interested in some specification of this model, in particular a variant of the Hull and White model (HW), a variant of the Cheyette model (Ch) and the Moreni and Pallavicini model (MP). The HW model [18] is the simplest one, and is obtained choosing

$$h(t) \doteq R\ , \quad q(u,T,x) \doteq Id\ , \quad a(s) \doteq a\ , \quad \kappa(T,x) \doteq \frac{1}{x} \tag{11}$$

where $a$ is a constant vector, and $R$ is the Cholesky decomposition of the correlation matrix that we want our $X_t$ vector to have. In this case we obtain $\sigma_t(u;T,x) = R\cdot e^{-a(u-t)}$, where the exponential is intended to be component-wise. Then we note that $X_t$ is a mean reverting Gaussian process while the $Y_t$ process is deterministic.

In order to model implied volatility smiles, we can add a stochastic volatility process to our model, as shown in [22]. In particular we can obtain a variant of the Ch model ([10]), considering a common square-root process for all the entries of $h$, as in [29]. More precisely we replace $h(t)$ in (11) with $h(t) \doteq \sqrt{v_t}R$. With $a$ and $R$ as before and $v_t$ being a process with the following dynamic:

$$dv_t = \eta\,(1 - v_t)\;dt + \nu_0\left(1 + (\nu_1 - 1)e^{-\nu_2 t}\right)\sqrt{v_t}\,dZ_t\;, \quad v_0 = \bar{v} \tag{12}$$

where $Z_t$ is a Brownian motion correlated to $W_t$. Obtaining as a volatility process $\sigma_t(u; T, x) = \sqrt{v_t}R \cdot e^{-a(u-t)}$.

As the last specification of the framework we consider the MP model which uses a different shift $k(T, x)$, and introduces a dependence on the tenor in the volatility process.

$$h(t) \doteq \sqrt{v_t}R\;, \quad q(u, T, x)^{i,i} \doteq e^{x\eta^i}\;, \quad a(s) \doteq a\;, \quad \kappa(T, x) \doteq \frac{e^{-\gamma x}}{x} \tag{13}$$

With $a$ and $R$ as before and $v_t$ being defined by (12). Here we have for the volatility $\sigma_t(u; T, x) = \sqrt{v_t}R \cdot e^{\eta x - a(u-t)}$.

To better appreciate the difference between the Ch model and the MP model one could compute the quantity

$$\beta_t(x_1, x_2; e) := \frac{1}{x_2}\log\left(\frac{\frac{1}{x_2} + E_t(t + x_2, x_2; e)}{\frac{1}{x_2} + F_t(t + x_2, x_2; e)}\right) - \frac{1}{x_1}\log\left(\frac{\frac{1}{x_1} + E_t(t + x_1, x_1; e)}{\frac{1}{x_1} + F_t(t + x_1, x_1; e)}\right)$$

which represents the time-normalized difference between two forward rates with different tenors and thus can be used as a proxy for the value of a basis swap. We have that in the HW and in the Ch models $\beta_t(x_1, x_2; e)$ is deterministic while in the MP model is a stochastic quantity. This suggests that the MP model should be able to better capture the dynamics of the basis between two rates with different tenors. We refer the reader to [3] for a more detailed analysis of the issue, and to [23] for calibration and valuation examples for the swaptions and cap/floor market.

## *4.2 Numerical Results*

We apply our framework to simple but relevant products: an IRS and a basis swap. We analyze the impact of the choice of an interest rate model on the portfolio valuation, in particular we measure the dependency of the price on the correlations between interest-rates and credit spreads, the so-called wrong-way risk. We model the market risks by simulating the following processes in a multiple-curve HJM model under the pricing measure $\mathbb{Q}$. The overnight rate $e_t$ and the LIBOR forward rates $F_t(T; e)$ are simulated according to the dynamics given in Sect. 4.1. Maintaining the same notation of the aforementioned section, we choose $N = 2$, and for our numerical experiments we use a HW model, a Ch model and an MP model, all calibrated to swaption at-the-money volatilities listed on the European market.

As we have already noted, the Ch model introduces a stochastic volatility and hence has an increased number of parameters with respect to the HW model. The MP model aims at better modeling the basis between rates with different tenors, while keeping the model parsimonious in terms of extra parameters with respect to the Ch model. In particular the HW model is able to reproduce the ATM quotes but is not able to correctly reproduce the volatility smile. On the other hand, the introduction of a stochastic volatility process helps in recovering the market data smile and thus the Ch and the MP models have similar results in properly fitting the smile. The detailed results of the calibration are available in [3].

For what concerns the credit part, the default intensities of the investor and the counterparty are given by two CIR++ processes $\lambda_t^i = y_t^i + \psi^i(t)$ under the $\mathbb{Q}^{T;e}$ measure, i.e. they follow

$$dy_t^i = \gamma^i(\mu^i - y_t^i)\,dt + \zeta^i\sqrt{y_t^i}\,dZ_t^i\ , \quad i \in \{I, C\}$$

where the two $Z^i$s are Brownian motions correlated with the $W^{T;e}$s, and they are calibrated to the market data shown in [4]. In particular, two different market settings are used in the numerical examples: the medium risk and the high risk settings. The correlations among the risky factors are induced by correlating the Brownian motions as in [8].

We now analyze the impact of wrong-way risk on the bilateral adjustment, namely CVA plus DVA, of IRS and basis swaps when collateralization is switched off, namely we want to evaluate Eq. (1) when $\alpha_t \doteq 0$. For an extended analysis see [3]. Wrong-way risk is expressed with respect to the correlation between the default intensities and a proxy of market risk, namely the short rate $e_t$.

In Fig. 1 we show the variation of the bilateral adjustment for a ten years IRS receiving a fix rate yearly and paying 6 m Libor twice a year and for a ten years basis swap receiving 3 m Libor plus spread and paying 6 m Libor. It is clear that for a product like the IRS, not subject to the basis dynamic, we have that the big difference among the models is the presence of a stochastic volatility. In fact we can see that the Ch model and the MP model are almost indistinguishable while the results of the HW model are different from the stochastic volatility ones. Moreover we can

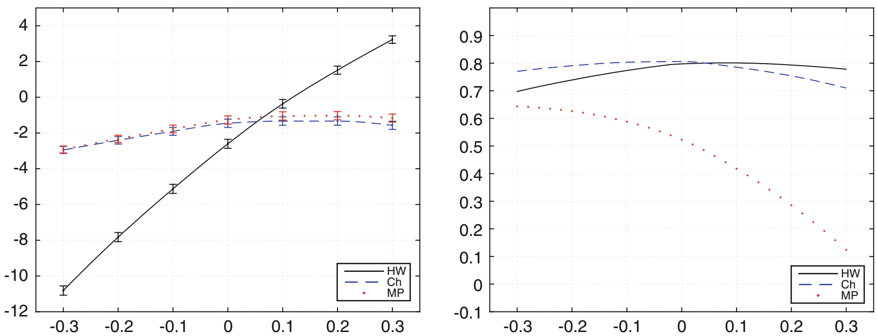


**Fig. 1** Wrong-way risk for different models. On the horizontal axis correlation among credit and market risks; on the vertical axis the bilateral adjustment, namely CVA + DVA, in basis points. *Left panel* a 10y IRS receiving a fix rate and paying 6 m Libor. *Right panel* a 10y basis swap receiving 3 m Libor plus spread and paying 6 m Libor. Montecarlo error is displayed where significant

observe that all the models have the same trend, i.e. the bilateral adjustment grows as correlation increase. In fact this can be explained by the fact that a higher correlation means that the deal will be more profitable when it will be more risky (since we are receiving the fixed rate and paying the floating one), hence the bilateral adjustment will be bigger.

In the case of a basis swap instead, we see that, as said before, the HW model and the Ch model do not have a basis dynamic and hence the curve represented is almost flat. On the other hand the MP model is able to capture the dynamics of the basis and hence we can see that the more the overnight rate is correlated with the credit risk the smaller the bilateral adjustment becomes.

We conclude by pointing out that our analysis will be extended to partially collateralized deals in future work. In such a context funding costs enter the picture in a more comprehensive way. Some initial suggestions in this respect were given in [24].

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# A Generalized Intensity-Based Framework for Single-Name Credit Risk

**Frank Gehmlich and Thorsten Schmidt**

**Abstract** The intensity of a default time is obtained by assuming that the default indicator process has an absolutely continuous compensator. Here we drop the assumption of absolute continuity with respect to the Lebesgue measure and only assume that the compensator is absolutely continuous with respect to a general $\sigma$-finite measure. This allows for example to incorporate the Merton-model in the generalized intensity-based framework. We propose a class of generalized Merton models and study absence of arbitrage by a suitable modification of the forward rate approach of Heath–Jarrow–Morton (1992). Finally, we study affine term structure models which fit in this class. They exhibit stochastic discontinuities in contrast to the affine models previously studied in the literature.

**Keywords** Credit risk · HJM · Forward-rate · Structural approach · Reduced-form approach · Stochastic discontinuities

## 1 Introduction

The two most common approaches to credit risk modeling are the *structural* approach, pioneered in the seminal work of Merton [23], and the *reduced-form* approach which can be traced back to early works of Jarrow, Lando, and Turnbull [18, 22] and to [1].

Default of a company happens when the company is not able to meet its obligations. In many cases the debt structure of a company is known to the public, such that default happens with positive probability at times which are known a priori. This, however, is excluded in the intensity-based framework and it is the purpose of this article to put forward a generalization which allows to incorporate such effects. Examples in the literature are, e.g., structural models like [13, 14, 23]. The recently

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missed coupon payment by Argentina is an example for such a credit event as well as the default of Greece on the 1st of July 2015.[1]

It is a remarkable observation of [2] that it is possible to extend the reduced-form approach beyond the class of intensity-based models. The authors study a class of first-passage time models under a filtration generated by a Brownian motion and show its use for pricing and modeling credit risky bonds. Our goal is to start with even weaker assumptions on the default time and to allow for jumps in the compensator of the default time at deterministic times. From this general viewpoint it turns out, surprisingly, that previously used HJM approaches lead to arbitrage: the whole term structure is absolutely continuous and cannot compensate for points in time bearing a positive default probability. We propose a suitable extension with an additional term allowing for discontinuities in the term structure at certain random times and derive precise drift conditions for an appropriate no-arbitrage condition. The related article [12] only allows for the special case of finitely many risky times, an assumption which is dropped in this article.

The structure of this article is as follows: in Sect. 2, we introduce the general setting and study drift conditions in an extended HJM-framework which guarantee absence of arbitrage in the bond market. In Sect. 3 we study a class of affine models which are stochastically discontinuous. Section 4 concludes.

## 2 A General Account on Credit Risky Bond Markets

Consider a filtered probability space $(\Omega, \mathscr{A}, \mathbb{G}, P)$ with a filtration $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ (the *general* filtration) satisfying the usual conditions, i.e. it is right-continuous and $\mathscr{G}_0$ contains the $P$-null sets $N_0$ of $\mathscr{A}$. Throughout, the probability measure $P$ denotes the objective measure. As we use tools from stochastic analysis, all appearing filtrations shall satisfy the usual conditions. We follow the notation from [17] and refer to this work for details on stochastic processes which are not laid out here.

The filtration $\mathbb{G}$ contains all available information in the market. The default of a company is public information and we therefore assume that the default time $\tau$ is a $\mathbb{G}$-stopping time. We denote the *default indicator process* $H$ by

$$H_t = 1_{\{t\geq\tau\}}, \quad t \geq 0,$$

such that $H_t = 1_{[\![\tau,\infty[\![}(t)$ is a right-continuous, increasing process. We will also make use of the *survival process* $1 - H = 1_{[\![0,\tau[\![}$. The following remark recalls the essentials of the well-known intensity-based approach.

[1] Argentina's missed coupon payment on \$29 billion debt was voted a credit event by the International Swaps and Derivatives Association, see the announcements in [16, 24]. Regarding the failure of 1.5 Billion EUR of Greece on a scheduled debt repayment to the International Monetary fund, see e.g. [9].

*Remark 1* (*The intensity-based approach*) The intensity-based approach consists of two steps: first, denote by $\mathbb{H} = (\mathscr{H}_t)_{t\geq 0}$ the filtration generated by the default indicator, $\mathscr{H}_t = \sigma(H_s : 0 \leq s \leq t) \vee N_0$, and assume that there exists a sub-filtration $\mathbb{F}$ of $\mathbb{G}$, i.e. $\mathscr{F}_t \subset \mathscr{G}_t$ holds for all $t \geq 0$ such that

$$\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t, \quad t \geq 0. \tag{1}$$

Viewed from this perspective, $\mathbb{G}$ is obtained from the default information $\mathbb{H}$ by a *progressive enlargement*[2] with the filtration $\mathbb{F}$. This assumption opens the area for the largely developed field of enlargements of filtration with a lot of powerful and quite general results.

Second, the following key assumption specifies the default intensity: assume that there is an $\mathbb{F}$-progressive process $\lambda$, such that

$$P(\tau > t|\mathscr{F}_t) = \exp\Big(-\int_0^t \lambda_s ds\Big), \quad t \geq 0. \tag{2}$$

It is immediate that the inclusion $\mathscr{F}_t \subset \mathscr{G}_t$ is strict under existence of an intensity, i.e. $\tau$ is not an $\mathbb{F}$-stopping time. Arbitrage-free pricing can be achieved via the following result: Let $Y$ be a non-negative random variable. Then, for all $t \geq 0$,

$$E[1_{\{\tau>t\}}Y|\mathscr{G}_t] = 1_{\{\tau>t\}}e^{\int_0^t \lambda_s ds}E[1_{\{\tau>t\}}Y|\mathscr{F}_t].$$

Of course, this result holds also when a pricing measure $Q$ is used instead of $P$. For further literature and details we refer for example to [11], Chap. 12, and to [3].

## 2.1 The Generalized Intensity-Based Framework

The default indicator process $H$ is a bounded, cádlág, and increasing process, hence a submartingale of class (D), that is, the family $(X_T)$ over all stopping times $T$ is uniformly integrable. By the Doob–Meyer decomposition,[3] the process

$$M_t = H_t - \Lambda_t, \quad t \geq 0 \tag{3}$$

is a true martingale where $\Lambda$ denotes the dual $\mathbb{F}$-predictable projection, also called compensator, of $H$. As 1 is an absorbing state, $\Lambda_t = \Lambda_{t\wedge\tau}$. To keep the arising technical difficulties at a minimum, we assume that there is an increasing process $A$ such that

[2] Note that here $\mathbb{G}$ is right-continuous and $P$-complete by assumption which is a priori not guaranteed by (1). One can, however, use the right-continuous extension and we refer to [15] for a precise treatment and for a guide to the related literature.

[3] See [20], Theorem 1.4.10.

$$\Lambda_t = \int_0^{t \wedge \tau} \lambda_s dA(s), \quad t \geq 0, \tag{4}$$

with a non-negative and predictable process $\lambda$. The process $\lambda$ is called *generalized intensity* and we refer to Chap. VIII.4 of [5] for a more detailed treatment of generalized intensities (or, equivalently, dual predictable projections) in the context of point processes.

Note that with $\Delta M \leq 1$ we have that $\Delta \Lambda = \lambda_s \Delta A(s) \leq 1$. Whenever $\lambda_s \Delta A(s) > 0$, there is a positive probability that the company defaults at time $s$. We call such times *risky times*, i.e. predictable times having a positive probability of a default occurring right at that time. Note that under our assumption (4), all risky times are deterministic. The relationship between $\Delta \Lambda(s)$ and the default probability at time $s$ will be clarified in Example 3.

## 2.2 An Extension of the HJM Approach

A credit risky bond with maturity $T$ is a contingent claim promising to pay one unit of currency at $T$. The price of the bond with maturity $T$ at time $t \leq T$ is denoted by $P(t, T)$. If no default occurred prior to or at $T$ we have that $P(T, T) = 1$. We will consider zero recovery, i.e. the bond loses its total value at default, such that $P(t, T) = 0$ on $\{t \geq \tau\}$. The family of stochastic processes $\{(P(t, T)_{0 \leq t \leq T}), T \geq 0\}$ describes the evolution of the *term structure* $T \mapsto P(., T)$ over time.

Besides the bonds there is a *numéraire* $X^0$, which is a strictly positive, adapted process. We make the weak assumption that $\log X^0$ is absolutely continuous, i.e. $X_t^0 = \exp(\int_0^t r_s ds)$ with a progressively measurable process $r$, called the short rate. For practical applications one would use the overnight index swap (OIS) rate for constructing such a numéraire.

The aim of the following is to extend the HJM approach in an appropriate way to the generalized intensity-based framework in order to obtain arbitrage-free bond prices. First approaches in this direction were [7, 25] and a rich source of literature is again [3]. Absence of arbitrage in such an infinite dimensional market can be described in terms of no asymptotic free lunch (NAFL) or the more economically meaningful no asymptotic free lunch with vanishing risk, see [6, 21].

Consider a pricing measure $Q^* \sim P$. Our intention is to find conditions which render $Q^*$ an equivalent local martingale measure. In the following, only occasionally the measure $P$ will be used, such that from now on, all appearing terms (like martingales, almost sure properties, etc.) are to be considered with respect to $Q^*$.

To ensure that the subsequent analysis is meaningful, we make the following technical assumption.

**Assumption 2.1** The generalized default intensity $\lambda$ is non-negative, predictable, and $A$-integrable on $[0, T^*]$:

$$\int_0^{T^*} \lambda_s dA(s) < \infty, \quad Q^*\text{-a.s.}$$

Moreover, $A$ has vanishing singular part, i.e.

$$A(t) = t + \sum_{0<s\leq t} \Delta A(s). \tag{5}$$

The representation (5) of $A$ is without loss of generality: indeed, if the continuous part $A^c$ is absolutely continuous, i.e. $A^c(t) = \int_0^t a(s)ds$, replacing $\lambda_s$ by $\lambda_s a(s)$ gives the compensator of $H$ with respect to $\tilde{A}$ whose continuous part is $t$.

Next, we aim at building an arbitrage-free framework for bond prices. In the generalized intensity-based framework, the (HJM) approach does allow for arbitrage opportunities at risky times. We therefore consider the following generalization: consider a $\sigma$-finite (deterministic) measure $\nu$. We could be general on $\nu$, allowing for an absolutely continuous, a singular continuous, and a pure-jump part. However, for simplicity, we leave the singular continuous part aside and assume that

$$\nu = \nu^{ac} + \nu^d$$

where $\nu^{ac}(ds) = ds$ and $\nu^d$ distributes mass only to points, i.e. $\nu^d(A) = \sum_{i\geq 1} w_i \delta_{u_i}(A)$, for $0 < u_1 < u_2 < \cdots$ and positive weights $w_i > 0$, $i \geq 1$; here $\delta_u$ denotes the Dirac measure at $u$. Moreover, we assume that defaultable bond prices are given by

$$\begin{aligned} P(t,T) &= 1_{\{\tau>t\}} \exp\left( -\int_t^T f(t,u)\nu(du)\right) \\ &= 1_{\{\tau>t\}} \exp\left( -\int_t^T f(t,u)du - \sum_{i\geq 1} 1_{\{u_i\in(t,T]\}} w_i f(t,u_i)\right), \quad 0\leq t\leq T\leq T^*. \end{aligned} \tag{6}$$

The sum in the last line gives the extension over the (HJM) approach which allows us to deal with risky times in an arbitrage-free way.

The family of processes $(f(t,T))_{0\leq t\leq T}$ for $T \in [0, T^*]$ are assumed to be Itô processes satisfying

$$f(t,T) = f(0,T) + \int_0^t a(s,T)ds + \int_0^t b(s,T)\cdot dW_s \tag{7}$$

with an $n$-dimensional $Q^*$-Brownian motion $W$.

Denote by $\mathscr{B}$ the Borel $\sigma$-field over $\mathbb{R}$.

**Assumption 2.2** We require the following technical assumptions:

(i) the initial forward curve is measurable, and integrable on $[0, T^*]$:

$$\int_0^{T^*} |f(0, u)| < \infty, \quad Q^*\text{-a.s.},$$

(ii) the *drift parameter* $a(\omega, s, t)$ is $\mathbb{R}$-valued $\mathscr{O} \otimes \mathscr{B}$-measurable and integrable on $[0, T^*]$:

$$\int_0^{T^*} \int_0^{T^*} |a(s, u)| ds\, \nu(du) < \infty, \quad Q^*\text{-a.s.},$$

(iii) the *volatility parameter* $b(\omega, s, t)$ is $\mathbb{R}^n$-valued, $\mathscr{O} \otimes \mathscr{B}$-measurable, and

$$\sup_{s,t \le T^*} \| b(s, t) \| < \infty, \quad Q^*\text{-a.s.}$$

(iv) it holds that

$$0 \le \lambda(u_i) \Delta A(u_i) < w_i, \quad i \ge 1.$$

Set

$$\begin{aligned}
\bar{a}(t, T) &= \int_t^T a(t, u)\nu(du), \\
\bar{b}(t, T) &= \int_t^T b(t, u)\nu(du), \\
H'(t) &= \int_0^t \lambda_s ds - \sum_{u_i \le t} \log\Big(\frac{w_i - \lambda_{u_i} \Delta A(u_i)}{w_i}\Big).
\end{aligned} \tag{8}$$

The following proposition gives the desired drift condition in the generalized Merton models.

**Theorem 1** *Assume that Assumptions 2.1 and 2.2 hold. Then $Q^*$ is an ELMM if and only if the following conditions hold: $\{s : \Delta A(s) \neq 0\} \subset \{u_1, u_2, \dots\}$, and*

$$\int_0^t f(s, s)\nu(ds) = \int_0^t r_s ds + H'(t), \tag{9}$$

$$\bar{a}(t, T) = \frac{1}{2} \| \bar{b}(t, T) \|^2, \tag{10}$$

*for $0 \le t \le T \le T^*$ $dQ^* \otimes dt$-almost surely on $\{t < \tau\}$.*

The first condition, (9), can be split in the continuous and pure-jump part, such that (9) is equivalent to

$$f(t,t) = r_s + \lambda_s$$
$$f(t,u_i) = \log \frac{w_i}{w_i - \lambda(u_i)\Delta A(u_i)} \geq 0.$$

The second relation states explicitly the connection of the forward rate at a risky time $u_i$ to the probability $Q^*(\tau = u_i | \mathscr{F}_{u_i-})$, given that $\tau \geq u_i$, of course. It simplifies moreover, if $\Delta A(u_i) = w_i$ to

$$f(t,u_i) = -\log(1 - \lambda(u_i)). \tag{11}$$

For the proof we first provide the canonical decomposition of

$$J(t,T) := \int_t^T f(t,u)\nu(du), \quad 0 \leq t \leq T.$$

**Lemma 1** *Assume that Assumption 2.2 holds. Then, for each $T \in [0,T^*]$ the process $(J(t,T))_{0\leq t\leq T}$ is a special semimartingale and*

$$J(t,T) = \int_0^T f(0,u)\nu(du) + \int_0^t \bar{a}(u,T)du + \int_0^t \bar{b}(u,T)dW_u - \int_0^t f(u,u)\nu(du).$$

*Proof* Using the stochastic Fubini Theorem (as in [26]), we obtain

$$\begin{aligned}
J(t,T) &= \int_t^T \left( f(0,u) + \int_0^t a(s,u)ds + \int_0^t b(s,u)dW_s \right) \nu(du) \\
&= \int_0^T f(0,u)\nu(du) + \int_0^t \int_s^T a(s,u)\nu(du)ds + \int_0^t \int_s^T b(s,u)\nu(du)dW_s \\
&\quad - \int_0^t f(0,u)\nu(du) - \int_0^t \int_s^t a(s,u)\nu(du)ds - \int_0^t \int_s^t b(s,u)\nu(du)dW_s \\
&= \int_0^T f(0,u)\nu(du) + \int_0^t \bar{a}(s,T)ds + \int_0^t \bar{b}(s,T)dW_s \\
&\quad - \int_0^t \left( f(0,u) - \int_0^u a(s,u)ds - \int_0^u b(s,u)dW_s \right) \nu(du),
\end{aligned}$$

and the claim follows.

*Proof* (*Proof of Theorem 1*) Set, $E(t) = 1_{\{\tau>t\}}$, and $F(t,T) = \exp\left(-\int_t^T f(t,u) \nu(du)\right)$, such that $P(t,T) = E(t)F(t,T)$. Integration by parts yields that

$$dP(t,T) = F(t-,T)dE(t) + E(t-)dF(t,T) + d[E,F(.,T)]_t =: (1') + (2') + (3'). \tag{12}$$

In view of (1′), we obtain from (4), that

$$E(t) + \int_0^{t\wedge\tau} \lambda_s dA(s) =: M_t^1 \tag{13}$$

is a martingale. Regarding (2′), note that from Lemma 1 we obtain by Itô's formula that

$$\begin{aligned}\frac{dF(t,T)}{F(t-,T)} &= \Big(f(t,t) - \bar{a}(t,T) + \frac{1}{2} \parallel \bar{b}(t,T) \parallel^2\Big) dt \\ &\quad + \sum_{i\geq 0} \big(e^{f(t,t)} - 1\big)\, w_i \delta_{u_i}(dt) + dM_t^2,\end{aligned} \tag{14}$$

with a local martingale $M^2$. For the remaining term (3′), note that

$$\begin{aligned}\sum_{0<s\leq t} \Delta E(s)\Delta F(s,T) &= \int_0^t F(s-,T)(e^{f(s,s)} - 1)\nu(\{s\})dE(s) \\ &= \int_0^t F(s-,T)(e^{f(s,s)} - 1)\nu(\{s\})dM_s^1 \\ &\quad - \int_0^{t\wedge\tau} F(s-,T)(e^{f(s,s)} - 1)\nu(\{s\})\lambda_s dA(s).\end{aligned} \tag{15}$$

Inserting (14) and (15) into (12) we obtain

$$\begin{aligned}\frac{dP(t,T)}{P(t-,T)} &= -\lambda_t dA(t) \\ &\quad + \Big(f(t,t) - \bar{a}(t,T) + \frac{1}{2} \parallel \bar{b}(t,T) \parallel^2\Big) dt \\ &\quad + \sum_{i\geq 0} \big(e^{f(t,t)} - 1\big)\, w_i \delta_{u_i}(dt) \\ &\quad - \int_{\mathbb{R}} \nu(\{t\})(e^{f(t,t)} - 1)\lambda_t dA(t) + dM_t^3\end{aligned}$$

with a local martingale $M^3$. We obtain a $Q^*$-local martingale if and only if the drift vanishes. Next, we can separate between absolutely continuous and discrete part. The absolutely continuous part yields (10) and $f(t,t) = r_t + \lambda_t \ dQ^* \otimes dt$-almost surely. It remains to compute the discontinuous part, which is given by

$$\sum_{i:u_i\leq t} P(u_i-,T)(e^{f(u_i,u_i)} - 1)w_i - \sum_{0<s\leq t} P(s-,T)e^{f(s,s)}\lambda_s \Delta A(s),$$

for $0 \le t \le T \le T^*$. This yields $\{s : \Delta A(s) \ne 0\} \subset \{u_1, u_2, \dots\}$. The discontinuous part vanishes if and only if

$$1_{\{u_i \le T^* \wedge \tau\}} e^{-f(u_i,u_i)} w_i = 1_{\{u_i \le T^* \wedge \tau\}} \Big( w_i - \lambda_{u_i} \Delta A(u_i) \Big), \quad i \ge 1,$$

which is equivalent to

$$1_{\{u_i \le T^* \wedge \tau\}} f(u_i, u_i) = - 1_{\{u_i \le T^* \wedge \tau\}} \log \frac{w_i - \lambda_{u_i} \Delta A(u_i)}{w_i}, \quad i \ge 1.$$

We obtain (9) and the claim follows.

*Example 1* (*The Merton model*) The paper [23] considers a simple capital structure of a firm, consisting only of equity and a zero-coupon bond with maturity $U > 0$. The firm defaults at $U$ if the total market value of its assets is not sufficient to cover the liabilities.

We are interested in setting up an arbitrage-free market for credit derivatives and consider a market of defaultable bonds $P(t, T)$, $0 \le t \le T \le T^*$ with $0 < U \le T^*$ as basis for more complex derivatives. In a stylized form the Merton model can be represented by a Brownian motion $W$ denoting the normalized logarithm of the firm's assets, a constant $K > 0$ and the default time

$$\tau = \begin{cases} U & \text{if } W_U \le K \\ \infty & \text{otherwise.} \end{cases}$$

Assume for simplicity a constant interest rate $r$ and let $\mathbb{F}$ be the filtration generated by $W$. Then $P(t, T) = e^{-r(T-t)}$ whenever $T < U$ because these bonds do not carry default risk. On the other hand, for $t < U \le T$,

$$P(t, T) = e^{-r(T-t)} E^*[1_{\{\tau > T\}} | \mathscr{F}_t] = e^{-r(T-t)} E^*[1_{\{\tau = \infty\}} | \mathscr{F}_t] = e^{-r(T-t)} \Phi\Big( \frac{W_t - K}{\sqrt{U - t}} \Big),$$

where $\Phi$ denotes the cumulative distribution function of a standard normal random variable and $E^*$ denotes the expectation with respect to $Q^*$. For $t \to U$ we recover $P(U, U) = 1_{\{\tau = \infty\}}$. The derivation of representation (6) with $\nu(du) := du + \delta_U(du)$ is straightforward. A simple calculation with

$$P(t, T) = 1_{\{\tau > t\}} \exp\Big( - \int_t^T f(t, u) du - f(t, U) 1_{\{t < U \le T\}} \Big) \tag{16}$$

yields $f(t, T) = r$ for $T \ne U$ and

$$f(t, U) = - \log \Phi\Big( \frac{W_t - K}{\sqrt{U - t}} \Big).$$

By Itô's formula we obtain

$$b(t, U) = -\frac{\varphi\left(\frac{W_t - K}{\sqrt{U-t}}\right)}{\Phi\left(\frac{W_t - K}{\sqrt{U-t}}\right)}(U-t)^{-1/2},$$

and indeed, $a(t, U) = \frac{1}{2}b^2(t, U)$. Note that the conditions for Proposition 1 hold and, the market consisting of the bonds $P(t, T)$ satisfies NAFL, as expected. More flexible models of arbitrage-free bond prices can be obtained if the market filtration $\mathbb{F}$ is allowed to be more general, as we show in Sect. 3 on affine generalized Merton models.

*Example 2* (*An extension of the Black–Cox model*) The model suggested in [4] uses a first-passage time approach to model credit risk. Default happens at the first time, when the firm value falls below a pre-specified boundary, the default boundary. We consider a stylized version of this approach and continue the Example 1. Extending the original approach, we include a zero-coupon bond with maturity $U$. The reduction of the firm value at $U$ is equivalent to considering a default boundary with an upward jump at that time. Hence, we consider a Brownian motion $W$ and the default boundary

$$D(t) = D(0) + K 1_{\{U \geq t\}}, \quad t \geq 0,$$

with $D(0) < 0$, and let default be the first time when $W$ hits $D$, i.e.

$$\tau = \inf\{t \geq 0 : W_t \leq D(t)\}$$

with the usual convention that $\inf\ \emptyset = \infty$. The following lemma computes the default probability in this setting and the forward rates are directly obtained from this result together with (16). The filtration $\mathbb{G} = \mathbb{F}$ is given by the natural filtration of the Brownian motion $W$ after completion. Denote the random sets

$$\Delta_1 := \left\{(x, y) \in \mathbb{R}^2 : x\sqrt{T-U} \leq D(U) - \left(y\sqrt{U-t} + W_t\right), y\sqrt{U-t} + W_t > D(0)\right\}$$

$$\Delta_2 := \left\{(x, y) \in \mathbb{R}^2 : x\sqrt{T-U} \leq D(U) - \left(y\sqrt{U-t} + 2D(0) - W_t\right), \right.$$
$$\left. y\sqrt{U-t} + D(0) - W_t > 0\right\}.$$

**Lemma 2** *Let $D(0) < 0$, $U > 0$ and $D(U) \geq D(0)$. For $0 \leq t < U$, it holds on $\{\tau > t\}$, that*

$$P(\tau > T|\mathscr{F}_t) = 1 - 2\Phi\left(\frac{D(0) - W_t}{\sqrt{T-t}}\right) - 1_{\{T \geq U\}}2(\Phi_2(\Delta_1) - \Phi_2(\Delta_2)), \quad (17)$$

*where $\Phi_2$ is the distribution of a two-dimensional standard normal distribution and the sets $\Delta_t = \Delta_t(D),\ t \geq U$ are given by*

$$\Delta_t = \Big\{(x,y) \in \mathbb{R}^2 : x\sqrt{T-U} + y\sqrt{U} \leq -D(U), \Big\}.$$

*For $t \geq U$ it holds on $\{\tau > t\}$, that*

$$P(\tau > T|\mathscr{F}_t) = 1 - 2\Phi\bigg(\frac{D(U) - W_t}{\sqrt{T-t}}\bigg).$$

*Proof* The first part of (17) where $T < U$ follows directly from the reflection principle and the property that $W$ has independent and stationary increments. Next, consider $0 \leq t < U \leq T$. Then, on $\{W_U > D(U)\}$,

$$P(\inf_{[U,T]} W > D(U)|\mathscr{F}_U) = 1 - 2\Phi\bigg(\frac{D(U) - W_U}{\sqrt{T-U}}\bigg). \tag{18}$$

Moreover, on $\{W_t > D(0)\}$ it holds for $x > D(0)$ that

$$\begin{aligned} P(\inf_{[0,U]} W > D(0), W_U > x|\mathscr{F}_t) &= P(W_U > x|\mathscr{F}_t) - P(W_U < x, \inf_{[0,U]} W \leq D(0)|\mathscr{F}_t) \\ &= \Phi\bigg(\frac{W_t - x}{\sqrt{U-t}}\bigg) - \Phi\bigg(\frac{2D(0) - x - W_t}{\sqrt{U-t}}\bigg). \end{aligned}$$

Hence, $E[g(W_U)1_{\{\inf_{[0,U]} W > D(0)\}}|\mathscr{F}_t] = 1_{\{\inf_{[0,t]} W > D(0)\}} \int_{D(0)}^{\infty} g(x) f_t(x) dx$ with density

$$f_t(x) = 1_{\{x > D(0)\}} \frac{1}{\sqrt{U-t}} \bigg[\phi\bigg(\frac{W_t - x}{\sqrt{U-t}}\bigg) - \phi\bigg(\frac{2D(0) - x - W_t}{\sqrt{U-t}}\bigg)\bigg].$$

Together with (18) this yields on $\{\inf_{[0,t]} W > D(0)\}$

$$\begin{aligned} P(\inf_{[0,T]}(W - D) > 0|\mathscr{F}_t) &= \int_{D(0)}^{\infty} \bigg[1 - 2\Phi\bigg(\frac{D(U) - x}{\sqrt{T-U}}\bigg)\bigg] f_t(x) dx \\ &= P(\inf_{[t,T]} W > D(0)|\mathscr{F}_t) - 2\int_{D(0)}^{\infty} \Phi\bigg(\frac{D(U) - x}{\sqrt{T-U}}\bigg) f_t(x) dx. \end{aligned}$$

It remains to compute the integral. Regarding the first part, letting $\xi$ and $\eta$ be independent and standard normal, we obtain that

$$\begin{aligned} &\int_{D(0)}^{\infty} \Phi\bigg(\frac{D(U) - x}{\sqrt{T-U}}\bigg) \frac{1}{\sqrt{U-t}} \phi\Big(\frac{x - W_t}{\sqrt{U-t}}\Big) dx \\ &= P_t\Big(\sqrt{T-U}\xi \leq D(U) - (\sqrt{U-t}\eta + W_t), \sqrt{U-t}\eta + W_t > D(0)\Big) \\ &= \Phi_2(\Delta_1), \end{aligned}$$

where we abbreviate $P_t(\cdot) = P(\cdot|\mathscr{F}_t)$. In a similar way,

$$\int_{D(0)}^{\infty} \Phi\Big(\frac{D(U)-x}{\sqrt{T-U}}\Big)\frac{1}{\sqrt{U-t}}\phi\Big(\frac{x-(2D(0)-W_t)}{\sqrt{U-t}}\Big)dx$$
$$= P_t\Big(\sqrt{T-U}\xi \le D(U) - (\sqrt{U-t}\eta + 2D(0) - W_t), \sqrt{U-t}\eta + D(0) - W_t > 0\Big)$$
$$= \Phi_2(\Delta_2)$$

and we conclude.

## 3 Affine Models in the Generalized Intensity-Based Framework

Affine processes are a well-known tool in the financial literature and one reason for this is their analytical tractability. In this section we closely follow [12] and shortly state the appropriate affine models which fit the generalized intensity framework. For proofs, we refer the reader to this paper.

The main point is that affine processes in the literature are assumed to be *stochastically continuous* (see [8, 10]). Due to the discontinuities introduced in the generalized intensity-based framework, we propose to consider *piecewise continuous affine processes*.

*Example 3* Consider a non-negative integrable function $\lambda$, a constant $\lambda' \ge 0$ and a deterministic time $u > 0$. Set

$$K(t) = \int_0^t \lambda(s)ds + 1_{\{t \ge u\}}\kappa, \quad t \ge 0.$$

Let the default time $\tau$ be given by $\tau = \inf\{t \ge 0 : K_t \ge \zeta\}$ with a standard exponential-random variable $\zeta$. Then $P(\tau = u) = 1 - e^{-\kappa} =: \lambda'$. Considering $\nu(ds) = ds + \delta_u(ds)$ with $u_1 = u$ and $w_1 = 1$, we are in the setup of the previous section. The drift condition (9) holds, if

$$f(u,u) = -\log(1-\lambda') = \kappa.$$

Note, however, that $K$ is not the compensator of $H$. Indeed, the compensator of $H$ equals $\Lambda_t = \int_0^{t\wedge\tau} \lambda(s)ds + 1_{\{t\ge u\}}\lambda'$, see [19] for general results in this direction.

The purpose of this section is to give a suitable extension of the above example involving affine processes. Recall that we consider a $\sigma$-finite measure

$$\nu(du) = du + \sum_{i\ge 1} w_i \delta_{u_i}(du),$$

as well as $A(u) = u + \sum_{i\geq 1} 1_{\{u\geq u_i\}}$. The idea is to consider an affine process $X$ and study arbitrage-free doubly stochastic term structure models where the compensator $\Lambda$ of the default indicator process $H = 1_{\{\cdot\leq\tau\}}$ is given by

$$\Lambda_t = \int_0^t \Big(\phi_0(s) + \psi_0(s)^\top \cdot X_s\Big)ds + \sum_{i\geq 1} 1_{\{t\geq u_i\}}\Big(1 - e^{-\phi_i - \psi_i^\top \cdot X_{u_i}}\Big). \tag{19}$$

Note that by continuity of $X$, $\Lambda_t(\omega) < \infty$ for almost all $\omega$. To ensure that $\Lambda$ is non-decreasing we will require that $\phi_0(s) + \psi_0(s)^\top \cdot X_s \geq 0$ for all $s \geq 0$ and $\phi_i + \psi_i^\top \cdot X_{u_i} \geq 0$ for all $i \geq 1$.

Consider a state space in canonical form $\mathscr{X} = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ for integers $m, n \geq 0$ with $m + n = d$ and a $d$-dimensional Brownian motion $W$. Let $\mu$ and $\sigma$ be defined on $\mathscr{X}$ by

$$\mu(x) = \mu_0 + \sum_{i=1}^d x_i \mu_i, \tag{20}$$

$$\frac{1}{2}\sigma(x)^\top \sigma(x) = \sigma_0 + \sum_{i=1}^d x_i \sigma_i, \tag{21}$$

where $\mu_0, \mu_i \in \mathbb{R}^d$, $\sigma_0, \sigma_i \in \mathbb{R}^{d\times d}$, for all $i \in \{1, \dots, d\}$. We assume that the parameters $\mu^i$, $\sigma^i$, $i = 0, \dots, d$ are admissible in the sense of Theorem 10.2 in [11]. Then the continuous, unique strong solution of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{22}$$

is an *affine* process $X$ on the state space $\mathscr{X}$, see Chap. 10 in [11] for a detailed exposition.

We call a bond-price model *affine* if there exist functions $A : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $B : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ such that

$$P(t, T) = 1_{\{\tau > t\}} e^{-A(t,T) - B(t,T)^\top \cdot X_t}, \tag{23}$$

for $0 \leq t \leq T \leq T^*$. We assume that $A(., T)$ and $B(., T)$ are right-continuous. Moreover, we assume that $t \mapsto A(t, .)$ and $t \mapsto B(t, .)$ are differentiable from the right and denote by $\partial_t^+$ the right derivative. For the convenience of the reader we state the following proposition giving sufficient conditions for absence of arbitrage in an affine generalized intensity-based setting. It extends [12] where only finitely many risky times were treated.

**Proposition 1** *Assume that* $\phi_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\psi_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ *are continuous,* $\psi_0(s) + \psi_0(s)^\top \cdot x \geq 0$ *for all* $s \geq 0$ *and* $x \in \mathscr{X}$ *and the constants* $\phi_i \in \mathbb{R}$ *and* $\psi_i \in \mathbb{R}^d$, $i \geq 1$ *satisfy* $\phi_i + \psi_i^\top \cdot x \geq 0$ *for all* $1 \leq i \leq n$ *and* $x \in \mathscr{X}$ *as well as* $\sum_{i\geq 1} |w_i|(|\phi_i| + |\psi_{i,1}| + \dots + |\psi_{i,d}|) < \infty$. *Moreover, let the functions* $A : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and*

$B : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ *be the unique solutions of*

$$\begin{aligned} A(T, T) &= 0 \\ A(u_i, T) &= A(u_i-, T) - \phi_i w_i \\ -\partial_t^+ A(t, T) &= \phi_0(t) + \mu_0^\top \cdot B(t, T) - B(t, T)^\top \cdot \sigma_0 \cdot B(t, T), \end{aligned} \tag{24}$$

*and*

$$\begin{aligned} B(T, T) &= 0 \\ B_k(u_i, T) &= B_k(u_i-, T) - \psi_{i,k} w_i \\ -\partial_t^+ B_k(t, T) &= \psi_{0,k}(t) + \mu_k^\top \cdot B(t, T) - B(t, T)^\top \cdot \sigma_k \cdot B(t, T), \end{aligned} \tag{25}$$

*for* $0 \leq t \leq T$. *Then, the doubly-stochastic affine model given by* (19) *and* (23) *satisfies NAFL.*

*Proof* By construction,

$$\begin{aligned} A(t, T) &= \int_t^T a'(t, u)du + \sum_{i: u_i \in (t,T]} \phi_i w_i \\ B(t, T) &= \int_t^T b'(t, u)du + \sum_{i: u_i \in (t,T]} \psi_i w_i \end{aligned}$$

with suitable functions $a'$ and $b'$ and $a'(t, t) = \phi_0(t)$ as well as $b'(t, t) = \psi_0(t)$. A comparison of (23) with (6) yields the following: on the one hand, for $T = u_i \in \mathscr{U}$, we obtain $f(t, u_i) = \phi_i + \psi_i^\top \cdot X_t$. Hence, the coefficients $a(t, T)$ and $b(t, T)$ in (7) for $T = u_i \in \mathscr{U}$ compute to $a(t, u_i) = \psi_i^\top \cdot \mu(X_t)$ and $b(t, u_i) = \psi_i^\top \cdot \sigma(X_t)$.

On the other hand, for $T \notin \mathscr{U}$ we obtain that $f(t, T) = a'(t, T) + b'(t, T)^\top \cdot X_t$. Then, the coefficients $a(t, T)$ and $b(t, T)$ can be computed as follows: applying Itô's formula to $f(t, T)$ and comparing with (7) yields that

$$\begin{aligned} a(t, T) &= \partial_t a'(t, T) + \partial_t b'(t, T)^\top \cdot X_t + b'(t, T)^\top \cdot \mu(X_t) \\ b(t, T) &= b'(t, T)^\top \cdot \sigma(X_t). \end{aligned} \tag{26}$$

Set $\bar{a}'(t, T) = \int_t^T a'(t, u)du$ and $\bar{b}'(t, T) = \int_t^T b'(t, u)du$ and note that,

$$\int_t^T \partial_t a'(t, u)du = \partial_t \bar{a}'(t, T) + a'(t, t).$$

As $\partial_t^+ A(t, T) = \partial_t \bar{a}'(t, T)$, and $\partial_t^+ B(t, T) = \partial_t \bar{b}'(t, T)$, we obtain from (26) that

$$\begin{aligned}\bar{a}(t,T) &= \int_t^T a(t,u)\nu(du) = \int_t^T a(t,u)du + \sum_{u_i\in(t,T]} w_i\psi_i^\top \cdot \mu(X_t)\\ &= \partial_t^+ A(t,T) + a'(t,t) + \big(\partial_t^+ B(t,T) + b'(t,t)\big)^\top \cdot X_t + B(t,T)^\top \cdot \mu(X_t),\\ \bar{b}(t,T) &= \int_t^T b(t,u)\nu(du) = \int_t^T b(t,u)du + \sum_{u_i\in(t,T]} w_i\psi_i^\top \cdot \sigma(X_t)\\ &= B(t,T)^\top \cdot \sigma(X_t)\end{aligned}$$

for $0 \le t \le T \le T^*$. We now show that under our assumptions, the drift conditions (9) and (10) hold: Observe that, by Eqs. (24), (25), and the affine specification (20), and (21), the drift condition (10) holds. Moreover, from (11),

$$\Delta H'(u_i) = \phi_i + \psi_i^\top \cdot X_{u_i}$$

and $\lambda_s = \phi_0(s) + \psi_0(s)^\top \cdot X_s$ by (19). We recover $\Delta\Lambda_{u_i} = 1 - \exp(-\phi_i - \psi_i^\top \cdot X_{u_i})$ taking values in $[0, 1)$ by assumption. Hence, (9) holds and the claim follows.

*Example 4* In the one-dimensional case we consider $X$, given as solution of

$$dX_t = (\mu_0 + \mu_1 X_t)dt + \sigma\sqrt{X_t}dW_t, \quad t \ge 0.$$

Consider only one risky time $u_1 = 1$ and let $\phi_0 = \phi_1 = 0$, $\psi_0 = 1$, such that

$$\Lambda = \int_0^t X_s ds + 1_{\{u\ge 1\}}(1 - e^{-\psi_1 X_1}).$$

Hence the probability of having no default at time 1 just prior to 1 is given by $e^{-\psi_1 X_1}$, compare Example 3.

An arbitrage-free model can be obtained by choosing $A$ and $B$ according to Proposition 1 which can be immediately achieved using Lemma 10.12 from [11] (see in particular Sect. 10.3.2.2 on the CIR short-rate model): denote $\theta = \sqrt{\mu_1^2 + 2\sigma^2}$ and

$$\begin{aligned}L_1(t) &= 2(e^{\theta t} - 1),\\ L_2(t) &= \theta(e^{\theta t} + 1) + \mu_1(e^{\theta t} - 1),\\ L_3(t) &= \theta(e^{\theta t} + 1) - \mu_1(e^{\theta t} - 1),\\ L_4(t) &= \sigma^2(e^{\theta t} - 1).\end{aligned}$$

Then

$$A_0(s) = \frac{2\mu_0}{\sigma^2}\log\left(\frac{2\theta e^{\frac{(\sigma-\mu_1)t}{2}}}{L_3(t)}\right), \quad B_0(s) = -\frac{L_1(t)}{L_3(t)}$$

are the unique solutions of the Riccati equations $B_0' = \sigma^2 B_0^2 - \mu_1 B_0$ with boundary condition $B_0(0) = 0$ and $A_0' = -\mu_0 B_0$ with boundary condition $A_0(0) = 0$. Note that with $A(t,T) = A_0(T-t)$ and $B(t,T) = B_0(T-t)$ for $0 \le t \le T < 1$, the conditions of Proposition 1 hold. Similarly, for $1 \le t \le T$, choosing $A(t,T) = A_0(T-t)$ and $B(t,T) = B_0(T-t)$ implies again the validity of (24) and (25). On the other hand, for $0 \le t < 1$ and $T \ge 1$ we set $u(T) = B(1,T) + \psi_1 = B_0(T-1) + \psi_1$, according to (25), and let

$$A(t,T) = \frac{2\mu_0}{\sigma^2} \log\left( \frac{2\theta e^{\frac{(\sigma-\mu_1)(1-t)}{2}}}{L_3(1-t) - L_4(1-t)u(T)} \right)$$

$$B(t,T) = -\frac{L_1(1-t) - L_2(1-t)u(T)}{L_3(1-t) - L_4(1-t)u(T)}.$$

It is easy to see that (24) and (25) are also satisfied in this case, in particular $\Delta A(1,T) = -\phi_1 = 0$ and $\Delta B(1,T) = -\psi_1$. Note that, while $X$ is continuous, the bond prices are not even stochastically continuous because they jump almost surely at $u_1 = 1$. We conclude by Proposition 1 that this affine model is arbitrage-free. ◇

## 4 Conclusion

In this article we studied a new class of dynamic term structure models with credit risk where the compensator of the default time may jump at predictable times. This framework was called generalized intensity-based framework. It extends existing theory and allows to include Merton's model, in a reduced-form model for pricing credit derivatives. Finally, we studied a class of highly tractable affine models which are only piecewise stochastically continuous.

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# Option Pricing and Sensitivity Analysis in the Lévy Forward Process Model

**Ernst Eberlein, M'hamed Eddahbi and Sidi Mohamed Lalaoui Ben Cherif**

**Abstract** The purpose of this article is to give a closed Fourier-based valuation formula for a caplet in the framework of the Lévy forward process model which was introduced in Eberlein and Özkan, Financ. Stochast. 9:327-348, 2005, [5]. Afterwards, we compute Greeks by two approaches which come from totally different mathematical fields. The first is based on the integration-by-parts formula, which lies at the core of the application of the Malliavin calculus to finance. The second consists in using Fourier-based methods for pricing derivatives as exposed in Eberlein, Quantitative Energy Finance, 2014, [3]. We illustrate the results in the case where the jump part of the underlying model is driven by a time-inhomogeneous Gamma process and alternatively by a Variance Gamma process.

**Keywords** Option valuation · Lévy forward process model · Fourier transform · Time-inhomogeneous Lévy processes · Malliavin calculus · Greeks and sensitivity analysis

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# 1 Introduction

To compute expectations which arise as prices of derivative products is a key issue in quantitative finance. The effort which is necessary to get these values depends to a high degree on the sophistication of the model approach which is used. Simple models such as the classical geometric Brownian motion lead to easy-to-evaluate formulas for expectations but entail at the same time a high model risk. As has been shown in numerous studies, the empirical return distributions which one can observe are far from normality. This is true for all categories of financial markets: equity, fixed income, foreign exchange as well as credit markets (see e.g. Eberlein and Keller (1995) [4] for the analysis of stock price data and Eberlein and Kluge (2007) [7] for data from fixed income markets). A first step to reduce model risk and to improve the performance of the model consists in introducing volatility as a stochastic quantity. Some of the stochastic volatility models became quite popular. Nevertheless one must be aware that the distributions which diffusion processes with non-deterministic coefficients generate on a given time horizon are not known. They can only be determined approximately on the basis of simulations of process paths. In order to get more realistic distributions, an excellent choice is to replace the driving Brownian motion in classical models by a suitably chosen Lévy process. This can also be interpreted in the sense that instead of making volatility stochastic one can go over to a stochastic clock. The reason is that many Lévy processes can be obtained as time-changed Brownian motions. For example, the Variance Gamma process results when one replaces linear time by a Gamma process as subordinator. Of course, one can also consider both: a more powerful driver and stochastic volatility.

Lévy processes are in a one-to-one correspondence to the rich class of infinitely divisible distributions and at the same time analytically well tractable. Due to the higher number of available parameters, this class of distributions is flexible enough to allow a much better fit to empirical return distributions. The systematic error which results from the assumption of normality is avoided. The generating distribution of a Lévy process shows up as the distribution of increments of length one. Consequently, any distribution which one gets by fitting a parametrized subclass to empirical return data can be implemented not only approximately but exactly into Lévy-driven models. Suitably parametrized model classes which have been used successfully so far are driven by generalized hyperbolic, normal inverse Gaussian (NIG), or Variance Gamma (VG) processes, just to mention a few.

As noted above, advanced models with superior statistical properties require more demanding numerical methods. Efficient and accurate algorithms are crucial in this context, in particular for calibration purposes. For pricing of derivatives the historical distribution, which can be derived from price data of the underlying and which is used for risk management, is of less interest. Calibration usually means to estimate the risk-neutral distribution parameters. In other words, one exploits price data of derivatives. In most cases this is given in terms of volatilities. Whereas years ago calibration was usually done overnight, many trading desks recalibrate nowadays on an intraday basis. During a calibration procedure in each iteration step a large number of model prices have to be computed and compared to market prices. A method which almost always works to get the corresponding expectations is Monte Carlo simulation. Its disadvantage is that it is computationally intensive and therefore too slow for many purposes. Another classical approach is to represent prices as solutions of partial differential equations (PDEs) which in the case of Lévy processes with jumps become partial integro–differential equations (PIDEs). This approach, which is based on the Feynman–Kac formula, applies to a wide range of valuation problems, in particular it allows to compute prices of American options as well. Nevertheless, the numerical solution of PIDEs rests on sophisticated discretization methods and corresponding programs. In this paper we concentrate on the third, namely the Fourier-based approach.

To manage portfolios of derivatives, traders have to understand how sensitive prices of derivative products are with respect to changes in the underlying parameters. For this purpose they need to know the Greeks which are given by the partial derivatives of the pricing functional with respect to those parameters. Usually Greeks are estimated by means of a finite difference approximation. Two kinds of errors are produced this way: the first one comes from the approximation of the derivative by a finite difference and the second one results from the numerical computation of the expectation. To eliminate one of the sources of error, Fournié et al. (1999) [9] adopted a new approach which consists in shifting the differential operator from the pricing functional to the diffusion kernel. This procedure results in an expectation operator applied to the payoff multiplied by a random weight function.

In the following we focus on a discrete tenor interest rate model which has been introduced in Eberlein and Özkan (2005) [5]. This so-called Lévy forward process model is driven by a time-inhomogeneous Lévy process and is developed on the basis of a backward induction that is necessary to get the LIBOR rates in a convenient homogeneous form. A major advantage of the forward process approach is that it is invariant under the measure change in the sense that the driving process remains a time-inhomogeneous Lévy process. Moreover, the measure changes do not only have the invariance property but in addition they are analytically and consequently also numerically much simpler compared to the corresponding measure changes in the so-called LIBOR model. The reason is that in each induction step the forward process itself represents up to a norming constant the density process on which the measure change is based. As a consequence, any approximation such as the 'frozen drift' approximation or more sophisticated versions of it are completely avoided. This means that the approximation error with which one has to struggle in the LIBOR approach does not show up in the forward process approach.

Another important aspect is that in the latter model the increments of the driving process translate directly into increments of the LIBOR rates. This is not the case for the LIBOR model where the increments of the LIBOR rates are proportional to the corresponding increments of the driving process scaled with the current value of the LIBOR rate. Expressed in terms of the terminology which will be developed in Sects. 2 and 3 this means that in the Lévy LIBOR model

$$L(t+\Delta t, T_k) - L(t, T_k) \sim L(t, T_k)\left(L_{t+\Delta t}^{T_{k+1}} - L_t^{T_{k+1}}\right), \tag{1}$$

whereas in the Lévy forward process model

$$L(t+\Delta t, T_k) - L(t, T_k) \sim \delta_k^{-1}\left(L_{t+\Delta t}^{T_{k+1}} - L_t^{T_{k+1}}\right). \tag{2}$$

The fact that the increments of the LIBOR rate process do not depend on current LIBOR values, translates into increased flexibility and a superior model performance of the forward process approach.

In addition to the differences in mathematical properties there is a fundamental economic difference. The forward process approach allows for negative interest rates as well as for negative starting values. This is of crucial importance in particular in the current economic environment where negative rates are common. Models where by construction interest rates stay strictly positive are not able to produce realistic valuations for a large collection of interest rate derivatives in a deflationary or near-deflationary environment.

As far as the calculation of Greeks in this setting is concerned, we refer to Glasserman and Zhao (1999) [12], Glasserman (2004) [11], and Fries (2007) [10] where some treatment of this issue is given. The classical diffusion-based LIBOR market model offers a high degree of analytical tractability. However, this model cannot reproduce the phenomenon of changing volatility smiles along the maturity axis. In order to gain more flexibility in a first step one can replace the driving Brownian motion by a (time-homogeneous) Lévy process. However, one observes that the shape of the volatility surface produced by cap and floor prices is too sophisticated in order to be matched with sufficient accuracy by a model which is driven by a time-homogeneous process. To achieve a more accurate calibration of the model across different strikes and maturities one has to use the more flexible class of time-inhomogeneous Lévy processes (see e.g. Eberlein and Özkan (2005) [5] and Eberlein and Kluge (2006) [6]). Graphs in the latter paper show in particular that interest rate models driven by time-inhomogeneous Lévy processes are able to reproduce implied volatility curves (smiles) observed in the market across all maturities with high accuracy. If one restricts the approach to (time-homogeneous) Lévy processes as drivers, the smiles flatten out too fast at longer maturities. Consequently, we have analytical—the invariance under measure changes—as well as statistical reasons to choose time-inhomogeneous Lévy processes as drivers. In implementations of the model already a rather mild form of time-inhomogeneity turns out to be sufficient. Typically one has to glue together three pieces of (time-homogeneous) Lévy processes in order to cover the full range of maturities with sufficient accuracy. In terms of parameters this means that instead of three or four one uses nine or twelve parameters.

The first goal of this paper is to give a closed Fourier-based valuation formula for a caplet in the framework of the Lévy forward process model. The second aim is to study sensitivities. We discuss two approaches for this purpose. The first is based on the integration-by-parts formula, which lies at the core of the application of the Malliavin calculus to finance as developed in Fournié et al. (1999) [9], León et al.

(2002) [14], Petrou (2008) [17], Yablonski (2008) [19]. This approach is appropriate if the driving process has a diffusion component. The second approach which covers purely discontinuous drivers as well relies on Fourier-based methods for pricing derivatives. For a survey of Fourier-based methods see Eberlein (2014) [3]. We illustrate the result by applying the formula to the pricing of a caplet where the jump-part of the underlying model is driven by a time-inhomogeneous Gamma process and alternatively by a Variance Gamma process.

## 2 The Lévy Forward Process Model

Let $0 = T_0 < T_1 < \cdots < T_{n-1} < T_n = T^*$ denote a discrete tenor structure and set $\delta_k = T_{k+1} - T_k$ for all $k \in \{0, \ldots, n-1\}$. Because we proceed by backward induction, let us use the notation $T_i^* := T_{n-i}$ and $\delta_i^* = \delta_{n-i}$ for $i \in \{1, \ldots, n\}$. For zero-coupon bond prices $B(t, T_i^*)$ and $B(t, T_{i-1}^*)$, the forward process is defined by

$$F(t, T_i^*, T_{i-1}^*) = \frac{B(t, T_i^*)}{B(t, T_{i-1}^*)}. \tag{3}$$

Hence, modeling forward processes means specifying the dynamics of ratios of successive bond prices. Let $(\Omega; \mathscr{F}=\mathscr{F}_{T^*}; \mathbb{F}; \mathbb{P}_{T^*})$ be a complete stochastic basis where $\mathbb{P}_{T^*}$ should be regarded as the forward martingale measure for the settlement date $T^* > 0$ and the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T^*]}$ satisfies the usual conditions. Consider a time-inhomogeneous Lévy process $L^{T^*}$ defined on $(\Omega; \mathscr{F} = \mathscr{F}_{T^*}; \mathbb{F}; \mathbb{P}_{T^*})$ starting at 0 with local characteristics $(b^{T^*}, c, F^{T^*})$ such that the drift term $b_s^{T^*} \in \mathbb{R}$, the volatility coefficient $c_s$ and the Lévy measure $F_s^{T^*}$ satisfy the following conditions

$$\exists\, \sigma > 0, \quad \forall\, s \in [0, T^*]: \quad c_s \; > \; \sigma, \quad F_s^{T^*}(\{0\}) \; = \; 0 \tag{4}$$

and

$$\int_0^{T^*} \left( |b_s^{T^*}| + |c_s| + \int_{\mathbb{R}} \left(|x|^2 \wedge 1\right) F_s^{T^*}(dx) \right) ds < \infty. \tag{5}$$

We impose as usual a further integrability condition. Note that the processes which we will define later, are by construction martingales and therefore every single random variable has to be integrable.

**Assumption 2.1** ($\mathbb{EM}$) There exists a constant $M > 1$ such that

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux) F_s^{T^*}(dx) ds < \infty, \quad \forall\, u \in [-M, M]. \tag{6}$$

Under $(\mathbb{EM})$ the random variable $L_t^{T^*}$ has a finite expectation and its law is given by the characteristic function

$$\mathbb{E}\left[e^{\mathrm{i}uL_t^{T^*}}\right] = \exp\left(\int_0^t \left(\mathrm{i}ub_s^{T^*} - \frac{1}{2}u^2 c_s + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\right) F_s^{T^*}(dx)\right) ds\right). \quad (7)$$

Furthermore, the process $L^{T^*}$ is a special semimartingale, and thus its canonical representation has the simple form

$$L_t^{T^*} = \int_0^t b_s^{T^*} ds + \int_0^t \sqrt{c_s} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x \widetilde{\mu}^{L^{T^*}}(ds, dx), \quad (8)$$

where $(W_t^{T^*})_{t\geq 0}$ is a $\mathbb{P}_{T^*}$-standard Brownian motion and $\widetilde{\mu}^{L^{T^*}} := \mu^{L^{T^*}} - \nu^{T^*}$ is the $\mathbb{P}_{T^*}$-compensated random measure of jumps of $L^{T^*}$. As usual, $\mu^{L^{T^*}}$ denotes the random measure of jumps of $L^{T^*}$ and $\nu^{T^*}(ds, dx) := F_s^{T^*}(dx)ds$ the $\mathbb{P}_{T^*}$-compensator of $\mu^{L^{T^*}}$. We denote by $\theta_s$ the cumulant function associated with the process $L^{T^*}$ as given in (8) with local characteristics $(b^{T^*}, c, F^{T^*})$, that is, for appropriate $z \in \mathbb{C}$

$$\theta_s(z) = zb_s^{T^*} + \frac{z^2}{2} c_s + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\right) F_s^{T^*}(dx), \quad (9)$$

where $c$ and $F^{T^*}$ are free parameters, whereas the drift characteristic $b^{T^*}$ will later be chosen to guarantee that the forward process is a martingale. The following ingredients are needed.

**Assumption 2.2** $(\mathbb{LR}.1)$ For any maturity $T_i^*$ there is a bounded, deterministic function $\lambda(\cdot, T_i^*) : [0, T^*] \longmapsto \mathbb{R}$ which represents the volatility of the forward process $F(\cdot, T_i^*, T_{i-1}^*)$. These functions satisfy

$\lambda(s, T_i^*) > 0$, $\forall s \in [0, T_i^*]$ and $\lambda(s, T_i^*) = 0$ for $s > T_i^*$ for any maturity $T_i^*$, $\sum_{i=1}^{n-1} \lambda(s, T_i^*) \leq M$, $\forall s \in [0, T^*]$ where $M$ is the constant from Assumption $(\mathbb{EM})$.

**Assumption 2.3** $(\mathbb{LR}.2)$ The initial term structure of zero-coupon bond prices $B(0, T_i^*)$ is strictly positive for all $i \in \{1, \dots, n\}$.

We begin to construct the forward process with the most distant maturity and postulate

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*}\right). \quad (10)$$

One forces this process to become a $\mathbb{P}_{T^*}$-martingale by choosing $b^{T^*}$ such that

$$\int_0^t \lambda(s, T_1^*) b_s^{T^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_{\mathbb{R}} \left(e^{x\lambda(s,T_1^*)} - 1 - x\lambda(s, T_1^*)\right) \nu^{T^*}(ds, dx). \quad (11)$$

Then the forward process $F(\cdot, T_1^*, T^*)$ can be given as a stochastic exponential

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \mathscr{E}_t \left(Z(\cdot, T_1^*)\right) \quad (12)$$

with

$$Z(t, T_1^*) = \int_0^t \sqrt{c_s} \lambda(s, T_1^*) dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} (e^{x\lambda(s,T_1^*)} - 1) \widetilde{\mu}^{L^{T^*}}(ds, dx). \quad (13)$$

Since the forward process $F(\cdot, T_1^*, T^*)$ is a $\mathbb{P}_{T^*}$-martingale, we can use it as a density process and define the forward martingale measure $\mathbb{P}_{T_1^*}$ by setting

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = \frac{F(T_1^*, T_1^*, T^*)}{F(0, T_1^*, T^*)} = \mathscr{E}_{T_1^*} \left(Z(\cdot, T_1^*)\right). \quad (14)$$

By the semimartingale version of Girsanov's theorem (see Jacod and Shiryaev (1987) [13])

$$W_t^{T_1^*} := W_t^{T^*} - \int_0^t \sqrt{c_s} \lambda(s, T_1^*) ds \quad (15)$$

is a $\mathbb{P}_{T_1^*}$-standard Brownian motion and

$$\nu^{T_1^*}(dt, dx) := e^{x\lambda(s,T_1^*)} \nu^{T^*}(dt, dx) = e^{x\lambda(s,T_1^*)} F_s^{T^*}(dx) ds \quad (16)$$

is the $\mathbb{P}_{T_1^*}$-compensator of $\mu^{L^{T^*}}$.

Continuing this way one gets the forward processes $F(\cdot, T_i^*, T_{i-1}^*)$ such that for all $i \in \{1, \ldots, n\}$

$$F(t, T_i^*, T_{i-1}^*) = F(0, T_i^*, T_{i-1}^*) \exp\left(\int_0^t \lambda(s, T_i^*) dL_s^{T_{i-1}^*}\right). \quad (17)$$

The drift term $b^{T_{i-1}^*}$ is chosen in such a way that the forward process $F(\cdot, T_i^*, T_{i-1}^*)$ becomes a martingale under the forward measure $\mathbb{P}_{T_{i-1}^*}$, that is

$$\int_0^t \lambda(s, T_i^*) b_s^{T_{i-1}^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_i^*) ds - \int_0^t \int_{\mathbb{R}} \left(e^{x\lambda(s,T_i^*)} - 1 - x\lambda(s, T_i^*)\right) \nu^{T_{i-1}^*}(ds, dx). \quad (18)$$

We propose the following choice for the functions $b^{T^*_{i-1}}$ for all $i \in \{1, \ldots, n\}$

$$\begin{cases} b_s^{T^*_{i-1}} = -\frac{c_s}{2}\lambda(s, T^*_i) - \int_{\mathbb{R}} \left( \frac{e^{x\lambda(s,T^*_i)} - 1}{\lambda(s, T^*_i)} - x \right) F_s^{T^*_{i-1}}(dx), \; 0 \le s < T^*_i \\ b_s^{T^*_{i-1}} = 0, \qquad s \ge T^*_i. \end{cases} \tag{19}$$

The driving process $L^{T^*_{i-1}}$ becomes therefore

$$\begin{aligned} L_t^{T^*_{i-1}} = & -\int_0^t \left( \frac{c_s}{2}\lambda(s, T^*_i) + \int_{\mathbb{R}} \left( \frac{e^{x\lambda(s,T^*_i)} - 1}{\lambda(s, T^*_i)} - x \right) F_s^{T^*_{i-1}}(dx) \right) ds \\ & + \int_0^t \sqrt{c_s} dW_s^{T^*_{i-1}} + \int_0^t \int_{\mathbb{R}} x(\mu^{T^*} - \nu^{T^*_{i-1}})(ds, dx) \end{aligned} \tag{20}$$

under the successive forward measures $\mathbb{P}_{T^*_i}$ which are given by the recursive relation

$$\begin{cases} \frac{d\mathbb{P}_{T^*_i}}{d\mathbb{P}_{T^*_{i-1}}} = \frac{F(T^*_i, T^*_i, T^*_{i-1})}{F(0, T^*_i, T^*_{i-1})} = \mathscr{E}_{T^*_i}\left(Z(\cdot, T^*_i)\right), \quad i \in \{1, \ldots, n\} \\ \mathbb{P}_{T^*_0} = \mathbb{P}_{T^*} \end{cases} \tag{21}$$

with

$$Z(t, T^*_i) = \int_0^t \sqrt{c_s}\lambda(s, T^*_i) dW_s^{T^*_{i-1}} + \int_0^t \int_{\mathbb{R}} (e^{x\lambda(s,T^*_i)} - 1)\widetilde{\mu}^{L^{T^*_{i-1}}}(ds, dx), \tag{22}$$

where $(W_t^{T^*_{i-1}})_{t \ge 0}$ is a $\mathbb{P}_{T^*_{i-1}}$-standard Brownian motion such that

$$\begin{cases} W_t^{T^*_i} = W_t^{T^*_{i-1}} - \int_0^t \sqrt{c_s}\lambda(s, T^*_i) ds, \quad i \in \{1, \ldots, n\} \\ W_t^{T^*_0} = W_t^{T^*}. \end{cases} \tag{23}$$

$\widetilde{\mu}_{i-1}^{L^{T^*}} := \mu^{L^{T^*}} - \nu^{T^*_{i-1}}$ is the $\mathbb{P}_{T^*_{i-1}}$-compensated random measure of jumps of $L^{T^*}$ and $\nu^{T^*_{i-1}}(ds, dx) = F_s^{T^*_{i-1}}(dx)ds$ is the $\mathbb{P}_{T^*_{i-1}}$-compensator of $\mu^{L^{T^*}}$ such that

$$\begin{cases} F_s^{T^*_i}(dx) = e^{x\lambda(s,T^*_i)} F_s^{T^*_{i-1}}(dx), \quad i \in \{1, \ldots, n\} \\ F_s^{T^*_0}(dx) = F_s^{T^*}(dx). \end{cases} \tag{24}$$

Setting $\Lambda^i(s) := \sum_{j=1}^{i} \lambda(s, T_j^*)$, we conclude that for all $i \in \{1, \ldots, n\}$

$$W_t^{T_i^*} = W_t^{T^*} - \int_0^t \sqrt{c_s}\Lambda^i(s)ds \tag{25}$$

and

$$F_s^{T_i^*}(dx) = \exp\left(x\Lambda^i(s)\right) F_s^{T^*}(dx). \tag{26}$$

Note that the coefficients $\sqrt{c_s}\Lambda^i(s)$ and $\exp(x\Lambda^i(s))$, which appear in this measure change, are deterministic functions and therefore the measure change is structure preserving, i.e. the driving process is still a time-inhomogeneous Lévy process after the measure change.

Since the forward process $F(\cdot, T_i^*, T_{i-1}^*)$ is by construction a $\mathbb{P}_{T_{i-1}^*}$-martingale, the process $\frac{F(\cdot, T_i^*, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)}$, which is the density process

$$\left.\frac{d\mathbb{P}_{T_i^*}}{d\mathbb{P}_{T_{i-1}^*}}\right|_{\mathscr{F}_t} = \frac{F(t, T_i^*, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)} \tag{27}$$

is a $\mathbb{P}_{T_{i-1}^*}$-martingale as well. By iterating the relation (21) we get on $\mathscr{F}_{T_{i-1}^*}$

$$\begin{aligned}\frac{d\mathbb{P}_{T_{i-1}^*}}{d\mathbb{P}_{T^*}} &= \frac{B(0, T^*)}{B(0, T_{i-1}^*)} \prod_{j=1}^{i-1} F(T_{i-1}^*, T_j^*, T_{j-1}^*) \\ &= \exp\left(\sum_{j=1}^{i-1} \int_0^{T_{i-1}^*} \lambda(s, T_j^*) dL_s^{T_{j-1}^*}\right). \end{aligned} \tag{28}$$

Applying Proposition III.3.8 of Jacod and Shiryaev (1987) [13], we see that its restriction to $\mathscr{F}_t$ for $t \in [0, T_i^*]$

$$\left.\frac{d\mathbb{P}_{T_i^*}}{d\mathbb{P}_{T^*}}\right|_{\mathscr{F}_t} = \frac{B(0, T^*)}{B(0, T_i^*)} \prod_{j=1}^{i} F(t, T_j^*, T_{j-1}^*) \tag{29}$$

is a $\mathbb{P}_{T^*}$-martingale.

## 3 Fourier-Based Methods for Option Pricing

We will derive an explicit valuation formula for standard interest rate derivatives such as caps and floors in the Lévy forward process model. Since floor prices can be derived from the corresponding put-call-parity relation we concentrate on caps. Recall that a cap is a sequence of call options on subsequent LIBOR rates. Each single option is called a caplet. The payoff of a caplet with strike rate $K$ and maturity $T_i^*$ is

$$\delta_i^* \left(L(T_i^*, T_i^*) - K\right)^+, \tag{30}$$

where the payment is made at time point $T_{i-1}^*$. The forward LIBOR rates $L(T_i^*, T_i^*)$ are the discretely compounded, annualized interest rates which can be earned from investment during a future interval starting at $T_i^*$ and ending at $T_{i-1}^*$ considered at the time point $T_i^*$. These rates can be expressed in terms of the forward prices as follows

$$L(T_i^*, T_i^*) = \frac{1}{\delta_i^*}\left(F(T_i^*, T_i^*, T_{i-1}^*) - 1\right). \tag{31}$$

Its time-0-price, denoted by $Cplt_0(T_i^*, K)$, is given by

$$Cplt_0(T_i^*, K) = B(0, T_{i-1}^*)\delta_i^* \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}}\left[\left(L(T_i^*, T_i^*) - K\right)^+\right]. \tag{32}$$

Instead of basing the pricing on the Lévy LIBOR model one can use the Lévy forward process approach (see Eberlein and Özkan (2005) [5]). It is then more natural to write the pricing formula (32) in the form

$$Cplt_0(T_i^*, K) = B(0, T_{i-1}^*)\mathbb{E}_{\mathbb{P}_{T_{i-1}^*}}\left[\left(F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i\right)^+\right], \tag{33}$$

where $\widetilde{K}_i := 1 + \delta_i^* K$. From (17), the forward process $F(\cdot, T_i^*, T_{i-1}^*)$ is given by

$$\begin{aligned} F(T_i^*, T_i^*, T_{i-1}^*) = F(0, T_i^*, T_{i-1}^*) &\exp\left(\int_0^{T_i^*} b_s^{T_{i-1}^*}\lambda(s, T_i^*)ds\right) \\ &\times \exp\left(\int_0^{T_i^*} \sqrt{c_s}\lambda(s, T_i^*)dW_s^{T_{i-1}^*}\right) \\ &\times \exp\left(\int_0^{T_i^*}\int_{\mathbb{R}} x\lambda(s, T_i^*)\widetilde{\mu}_{i-1}^{L^{T^*}}(ds, dx)\right). \end{aligned} \tag{34}$$

Using the relations (25) and (26) we obtain for $t \in [0, T_i^*]$

$$F(t, T_i^*, T_{i-1}^*) = F(0, T_i^*, T_{i-1}^*) \exp\left(\int_0^t \lambda(s, T_i^*)dL_s^{T^*} + d(t, T_i^*)\right), \tag{35}$$

where

$$d(t,T_i^*)=\int_0^t \lambda(s,T_i^*)\left[b_s^{T_{i-1}^*}-b_s^{T^*}-\Lambda^{i-1}(s)c_s\right]ds \\ -\int_0^t \lambda(s,T_i^*)\int_{\mathbb{R}} x\left(e^{x\Lambda^{i-1}(s)}-1\right)F_s^{T^*}(dx)ds. \tag{36}$$

Remember that on $\mathscr{F}_{T_{i-1}^*}$

$$\frac{d\mathbb{P}_{T_{i-1}^*}}{d\mathbb{P}_{T^*}}=\exp\left(\sum_{j=1}^{i-1}\int_0^{T_{i-1}^*}\lambda(s,T_j^*)dL_s^{T^*}+\sum_{j=1}^{i-1}d(T_{i-1}^*,T_j^*)\right). \tag{37}$$

Keeping in mind Assumption 2.2 ($\mathbb{LR}$.1), we find

$$\exp\left(-\sum_{j=1}^{i-1}d(T_{i-1}^*,T_j^*)\right)=\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\left(\int_0^{T_{i-1}^*}\Lambda^{i-1}(s)dL_s^{T^*}\right)\right]. \tag{38}$$

Using Proposition 8 in Eberlein and Kluge (2006) [6], we find

$$\exp\left(-\sum_{j=1}^{i-1}d(T_{i-1}^*,T_j^*)\right)=\exp\left(\int_0^{T_{i-1}^*}\theta_s\left(\Lambda^{i-1}(s)\right)ds\right). \tag{39}$$

Consequently,

$$\frac{d\mathbb{P}_{T_{i-1}^*}}{d\mathbb{P}_{T^*}}=\exp\left(\int_0^{T_{i-1}^*}\Lambda^{i-1}(s)dL_s^{T^*}-\int_0^{T_{i-1}^*}\theta_s\left(\Lambda^{i-1}(s)\right)ds\right). \tag{40}$$

Knowing that the process $\left(\frac{F(\cdot,T_i^*,T_{i-1}^*)}{F(0,T_i^*,T_{i-1}^*)}\right)$ is a $\mathbb{P}_{T_{i-1}^*}$-martingale, we reach

$$\exp(-d(T_i^*,T_i^*))=\mathbb{E}_{\mathbb{P}_{T_{i-1}^*}}\left[\exp\left(\int_0^{T_i^*}\lambda(s,T_i^*)dL_s^{T^*}\right)\right]. \tag{41}$$

Hence,

$$\begin{aligned}\exp(-d(T_i^*,T_i^*)) &\\ =\exp\left(-\int_0^{T_i^*}\theta_s\left(\Lambda^{i-1}(s)\right)ds\right)&\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\left(\int_0^{T_i^*}\Lambda^{i}(s)dL_s^{T^*}\right)\right]\\ =\exp\left(\int_0^{T_i^*}\left[\theta_s\left(\Lambda^{i}(s)\right)-\theta_s\left(\Lambda^{i-1}(s)\right)\right]ds\right)&. \end{aligned}\tag{42}$$

Thus,

$$d(T_i^*, T_i^*) = \int_0^{T_i^*} \left[-\theta_s\left(\Lambda^i(s)\right) + \theta_s\left(\Lambda^{i-1}(s)\right)\right] ds. \tag{43}$$

Define the random variable $X_{T_i^*}$ as the logarithm of $F(T_i^*, T_i^*, T_{i-1}^*)$. Therefore,

$$X_{T_i^*} = \ln\left(F(0, T_i^*, T_{i-1}^*)\right) + \int_0^{T_i^*} \lambda(s, T_i^*) dL_s^{T^*} + d(T_i^*, T_i^*). \tag{44}$$

**Proposition 3.1** *Suppose there is a real number $R \in (1, 1+\varepsilon)$ such that the moment-generating function of $X_{T_i^*}$ with respect to $\mathbb{P}_{T_{i-1}^*}$ is finite at $R$, i.e. $M_{X_{T_i^*}}(R) < \infty$, then*

$$\begin{aligned} \mathrm{Cplt}_0(T_i^*, K) = & \frac{\widetilde{K}_i B(0, T_{i-1}^*)}{2\pi} \int_{\mathbb{R}} \left\{ \left( \frac{F(0, T_i^*, T_{i-1}^*)}{\widetilde{K}_i} \right)^{R+\mathrm{i}u} \right. \\ & \times \exp\left( \int_0^{T_i^*} \int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)} \left[ \left( e^{(R+\mathrm{i}u)x\lambda(s,T_i^*)} - 1 \right) - (R+\mathrm{i}u)\left( e^{x\lambda(s,T_i^*)} - 1 \right) \right] F_s^{T^*}(dx) ds \right) \\ & \left. \times \exp\left( \int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s, T_i^*) ds \right) \right\} \frac{du}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}. \end{aligned} \tag{45}$$

*Proof* The time-0-price of the caplet with strike rate $K$ and maturity $T_i^*$ has the form

$$\begin{aligned} Cplt_0(T_i^*, K) &= B(0, T_{i-1}^*) \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ \left( e^{X_{T_i^*}} - \widetilde{K}_i \right)^+ \right] \\ &= B(0, T_{i-1}^*) \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ f\left( X_{T_i^*} \right) \right], \end{aligned} \tag{46}$$

where the function $f : \mathbb{R} \to \mathbb{R}^+$ is defined by $f(x) = (e^x - \widetilde{K}_i)^+$.

Applying Theorem 2.2 in Eberlein et al. (2010) [8] (by the definition of $X_{T_i^*}$ we have $s = 0$ here), we get

$$Cplt_0(T_i^*, K) = \frac{B(0, T_{i-1}^*)}{2\pi} \int_{\mathbb{R}} M_{X_{T_i^*}}(R+\mathrm{i}u) \hat{f}(-u+\mathrm{i}R) du, \tag{47}$$

where the Fourier transform $\hat{f}$ is given by

$$\hat{f}(-u+\mathrm{i}R) = \frac{\widetilde{K}_i^{1-R-\mathrm{i}u}}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)} \tag{48}$$

and the moment-generating function $M_{X_{T_i^*}}$ is given by

$$\begin{aligned}M_{X_{T_i^*}}(R+\mathrm{i}u) &= \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}}\left[\exp\left((R+\mathrm{i}u)X_{T_i^*}\right)\right]\\ &= \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}\exp\left((R+\mathrm{i}u)d(T_i^*,T_i^*)\right)\\ &\quad\times\mathbb{E}_{\mathbb{P}_{T_{i-1}^*}}\left[\exp\left(\int_0^{T_i^*}(R+\mathrm{i}u)\lambda(s,T_i^*)dL_s^{T^*}\right)\right].\end{aligned} \tag{49}$$

Making a change of measure, we find

$$M_{X_{T_i^*}}(R+\mathrm{i}u) = \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}\exp\left((R+\mathrm{i}u)d(T_i^*,T_i^*)\right) \times\frac{\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\left(\int_0^{T_i^*}\left((R+\mathrm{i}u)\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)dL_s^{T^*}\right)\right]}{\mathbb{E}_{\mathbb{P}_{T^*}}\left[\exp\left(\int_0^{T_i^*}\Lambda^{i-1}(s)dL_s^{T^*}\right)\right]}. \tag{50}$$

Using Proposition 8 in Eberlein and Kluge (2006) [6], we can prove easily that

$$\begin{aligned}M_{X_{T_i^*}}(R+\mathrm{i}u) &= \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}\\ &\quad\times\exp\left((R+\mathrm{i}u)\int_0^{T_i^*}\left[-\theta_s\left(\Lambda^i(s)\right)+\theta_s\left(\Lambda^{i-1}(s)\right)\right]ds\right)\\ &\quad\times\frac{\exp\left(\int_0^{T_i^*}\theta_s\left((R+\mathrm{i}u)\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)ds\right)}{\exp\left(\int_0^{T_i^*}\theta_s\left(\Lambda^{i-1}(s)\right)ds\right)}\\ &= \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}\exp\left(\int_0^{T_i^*}\theta_s\left((R+\mathrm{i}u)\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)ds\right)\\ &\quad\times\exp\left(\int_0^{T_i^*}\left[(-R-\mathrm{i}u)\theta_s\left(\Lambda^i(s)\right)-(1-R-\mathrm{i}u)\theta_s\left(\Lambda^{i-1}(s)\right)\right]ds\right).\end{aligned} \tag{51}$$

Taking into account the choice of the drift coefficient in (19), the cumulant function $\theta_s$ (see (9)) and the moment-generating function $M_{X_{T_i^*}}$, respectively, become

$$\begin{aligned}\theta_s(R+\mathrm{i}u) &= (R+\mathrm{i}u)\int_{\mathbb{R}}\left(\frac{e^{x(R+\mathrm{i}u)}-1}{R+\mathrm{i}u}-\frac{(e^{x\lambda(s,T_1^*)}-1)}{\lambda(s,T_1^*)}\right)F_s^{T^*}(dx)\\ &\quad+\frac{c_s}{2}(R+\mathrm{i}u)\left(R+\mathrm{i}u-\lambda(s,T_1^*)\right)\end{aligned} \tag{52}$$

and

$$M_{X_{T_i^*}}(R+\mathrm{i}u) = \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}\exp\left(\int_0^{T_i^*}\frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$

$$\times \exp\left(\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left(e^{(R+\mathrm{i}u)x\lambda(s,T_i^*)}-1\right)F_s^{T^*}(dx)ds\right)$$
$$\times \exp\left(-(R+\mathrm{i}u)\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left(e^{x\lambda(s,T_i^*)}-1\right)F_s^{T^*}(dx)ds\right). \quad (53)$$

Finally, from (48) and (53) we conclude that

$$Cplt_0(T_i^*,K)=\frac{\widetilde{K}_i B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}}\left\{\left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u}\right.$$
$$\times \exp\left(\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left[\left(e^{(R+\mathrm{i}u)x\lambda(s,T_i^*)}-1\right)-(R+\mathrm{i}u)\left(e^{x\lambda(s,T_i^*)}-1\right)\right]F_s^{T^*}(dx)ds\right)$$
$$\left.\times \exp\left(\int_0^{T_i^*}\frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)\right\}\frac{du}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}. \quad (54)$$

# 4 Sensitivity Analysis

## *4.1 Greeks Computed by the Malliavin Approach*

In this section we present an application of the Malliavin calculus to the computation of Greeks within the Lévy forward process model. We refer to the literature, for example Di Nunno et al. (2008) [2] as well as Nualart (2006) [15] for details on the theoretical aspects of Malliavin calculus. Another important reference is Yablonski (2008) [19]. See also the Appendix for a short presentation of definitions and results used in the sequel. The forward process $F(t,T_i^*,T_{i-1}^*)$ under the forward measures $\mathbb{P}_{T_{i-1}^*}$ can be written as stochastic exponential

$$F(t,T_i^*,T_{i-1}^*)=F(0,T_i^*,T_{i-1}^*)\mathscr{E}_t\left(Z(\cdot,T_i^*)\right) \quad (55)$$

with

$$Z(t,T_i^*)=\int_0^t\sqrt{c_s}\lambda(s,T_i^*)dW_s^{T_{i-1}^*}+\int_0^t\int_{\mathbb{R}}(e^{x\lambda(s,T_i^*)}-1)\widetilde{\mu}^{L^{T_{i-1}^*}}(ds,dx). \quad (56)$$

Expressed in a differential form we get the $\mathbb{P}_{T_{i-1}^*}$-dynamics

$$\frac{dF(t,T_i^*,T_{i-1}^*)}{F(t-,T_i^*,T_{i-1}^*)}=\sqrt{c_t}\lambda(t,T_i^*)dW_t^{T_{i-1}^*}+\int_{\mathbb{R}}(e^{x\lambda(t,T_i^*)}-1)\widetilde{\mu}^{L^{T_{i-1}^*}}(dt,dx), \quad (57)$$

where $F(t-,T_i^*,T_{i-1}^*)$ is the pathwise left limit of $F(\cdot,T_i^*,T_{i-1}^*)$ at the point $t$.

As in the classical Malliavin calculus we are able to associate the solution of (57) with the process $Y(t, T_i^*, T_{i-1}^*) := \frac{\partial F(t,T_i^*,T_{i-1}^*)}{\partial F(0,T_i^*,T_{i-1}^*)}$; called the first variation process of $F(t, T_i^*, T_{i-1}^*)$. The following proposition provides a simpler expression for the Malliavin derivative operator $D_{r,0}$ when applied to the forward process rates $F(t, T_i^*, T_{i-1}^*)$ (see Di Nunno et al. (2008) [2], Theorem 17.4 and Yablonski (2008) [19], Definition 17. for details). We will denote the domain of the operator $D_{r,0}$ in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$, meaning that $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables $\mathscr{S}$ (see (100) in the Appendix).

**Proposition 4.1** *Let $F(t, T_i^*, T_{i-1}^*)_{t\in[0,T^*]}$ be the solution of (57). Then $F(t, T_i^*, T_{i-1}^*) \in \mathbb{D}^{1,2}$ and the Malliavin derivative is given by*

$$\begin{aligned} &D_{r,0}F(t, T_i^*, T_{i-1}^*) \\ &\quad = Y(t, T_i^*, T_{i-1}^*)Y(r-, T_i^*, T_{i-1}^*)^{-1}F(r-, T_i^*, T_{i-1}^*)\lambda(r, T_i^*)\sqrt{c_r}\mathbf{1}_{\{r\le t\}}. \end{aligned} \quad (58)$$

### 4.1.1 Variation in the Initial Forward Price

In this section, we provide an expression for the *Delta*, the partial derivative of the expectation $Cplt_0(T_i^*, K)$ with respect to the initial condition $F(0, T_i^*, T_{i-1}^*)$ which is given by

$$\Delta(F(0, T_i^*, T_{i-1}^*)) = \frac{\partial Cplt_0(T_i^*, K)}{\partial F(0, T_i^*, T_{i-1}^*)}. \quad (59)$$

The derivative with respect to the initial LIBOR rate is then an easy consequence.

$$\begin{aligned} \Delta(L(0, T_i^*)) &= \frac{\partial Cplt_0(T_i^*, K)}{\partial L(0, T_i^*)} \\ &= \Delta(F(0, T_i^*, T_{i-1}^*))\frac{\partial F(0, T_i^*, T_{i-1}^*)}{\partial L(0, T_i^*)} \\ &= \delta_i^*\Delta(F(0, T_i^*, T_{i-1}^*)), \end{aligned} \quad (60)$$

since

$$L(0, T_i^*) = \frac{1}{\delta_i^*}\left(F(0, T_i^*, T_{i-1}^*) - 1\right). \quad (61)$$

Let us define the set

$$\widetilde{T}_i = \left\{h_i \in L^2([0, T_i^*]) : \int_0^{T_i^*} h_i(u)du = 1\right\}. \quad (62)$$

**Proposition 4.2** *For all functions $h_i \in \widetilde{T}_i$, we have*

$$\Delta(F(0, T_i^*, T_{i-1}^*)) = \frac{B(0, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)} \mathbb{E}_{\mathbb{P}_{T^*}} \Big[ \left(F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i\right)^+ \times \exp\left( \int_0^{T_i^*} \Lambda^{i-1}(s) dL_s^{T^*} - \int_0^{T_i^*} \theta_s\left(\Lambda^{i-1}(s)\right) ds \right) \times \left( \int_0^{T_i^*} \frac{h_i(u) dW_u^{T^*}}{\lambda(u, T_i^*)\sqrt{c_u}} - \int_0^{T_i^*} \frac{h_i(u)\Lambda^{i-1}(u) du}{\lambda(u, T_i^*)} \right) \Big]. \quad (63)$$

*Proof* We consider a more general payoff of the form $H(F(T_i^*, T_i^*, T_{i-1}^*))$ such that $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally integrable function satisfying

$$\mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ H(F(T_i^*, T_i^*, T_{i-1}^*))^2 \right] < \infty \cdot \quad (64)$$

First, assume that $H$ is a continuously differentiable function with compact support. Then we can differentiate inside the expectation and get

$$\begin{aligned} \Delta_H(F(0, T_i^*, T_{i-1}^*)) &:= \frac{\partial \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ H(F(T_i^*, T_i^*, T_{i-1}^*)) \right]}{\partial F(0, T_i^*, T_{i-1}^*)} \\ &= \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ H'(F(T_i^*, T_i^*, T_{i-1}^*)) \frac{\partial F(T_i^*, T_i^*, T_{i-1}^*)}{\partial F(0, T_i^*, T_{i-1}^*)} \right] \\ &= \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ H'(F(T_i^*, T_i^*, T_{i-1}^*)) Y(T_i^*, T_i^*, T_{i-1}^*) \right]. \end{aligned} \quad (65)$$

Using Proposition 4.1 we find for any $h_i \in \widetilde{T}_i$

$$Y(T_i^*, T_i^*, T_{i-1}^*) = \int_0^{T_i^*} D_{u,0} F(T_i^*, T_i^*, T_{i-1}^*) \frac{h_i(u) Y(u-, T_i^*, T_{i-1}^*) du}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}}. \quad (66)$$

From the chain rule (see Yablonski (2008) [19], Proposition 4.8) we find

$$\begin{aligned} \Delta_H(F(0, T_i^*, T_{i-1}^*)) &= \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \Bigg[ \int_0^{T_i^*} H'(F(T_i^*, T_i^*, T_{i-1}^*)) D_{u,0} F(T_i^*, T_i^*, T_{i-1}^*) \\ &\qquad \times \frac{h_i(u) Y(u-, T_i^*, T_{i-1}^*) du}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}} \Bigg] \\ &= \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \Bigg[ \int_0^{T_i^*} D_{u,0} H(F(T_i^*, T_i^*, T_{i-1}^*)) \\ &\qquad \times \frac{h_i(u) Y(u-, T_i^*, T_{i-1}^*) du}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}} \Bigg] \end{aligned}$$

$$= \mathbb{E}_{\mathbb{P}_{T^*_{i-1}}}\Bigg[\int_0^{T_i^*}\int_{\mathbb{R}} D_{u,x} H(F(T_i^*, T_i^*, T_{i-1}^*)) \times \frac{h_i(u)Y(u-, T_i^*, T_{i-1}^*) du \delta_0(dx)}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}}\Bigg], \tag{67}$$

where $\delta_0(dx)$ is the Dirac measure at 0.

By the definition of the Skorohod integral $\delta(\cdot)$ (see Yablonski (2008) [19], Sect. 5), we reach

$$\begin{aligned} &\Delta_H(F(0, T_i^*, T_{i-1}^*)) \\ &= \mathbb{E}_{\mathbb{P}_{T^*_{i-1}}}\left[H(F(T_i^*, T_i^*, T_{i-1}^*))\delta\left(\frac{h_i(\cdot)Y(\cdot-, T_i^*, T_{i-1}^*)\delta_0(\cdot)}{F(\cdot-, T_i^*, T_{i-1}^*)\lambda(\cdot, T_i^*)\sqrt{c_\cdot}}\right)\right]. \end{aligned} \tag{68}$$

However, the stochastic process

$$\left(\frac{h_i(u)Y(u-, T_i^*, T_{i-1}^*)}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}}\right)_{0\le u\le T_i^*} \tag{69}$$

is a predictable process, thus the Skorohod integral coincides with the Itô stochastic integral and we get

$$\begin{aligned} &\Delta_H(F(0, T_i^*, T_{i-1}^*)) \\ &= \mathbb{E}_{\mathbb{P}_{T^*_{i-1}}}\left[H(F(T_i^*, T_i^*, T_{i-1}^*))\int_0^{T_i^*}\frac{h_i(u)Y(u-, T_i^*, T_{i-1}^*)dW_u^{T^*_{i-1}}}{F(u-, T_i^*, T_{i-1}^*)\lambda(u, T_i^*)\sqrt{c_u}}\right]. \end{aligned} \tag{70}$$

By Lemma 12.28. p. 208 in Di Nunno et al. (2008) [2] the result (70) holds for any locally integrable function $H$ such that

$$\mathbb{E}_{\mathbb{P}_{T^*_{i-1}}}\left[H(F(T_i^*, T_i^*, T_{i-1}^*))^2\right] < \infty. \tag{71}$$

In particular, if one takes

$$H(F(T_i^*, T_i^*, T_{i-1}^*)) = B(0, T_{i-1}^*)\left(F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i\right)^+, \tag{72}$$

we can express the derivatives of the expectation $Cplt_0(T_i^*, K, \delta_i^*)$ with respect to the initial condition $F(0, T_i^*, T_{i-1}^*)$ in the form of a weighted expectation as follows

$$\begin{aligned} \Delta(F(0, T_i^*, T_{i-1}^*)) = B(0, T_{i-1}^*)\mathbb{E}_{\mathbb{P}_{T^*_{i-1}}}\Bigg[&\left(F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i\right)^+ \\ &\times \int_0^{T_i^*}\frac{h_i(u)Y(u-, T_i^*, T_{i-1}^*)dW_u^{T^*_{i-1}}}{\lambda(u, T_i^*)\sqrt{c_u}F(u-, T_i^*, T_{i-1}^*)}\Bigg]. \end{aligned} \tag{73}$$

We show easily that

$$Y(u-, T_i^*, T_{i-1}^*) = \frac{F(u-, T_i^*, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)}, \tag{74}$$

hence

$$\begin{aligned} &\Delta(F(0, T_i^*, T_{i-1}^*)) \\ &= \frac{B(0, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)} \mathbb{E}_{\mathbb{P}_{T_{i-1}^*}} \left[ \left( F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i \right)^+ \int_0^{T_i^*} \frac{h_i(u) dW_u^{T_{i-1}^*}}{\lambda(u, T_i^*)\sqrt{c_u}} \right]. \end{aligned} \tag{75}$$

In accordance with (25) we can write

$$W_t^{T_{i-1}^*} = W_t^{T^*} - \int_0^t \Lambda^{i-1}(s)\sqrt{c_s} ds. \tag{76}$$

By making a measure change using the fact (see (40)) that

$$\left.\frac{d\mathbb{P}_{T_{i-1}^*}}{d\mathbb{P}_{T^*}}\right|_{\mathscr{F}_{T_i^*}} = \exp\left( \int_0^{T_i^*} \Lambda^{i-1}(s) dL_s^{T^*} - \int_0^{T_i^*} \theta_s \left( \Lambda^{i-1}(s) \right) ds \right), \tag{77}$$

we end up with

$$\begin{aligned} \Delta(F(0, T_i^*, T_{i-1}^*)) = \frac{B(0, T_{i-1}^*)}{F(0, T_i^*, T_{i-1}^*)} \mathbb{E}_{\mathbb{P}_{T^*}} \Bigg[ & \left( F(T_i^*, T_i^*, T_{i-1}^*) - \widetilde{K}_i \right)^+ \\ & \times \exp\left( \int_0^{T_i^*} \Lambda^{i-1}(s) dL_s^{T^*} - \int_0^{T_i^*} \theta_s \left( \Lambda^{i-1}(s) \right) ds \right) \\ & \times \left( \int_0^{T_i^*} \frac{h_i(u) dW_u^{T^*}}{\lambda(u, T_i^*)\sqrt{c_u}} - \int_0^{T_i^*} \frac{h_i(u)\Lambda^{i-1}(u)}{\lambda(u, T_i^*)} du \right) \Bigg]. \end{aligned} \tag{78}$$

## 4.2 Greeks Computed by the Fourier-Based Valuation Method

Thanks to the Fourier-based valuation formula obtained in (45) and the structure of the forward process model as an exponential semimartingale, we can calculate readily the Greeks. We focus on the variation to the initial condition, i.e. Delta.

**Proposition 4.3** *Suppose there is a real number $R \in (1, 1+\varepsilon)$ such that the moment-generating function of $X_{T_i^*}$ with respect to $\mathbb{P}_{T_{i-1}^*}$ is finite at $R$, i.e. $M_{X_{T_i^*}}(R) < \infty$, then*

$$\Delta(F(0,T_i^*,T_{i-1}^*)) = \frac{B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}}\Bigg\{\left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u-1}$$
$$\times\exp\left(\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left(e^{(R+\mathrm{i}u)x\lambda(s,T_i^*)}-1\right)F_s^{T^*}(dx)ds\right)$$
$$\times\exp\left(-\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}(R+\mathrm{i}u)\left(e^{x\lambda(s,T_i^*)}-1\right)F_s^{T^*}(dx)ds\right)$$
$$\times\exp\left(\int_0^{T_i^*}\frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)\Bigg\}\frac{du}{R+\mathrm{i}u-1}. \quad (79)$$

*Proof* Based on the Sect. 4 in Eberlein et al. (2010) [8], this proposition can be shown easily.

## 4.3 Examples

### 4.3.1 Variance Gamma Process (VG)

We suppose that the jump component of the driving process $L^{T^*}$ (see (8)) is described by a Variance Gamma process with the Lévy density $\nu$ given by

$$\nu(dx) = F_{VG}(x)dx \quad (80)$$

such that

$$F_{VG}(x) := \frac{1}{\eta|x|}\exp\left(\frac{\theta}{\sigma^2}x - \frac{1}{\sigma}\sqrt{\frac{2}{\eta}+\frac{\theta^2}{\sigma^2}}|x|\right), \quad (81)$$

where $(\theta,\sigma,\eta)$ are the parameters such that $\theta\in\mathbb{R}$, $\sigma>0$ and $\eta>0$.

Let us put $B=\frac{\theta}{\sigma^2}$ and $C=\frac{1}{\sigma}\sqrt{\frac{2}{\eta}+\frac{\theta^2}{\sigma^2}}$ and get

$$F_{VG}(x) = \frac{\exp(Bx - C|x|)}{\eta|x|}. \quad (82)$$

In this case, the moment-generating function $M_{X_{T_i^*}}$ is given by

$$M_{X_{T_i^*}}(z) = \left(F(0,T_i^*,T_{i-1}^*)\right)^z\exp\left(\int_0^{T_i^*}\left(\frac{c_s z}{2}(z-1)\lambda^2(s,T_i^*)+I^{VG}(s,z)\right)ds\right), \quad (83)$$

where the generalized integral $I^{VG}(s,z)$ is given by

$$I^{VG}(s,z) := \int_{\mathbb{R}} \left( e^{x\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)} - e^{x\Lambda^{i-1}(s)} \right) F_{VG}(x)dx \\ - \int_{\mathbb{R}} z \left( e^{x\Lambda^{i}(s)} - e^{x\Lambda^{i-1}(s)} \right) F_{VG}(x)dx. \tag{84}$$

Now substituting $F_{VG}(x)$ by its explicit expression we get

$$\begin{aligned} I^{VG}(s,z) &= \int_{\mathbb{R}} \left( e^{x\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx - C|x|\right) \frac{dx}{\eta|x|} \\ &\quad - \int_{\mathbb{R}} z \left( e^{x\Lambda^{i}(s)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx - C|x|\right) \frac{dx}{\eta|x|} \\ &= \int_0^{+\infty} \left( e^{x\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx - Cx\right) \frac{dx}{\eta x} \\ &\quad - \int_0^{+\infty} z \left( e^{x\Lambda^{i}(s)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx - Cx\right) \frac{dx}{\eta x} \\ &\quad - \int_{-\infty}^{0} \left( e^{x\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)\right)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx + Cx\right) \frac{dx}{\eta x} \\ &\quad + \int_{-\infty}^{0} z \left( e^{x\Lambda^{i}(s)} - e^{x\Lambda^{i-1}(s)} \right) \exp\left(Bx + Cx\right) \frac{dx}{\eta x}, \end{aligned}$$

or

$$\begin{aligned} I^{VG}(s,z) &= \int_0^{+\infty} \left[ \frac{e^{\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)+B-C\right)x} - e^{\left(\Lambda^{i-1}(s)+B-C\right)x}}{\eta x} \right] dx \\ &\quad - \int_0^{+\infty} \left[ z \frac{e^{\left(\Lambda^{i}(s)+B-C\right)x} - e^{\left(\Lambda^{i-1}(s)+B-C\right)x}}{\eta x} \right] dx \\ &\quad - \int_{-\infty}^{0} \left[ \frac{e^{\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)+B+C\right)x} - e^{\left(\Lambda^{i-1}(s)+B+C\right)x}}{\eta x} \right] dx \\ &\quad + \int_{-\infty}^{0} \left[ z \frac{e^{\left(\Lambda^{i}(s)+B+C\right)x} - e^{\left(\Lambda^{i-1}(s)+B+C\right)x}}{\eta x} \right] dx \\ &= \int_0^{+\infty} \left[ \frac{e^{\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)+B-C\right)x} - e^{\left(\Lambda^{i-1}(s)+B-C\right)x}}{\eta x} \right] dx \\ &\quad - \int_0^{+\infty} \left[ z \frac{e^{\left(\Lambda^{i}(s)+B-C\right)x} - e^{\left(\Lambda^{i-1}(s)+B-C\right)x}}{\eta x} \right] dx \\ &\quad + \int_0^{+\infty} \left[ \frac{e^{-\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)+B+C\right)x} - e^{-\left(\Lambda^{i-1}(s)+B+C\right)x}}{\eta x} \right] dx \\ &\quad - \int_0^{+\infty} \left[ z \frac{e^{-\left(\Lambda^{i}(s)+B+C\right)x} - e^{-\left(\Lambda^{i-1}(s)+B+C\right)x}}{\eta x} \right] dx. \end{aligned}$$

Using the notations

$$\alpha_i(s,z) = -\left(z\lambda(s,T_i^*) + \Lambda^{i-1}(s) + B - C\right), \tag{85}$$

$$\beta_i(s) = -\left(\Lambda^{i-1}(s) + B - C\right), \tag{86}$$

$$\gamma_i(s) = -\left(\Lambda^{i}(s) + B - C\right), \tag{87}$$

we end up with

$$\begin{aligned} I^{VG}(s,z) = &\int_0^{+\infty} \left[ \frac{e^{-\alpha_i(s,z)x} - e^{-\beta_i(s)x}}{x} - z\frac{e^{-\gamma_i(s)x} - e^{-\beta_i(s)x}}{x} \right] dx \\ &+ \int_0^{+\infty} \left[ \frac{e^{-(2C-\alpha_i(s,z))x} - e^{-(2C-\beta_i(s))x}}{x} - z\frac{e^{-(2C-\gamma_i(s))x} - e^{-(2C-\beta_i(s))x}}{x} \right] dx. \end{aligned}$$

Using Frullani's integral (see for details Ostrowski (1949) [16]), we can show that, if $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ such that $\mathscr{R}e(\alpha) > 0, \mathscr{R}e(\beta) > 0$ and $\frac{\beta}{\alpha} \in \mathbb{C} \setminus \mathbb{R}^-$ where $\mathbb{R}^- = ]-\infty; 0]$,

$$I_{(\alpha,\beta)} := \int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx = Log\left(\frac{\beta}{\alpha}\right), \tag{88}$$

where $Log$ is the principal value of the logarithm. Consequently

$$\begin{aligned} I^{VG}(s,z) &= Log\left(\frac{\beta_i(s)}{\alpha_i(s,z)}\right) - zLog\left(\frac{\beta_i(s)}{\gamma_i(s)}\right) \\ &\quad + Log\left(\frac{2C - \beta_i(s)}{2C - \alpha_i(s,z)}\right) - zLog\left(\frac{2C - \beta_i(s)}{2C - \gamma_i(s)}\right) \\ &= Log\left(\frac{\beta_i(s)}{\alpha_i(s,z)}\right) + Log\left(\frac{2C - \beta_i(s)}{2C - \alpha_i(s,z)}\right) \\ &\quad - z\left(Log\left(\frac{\beta_i(s)}{\gamma_i(s)}\right) + Log\left(\frac{2C - \beta_i(s)}{2C - \gamma_i(s)}\right)\right) \\ &= Log\left(\frac{\beta_i(s)\,(2C - \beta_i(s))}{\alpha_i(s,z)\,(2C - \alpha_i(s,z))}\right) - zLog\left(\frac{\beta_i(s)\,(2C - \beta_i(s))}{\gamma_i(s)\,(2C - \gamma_i(s))}\right). \end{aligned}$$

The moment-generating function $M_{X_{T_i^*}}$ becomes

$$\begin{aligned} M_{X_{T_i^*}}(z) = &\left(F(0,T_i^*,T_{i-1}^*)\right)^z \exp\left(\int_0^{T_i^*} \frac{c_s z}{2}(z-1)\lambda^2(s,T_i^*)ds\right) \\ &\times \exp\left(\int_0^{T_i^*} Log\left(\frac{\beta_i(s)\,(2C - \beta_i(s))}{\alpha_i(s,z)\,(2C - \alpha_i(s,z))}\right) ds\right) \\ &\times \exp\left(-\int_0^{T_i^*} zLog\left(\frac{\beta_i(s)\,(2C - \beta_i(s))}{\gamma_i(s)\,(2C - \gamma_i(s))}\right) ds\right) \end{aligned}$$

or

$$M_{X_{T_i^*}}(R+\mathrm{i}u) = \left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u}$$
$$\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$
$$\times \exp\left(\int_0^{T_i^*} Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\alpha_i(s,R+\mathrm{i}u)\,(2C-\alpha_i(s,R+\mathrm{i}u))}\right)ds\right)$$
$$\times \exp\left(-\int_0^{T_i^*} (R+\mathrm{i}u)Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\gamma_i(s)\,(2C-\gamma_i(s))}\right)ds\right).$$

The valuation formula becomes

$$Cplt_0(T_i^*,K) = \frac{B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}} \frac{\widetilde{K}_i^{1-R-\mathrm{i}u} M_{X_{T_i^*}}(R+\mathrm{i}u)}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}du$$
$$= \frac{\widetilde{K}_i B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}} \left\{ \left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u}\right.$$
$$\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$
$$\times \exp\left(\int_0^{T_i^*} Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\alpha_i(s,R+\mathrm{i}u)\,(2C-\alpha_i(s,R+\mathrm{i}u))}\right)ds\right)$$
$$\left.\times \exp\left(-\int_0^{T_i^*} (R+\mathrm{i}u)Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\gamma_i(s)\,(2C-\gamma_i(s))}\right)ds\right)\right\} \frac{du}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}. \quad (89)$$

The *Delta* is given by

$$\Delta(F(0,T_i^*,T_{i-1}^*)) = \frac{B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}} \left\{ \left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u-1}\right.$$
$$\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$
$$\times \exp\left(\int_0^{T_i^*} Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\alpha_i(s,R+\mathrm{i}u)\,(2C-\alpha_i(s,R+\mathrm{i}u))}\right)ds\right)$$
$$\left.\times \exp\left(-\int_0^{T_i^*} (R+\mathrm{i}u)Log\left(\frac{\beta_i(s)\,(2C-\beta_i(s))}{\gamma_i(s)\,(2C-\gamma_i(s))}\right)ds\right)\right\} \frac{du}{R+\mathrm{i}u-1}. \quad (90)$$

### 4.3.2 Inhomogeneous Gamma Process (IGP)

We suppose that the jump component of the driving process $L^{T^*}$, is described by an inhomogeneous Gamma process (IGP), which has been introduced by Berman (1981) [1] as follows

**Definition 4.4** Let $A(t)$ be a nondecreasing function from $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $B > 0$. A Gamma process with shape function $A$ and scale parameter $B$ is a stochastic process $(L_t)_{t\geq 0}$ on $\mathbb{R}^+$ such that:

1. $L_0 = 0$;
2. Independent increments: for every increasing sequence of time points $t_0, \ldots, t_n$, the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent;
3. for $0 \leq s < t$, the distribution of the random variable $L_t - L_s$ is given by the Gamma distribution $\Gamma\,(A(t) - A(s); B)$.

We suppose that the shape function $A$ is differentiable, hence we can write

$$A(t) = A(0) + \int_0^t \dot{A}(s)ds \tag{91}$$

for all $t \in \mathbb{R}^+$ where $\dot{A}$ denotes the derivative of $A$. In this case, the Lévy density of the Gamma process $L$ is given by

$$F_s^G(x) = \dot{A}(s)\frac{e^{-Bx}}{x}\mathbf{1}_{\{x>0\}}. \tag{92}$$

The moment-generating function (53) has the form

$$\begin{aligned}
M_{X_{T_i^*}}(z) &= \left(F(0, T_i^*, T_{i-1}^*)\right)^z \exp\left(\int_0^{T_i^*} \frac{c_s z}{2}(z-1)\lambda^2(s, T_i^*)ds\right) \\
&\quad \times \exp\left(\int_0^{T_i^*}\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left[\left(e^{zx\lambda(s,T_i^*)} - 1\right) - z\left(e^{x\lambda(s,T_i^*)} - 1\right)\right] F_s^G(x)dxds\right) \\
&= \left(F(0, T_i^*, T_{i-1}^*)\right)^z \exp\left(\int_0^{T_i^*} \frac{c_s z}{2}(z-1)\lambda^2(s, T_i^*)ds\right) \\
&\quad \times \exp\left(\int_0^{T_i^*} \dot{A}(s)\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left(e^{zx\lambda(s,T_i^*)} - 1\right)\frac{e^{-Bx}}{x}\mathbf{1}_{\{x>0\}}dxds\right) \\
&\quad \times \exp\left(-z\int_0^{T_i^*} \dot{A}(s)\int_{\mathbb{R}} e^{x\Lambda^{i-1}(s)}\left(e^{x\lambda(s,T_i^*)} - 1\right)\frac{e^{-Bx}}{x}\mathbf{1}_{\{x>0\}}dxds\right) \\
&= \left(F(0, T_i^*, T_{i-1}^*)\right)^z \exp\left(\int_0^{T_i^*}\left(\frac{c_s z}{2}(z-1)\lambda^2(s, T_i^*) + \dot{A}(s)I^G(s,z)\right)ds\right),
\end{aligned}$$

where

$$\begin{aligned}
I^G(s,z) = &\int_0^{+\infty} \frac{e^{\left(z\lambda(s,T_i^*)+\Lambda^{i-1}(s)-B\right)x} - e^{\left(\Lambda^{i-1}(s)-B\right)x}}{x}dx \\
&- \int_0^{+\infty} z\frac{e^{\left(\Lambda^{i}(s)-B\right)x} - e^{\left(\Lambda^{i-1}(s)-B\right)x}}{x}dx.
\end{aligned}$$

Setting

$$\alpha_i(s,z) = -\left(z\lambda(s,T_i^*) + \Lambda^{i-1}(s) - B\right), \tag{93}$$

$$\beta_i(s) = -\left(\Lambda^{i-1}(s) - B\right), \tag{94}$$

$$\gamma_i(s) = -\left(\Lambda^{i}(s) - B\right) \tag{95}$$

and using Frullani's integral, we find that

$$\begin{aligned} I^G(s,z) &= \int_0^{+\infty} \left[ \frac{e^{-\alpha_i(s,z)x} - e^{-\beta_i(s)x}}{x} - z\frac{e^{-\gamma_i(s)x} - e^{-\beta_i(s)x}}{x} \right] dx \\ &= Log\left(\frac{\beta_i(s)}{\alpha_i(s,z)}\right) - zLog\left(\frac{\beta_i(s)}{\gamma_i(s)}\right) \\ &= Log\left(\frac{\Lambda^{i-1}(s) - B}{z\lambda(s,T_i^*) + \Lambda^{i-1}(s) - B}\right) - zLog\left(\frac{\Lambda^{i-1}(s) - B}{\Lambda^{i}(s) - B}\right). \end{aligned}$$

Therefore, we get the form

$$\begin{aligned} M_{X_{T_i^*}}(z) = \left(F(0,T_i^*,T_{i-1}^*)\right)^z &\exp\left(\int_0^{T_i^*} \frac{c_s z}{2}(z-1)\lambda^2(s,T_i^*)ds\right) \\ &\times \exp\left(\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s) - B}{z\lambda(s,T_i^*) + \Lambda^{i-1}(s) - B}\right)ds\right) \\ &\times \exp\left(-z\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s) - B}{\Lambda^{i}(s) - B}\right)ds\right) \end{aligned}$$

or

$$\begin{aligned} M_{X_{T_i^*}}(R+\mathrm{i}u) = &\left(F(0,T_i^*,T_{i-1}^*)\right)^{R+\mathrm{i}u} \\ &\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right) \\ &\times \exp\left(\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s) - B}{(R+\mathrm{i}u)\lambda(s,T_i^*) + \Lambda^{i-1}(s) - B}\right)ds\right) \\ &\times \exp\left(-(R+\mathrm{i}u)\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s) - B}{\Lambda^{i}(s) - B}\right)ds\right). \end{aligned}$$

The valuation formula becomes

$$\begin{aligned} Cplt_0(T_i^*,K) &= \frac{B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}} \frac{\widetilde{K}_i^{1-R-\mathrm{i}u} M_{X_{T_i^*}}(R+\mathrm{i}u)}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}du \\ &= \frac{\widetilde{K}_i B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}} \frac{du}{(R+\mathrm{i}u)(R+\mathrm{i}u-1)}\left\{\left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u}\right. \end{aligned}$$

$$\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$
$$\times \exp\left(\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s)-B}{(R+\mathrm{i}u)\lambda(s,T_i^*)+\Lambda^{i-1}(s)-B}\right)ds\right)$$
$$\times \exp\left(-\int_0^{T_i^*}(R+\mathrm{i}u)\dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s)-B}{\Lambda^{i}(s)-B}\right)ds\right)\Bigg\}. \quad (96)$$

The Greek *Delta* is given by

$$\Delta(F(0,T_i^*,T_{i-1}^*)) = \frac{B(0,T_{i-1}^*)}{2\pi}\int_{\mathbb{R}}\Bigg\{\left(\frac{F(0,T_i^*,T_{i-1}^*)}{\widetilde{K}_i}\right)^{R+\mathrm{i}u-1}$$
$$\times \exp\left(\int_0^{T_i^*} \frac{c_s}{2}(R+\mathrm{i}u)(R+\mathrm{i}u-1)\lambda^2(s,T_i^*)ds\right)$$
$$\times \exp\left(\int_0^{T_i^*} \dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s)-B}{(R+\mathrm{i}u)\lambda(s,T_i^*)+\Lambda^{i-1}(s)-B}\right)ds\right)$$
$$\times \exp\left(-\int_0^{T_i^*}(R+\mathrm{i}u)\dot{A}(s)Log\left(\frac{\Lambda^{i-1}(s)-B}{\Lambda^{i}(s)-B}\right)ds\right)\Bigg\}\frac{du}{R+\mathrm{i}u-1}. \quad (97)$$

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# A Appendix

## *A.1 Isonormal Lévy Process (ILP)*

Let $\mu$ and $\nu$ be $\sigma$-finite measures without atoms on the measurable spaces $(\mathrm{T}, \mathscr{A})$ and $(\mathrm{T}\times \mathrm{X}_0, \mathscr{B})$, respectively. Define a new measure

$$\pi(dt,dz) := \mu(dt)\delta_\Theta(dz) + \nu(dt,dz) \quad (98)$$

on a measurable space $(\mathrm{T}\times\mathrm{X}, \mathscr{G})$, where $\mathrm{X} = \mathrm{X}_0 \cup \{\Theta\}$, $\mathscr{G} = \sigma(\mathscr{A}\times\{\Theta\}, \mathscr{B})$ and $\delta_\Theta(dz)$ is the measure which gives mass one to the point $\Theta$. We assume that the Hilbert space $\mathrm{H} := \mathrm{L}^2(\mathrm{T}\times\mathrm{X}, \mathscr{G}, \pi)$ is separable.

**Definition A.1** We say that a stochastic process $\mathrm{L} = \{\mathrm{L}(h), h \in \mathrm{H}\}$ defined on a complete probability space $(\Omega, \mathscr{F}, P)$ is an isonormal Lévy process (or Lévy process on $H$) if the following conditions are satisfied:

1. The mapping $h \longrightarrow L(h)$ is linear;

2. $\mathrm{E}[e^{ixL(h)}] = \exp(\Psi(x,h))$, where $\Psi(x,h)$ is equal to

$$\int_{\mathrm{T}\times\mathrm{X}} \left( (e^{\mathrm{i}xh(t,z)} - 1 - \mathrm{i}xh(t,z))\mathbf{1}_{\mathrm{X}_0}(z) - \frac{1}{2}x^2h^2(t,z)\mathbf{1}_{\Theta}(z) \right) \pi(dt,dz). \quad (99)$$

## A.2 The Derivative Operator

Let $\mathscr{S}$ denote the class of smooth random variables, that is the class of random variables $\xi$ of the form

$$\xi = f(L(h_1), \dots, L(h_n)), \quad (100)$$

where $f$ belongs to $C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n$ are in $H$, and $n \geq 1$. The set $\mathscr{S}$ is dense in $L^p(\Omega)$ for any $p \geq 1$.

**Definition A.2** The stochastic derivative of a smooth random variable of the form (100) is the $H$-valued random variable $D\xi = \{D_{t,x}\xi, (t,x) \in T \times X\}$ given by

$$\begin{aligned} D_{t,x}\xi = & \sum_{k=1}^{n} \frac{\partial f}{\partial y_k}(L(h_1), \dots, L(h_n))h_k(t,x)\mathbf{1}_{\Theta}(x) \\ & + (f(L(h_1) + h_1(t,x), \dots, L(h_n) + h_n(t,x)) \\ & - f(L(h_1), \dots, L(h_n)))\, \mathbf{1}_{X_0}(x). \end{aligned} \quad (101)$$

We will consider $D\xi$ as an element of $L^2(T \times X \times \Omega) \cong L^2(\Omega; H)$; namely $D\xi$ is a random process indexed by the parameter space $T \times X$.

1. If the measure $\nu$ is zero or $h_k(t,x) = 0$, $k = 1, \dots, n$ when $x \neq \Theta$ then $D\xi$ coincides with the Malliavin derivative (see, e.g. Nualart (2006) [15] Definition 1.2.1 p.38).
2. If the measure $\mu$ is zero or $h_k(t,x) = 0$, $k = 1, \dots, n$ when $x = \Theta$ then $D\xi$ coincides with the difference operator (see, e.g. Picard (1996) [18]).

## A.3 Integration by Parts Formula

**Theorem A.3** *Suppose that $\xi$ and $\eta$ are smooth random variables and $h \in H$. Then*

*1.*

$$\mathrm{E}[\xi L(h)] = \mathrm{E}[\langle D\xi; h\rangle_H]; \quad (102)$$

*2.*

$$\mathrm{E}[\xi\eta L(h)] = \mathrm{E}[\eta\,\langle D\xi; h\rangle_H] + \mathrm{E}[\xi\,\langle D\eta; h\rangle_H] + \mathrm{E}[\big\langle D\eta; h\mathbf{1}_{X_0} D\xi\big\rangle_H]. \quad (103)$$

As a consequence of the above theorem we obtain the following result:

The expression of the derivative $D\xi$ given in (101) does not depend on the particular representation of $\xi$ in (100).
The operator $D$ is closable as an operator from $L^2(\Omega)$ to $L^2(\Omega; H)$.

We will denote the closure of $D$ again by $D$ and its domain in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$.

## A.4 The Chain Rule

**Proposition A.4** (see Yablonski (2008), Proposition 4.8) *Suppose $F = (F_1, F_2, \ldots, F_n)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Let $\phi \in \mathscr{C}^1(\mathbb{R}^n)$ be a function with bounded partial derivatives such that $\phi(F) \in \mathrm{L}^2(\Omega)$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and*

$$D_{t,x}\phi(F) = \begin{cases} \sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i}(F) D_{t,\Theta} F_i; & x = \Theta \\ \phi(F_1 + D_{t,x}F_1, \ldots, F_n + D_{t,x}F_n) - \phi(F_1, \ldots, F_n); & x \neq \Theta. \end{cases} \quad (104)$$

## A.5 Regularity of Solutions of SDEs Driven by Time-Inhomogeneous Lévy Processes

We focus on a class of models in which the price of the underlying asset is given by the following stochastic differential equation (see Di Nunno et al. [2] and Petrou [17] for details)

$$\begin{aligned} dS_t &= b(t, S_{t-})dt + \sigma(t, S_{t-})dW_t \\ &\quad + \int_{\mathbb{R}_0} \varphi(t, S_{t-}, z)\tilde{N}(dt, dz), \\ S_0 &= x, \end{aligned} \quad (105)$$

where $\mathbb{R}_0 := \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$, $x \in \mathbb{R}^d$, $\{W_t, 0 \le t \le T\}$ is an $m$-dimensional standard Brownian motion, $\tilde{N}$ is a compensated Poisson random measure on $[0, T] \times \mathbb{R}_0$ with compensator $\nu_t(dz)dt$. The coefficients $b : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^m$ and $\varphi : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d \times \mathbb{R}$, are continuously differentiable with bounded derivatives and the family of positive measures $(\nu_t)_{t\in[0,T]}$ satisfies $\int_0^T (\int_{\mathbb{R}_0} (\|z\|^2 \wedge 1)\nu_t(dz))dt < \infty$ and $\nu_t(\{0\}) = 0$. The coefficients are assumed to satisfy the following linear growth condition

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 + \int_{\mathbb{R}_0} \|\varphi(t,x,z)\|^2 \nu_t(dz) \leq C(1+\|x\|^2), \qquad (106)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, where $C$ is a positive constant. Furthermore, we suppose that there exists a function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}_0} |\rho(z)|^2 \nu_t(dz) < \infty, \qquad (107)$$

and a positive constant $K$ such that

$$\|\varphi(t,x,z) - \varphi(t,y,z)\| \leq K|\rho(z)|\|x-y\|, \qquad (108)$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ and $z \in \mathbb{R}_0$.

In the sequel we provide a theorem which proves that under specific conditions the solution of a stochastic differential equation belongs to the domain $\mathbb{D}^{1,2}$.

**Theorem A.5** *Let $(S_t)_{t\in[0,T]}$ be the solution of (105). Then $S_t \in \mathbb{D}^{1,2}$ for all $t \in [0, T]$ and the derivative $D_{r,0}S_t$ satisfies the following linear equation*

$$\begin{aligned} D_{r,0}S_t = &\int_r^t \frac{\partial b}{\partial x}(u, S_{u-}) D_{r,0}S_{u-} du + \sigma(r, S_{r-}) \\ &+ \int_r^t \frac{\partial \sigma}{\partial x}(u, S_{u-}) D_{r,0}S_{u-} dW_u \\ &+ \int_r^t \int_{\mathbb{R}_0} \frac{\partial \varphi}{\partial x}(u, S_{u-}, y) D_{r,0}S_{u-} \tilde{N}(du, dy) \end{aligned} \qquad (109)$$

*for $0 \leq r \leq t$ a.e. and $D_{r,0}S_t = 0$ a.e. otherwise.*

As in the classical Malliavin calculus we are able to associate the solution of (105) to the first variation process $Y_t := \nabla_x S_t$. Then, we will also provide a specific expression for $D_{r,0}S_t$, the Wiener directional derivative of the $S_t$.

**Proposition A.6** *Let $(S_t)_{t\in[0,T]}$ be the solution of (105). Then the derivative satisfies the following equation*

$$D_{r,0}S_t = Y_t Y_{r-}^{-1} \sigma(r, S_{r-}) \mathbf{1}_{\{r\leq t\}} \;\; a.e. \qquad (110)$$

*where $(Y_t)_{t\in[0,T]}$ is the first variation process of $(S_t)_{t\in[0,T]}$.*

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# Inside the EMs Risky Spreads and CDS-Sovereign Bonds Basis

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**Abstract** The paper considers a no-arbitrage setting for pricing and relative value analysis of risky sovereign bonds. The typical case of an emerging market country (EM) that has bonds outstanding both in foreign hard currency (Eurobonds) and local soft currency (treasuries) is inspected. The resulting two yield curves give rise to a credit and currency spread that need further elaboration. We discuss their proper measurement and also derive and analyze the necessary no-arbitrage conditions that must hold. Then we turn attention to the CDS-Bond basis in this multi-curve environment. For EM countries the concept shows certain specifics both in theoretical background and empirical performance. The paper further focuses on analyzing these peculiarities. If the proper measurement of the basis in the standard case of only hard currency debt being issued is still problematic, the situation is much more complicated in a multi-curve setting when a further contingent claim on the sovereign risk in the face of local currency debt curve appears. We investigate the issue and provide relevant theoretical and empirical input.

**Keywords** HJM · Foreign debt · Domestic debt · Z-Spread · CDS-Bond basis

## 1 Introduction

Local currency debt of EM sovereigns became a hot topic both for practitioners and academics in the recent years. Major investment banks and asset managers consider it a separate asset class and publish regularly special local currency investment reports. A joint working group of IMF, WB, EBRD, and OECD demonstrated recently an official interest in a thorough investigation of this market segment and support for its development, thus forming a strict policy agenda. It was recognized that not only do the local bonds complete the market and thus bring market efficiency, but also

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they could act as a shock absorber to the volatile capital inflows. Furthermore, they provide flexibility to the governments in financing their budget deficit. However, these instruments are not well understood from a no-arbitrage point of view and a formal setting is lacking. Such a setting would provide not only a better picture for their inherent risk-return characteristics, but would also be an indispensable tool for market research and strategy. The aim of this paper is exactly to focus attention on the large set of open questions the local currency debt gives rise to and lay the ground for a formal relative value analysis with a special emphasis on the CDS-Bond basis.

The paper begins with our general modeling no-arbitrage approach under an HJM reduced credit risk setting. It serves as a basis and gives a financial engineering intuition about the nature of the problem. The default of the sovereign is represented as the first jump of a counting process. For the dynamics of the interest rates and the exchange rate we use jump diffusions controlling the jumps and correlations in a suitable way, so that we have high precision in capturing the structural macrofinancial effects. We derive the no-arbitrage conditions that must hold in that multi-curve environment and then analyze their informational content. Then we turn to an application related to correctly extracting the credit and currency spreads and measuring the CDS-Bond basis on a broader scope. This provides basic building blocks for relative value trades under presence of the local currency yield curve which could serve as an additional pillar.

The literature on integrating the foreign and domestic debt of a risky sovereign in a consistent way is at a nascent stage both from an academic and practitioners' point of view. Related technically but different in essence is the work of Ehlers and Schönbucher [9] who give a reduced form model for CDS of an obligor denominated in different currencies which accounts for dependence between the exchange rate and the credit spread. Eberlain and Koval [8] give a high generalization of the cross-currency term structure models, but similarly they deal only with hard currencies. Regarding the CDS-Bond basis, Berd et al. [2] provide a thorough analysis of the shortcomings of the Z-spread as a risky spread metric.[1] Alizalde et al. [10] further discuss the issue and provide extensive simulations. Interesting new measures for the basis are given in Bernhard and Mai [3] which need further elaboration and development. However, all these references deal with the single-curve case with an extension to the multi-curve case pending.

## 2 Local Currency Bonds No-Arbitrage HJM Setting

In this section we first lay the foundations in brief for pricing of risky debt in a general reduced form setting. Then we add the local currency debt into the picture and discuss the risky spreads. We conclude by derivation and analysis of the no-arbitrage conditions.

[1]The Z-spread represents a simple shift of the discounting risk-free curve so that the price of the risky bond is attained.

## 2.1 Risky Bonds Under Marked Point Process

The first task is to model default in a suitable way. We start with the most general formulation and then modify it appropriately. We consider a filtered probability space $(\Omega, (G_t)_{t\geq 0}, P)$ which supports an $n$-dimensional Brownian motion $W^P = (W_1, W_2, \ldots, W_n)$ under the objective probability measure $P$ and a marked point process $\mu : \left(\Omega, B(R^+), \varepsilon\right) \to R^+$ with markers $(\tau_i, X_i)$ representing the jump times and their sizes in a measurable space $(E, \varepsilon)$, where $E = [0, 1]$ and by $\varepsilon$ we denote the Borel subsets of $E$. We assume that $\mu(\omega; dt, dx)$ has a separable compensator of the form:

$$\upsilon : \left(\Omega, B(R^+), \varepsilon\right) \to R^+ \quad \text{and} \quad \upsilon\,(\omega; dt, dx) = h(\omega; t)F_t(\omega; dx)dt,$$

where $h(\omega; t) = \int_{R^+} \upsilon\,(\omega; t, dx)$ is a $G_t$ measurable intensity and the marks have a conditional distribution of the jumps of $F_t(\omega; dx)$. Thus, we have the identity $\int_E F_t\,(\omega; dx) = 1$. Furthermore, we can define the total loss function $L(\omega; t) = \int_0^t \int_E l(\omega; s, x)\mu(\omega; ds, dx)$ and the recovery $R(\omega; t) = 1 - \int_0^t \int_E l(\omega; s, x)\mu(\omega; ds, dx)$. The function $l(\omega; t, x)$ scales the marks in a suitable way, and having control over it, we can define it such that our model is tractable enough. We define also the sum of the jumps by $S(\omega; t) = \int_0^t \int_E x\mu(\omega; ds, dx)$ and their number by $N(\omega; t) = \int_0^t \int_E \mu(\omega; ds, dx)$.

Effectively, the marked point process as a sequence of random jumps $(\tau_i, X_i)$ is characterized by the probability measure $\mu(\omega; dt, dx)$, which gives the number of jumps with size $dx$ in a small time interval of $dt$. The compensator $\upsilon\,(\omega; t, dx)$ provides a full probability characterization of the process. It incorporates in itself two effects. On one hand, we have the intensity $h(\omega; t)dt$, which gives the conditional probability of jump of the process in a small time interval of $dt$ incorporating the whole market information up to $t$. On the other hand, we have the conditional distribution $F_t(\omega; dx)$ of the markers $X$ in case of a jump realization.

We can look at the jumps of the marked point process as sequential defaults of an obligor at random times $\tau_i$ that lead to losses $X_i$ at each of them. They can also be considered a set of restructuring events leading to losses for the creditors. Under this general setting, the prices of the riskless and risky bonds are given by:

- **Riskless bond**:

$$P(t, T) = E^Q\left(\exp\left(-\int_t^T r(s)ds\right)|G_t\right) = \exp\left(-\int_t^T f(t, s)ds\right) \tag{1}$$

- **Risky bond**:

$$P^*(t,T) = E^Q\left(\exp\left(-\int_t^T r(s)ds\right) R(\omega;T)|G_t\right)$$
$$= R(t)\exp\left(-\int_t^T f^*(t,s)ds\right), \tag{2}$$

where $r(t)$, $f(t,T)$, and $f^*(t,T)$ are the riskless spot, riskless forward, and risky forward rates respectively.

Depending on how we specify the convention of recovery, we can get further simplification of the formulas. However, this should be well motivated and come either from the legal definitions of the debt contracts or their economic grounding.

Under a recovery of market value (RMV) setting, default is a percentage mark down, $q$, from the previous recovery. So we have $R(\omega;\tau_i) = (1-q(\omega;\tau_i,X_i))$ $R(\omega;\tau_i-)$ and $l(\omega;\tau_i)$ has the form $l(\omega;\tau_i) = -q(\omega;\tau_i,X_i)\times R(\omega;\tau_i-)$. This definition allows us to write:

$$\mu(\omega,dt,dx) = \sum_{s>0} 1_{\{\Delta N(\omega,s)\neq 0\}}\delta_{(s,\Delta N(\omega,s))}(dt,dx)$$
$$dR(\omega;t) = -R(\omega;t)\int_E q\,(\omega;t,x)\,\mu(\omega;dt,dx);\; R(\omega;0)=1$$

and if we assume no jumps of the intensity and the risk-free rate at default times (contagion effects), we have no change for the risk-free bond pricing formula and for the risky one and as in [13] we get:

$$P^{*RMV}(t,T) = E^Q\left(\exp\left(-\int_t^T r(s)ds\right)R(\omega;T)|G_t\right)$$
$$= R(t)E^Q\left(\exp\left(-\int_t^T (r(s)+h(s)\int_E q\,(\omega;s,x)\,F_s(dx))ds\right)|G_t\right)$$
$$= R(t)\exp\left(-\int_t^T f^{*RMV}(t,s)ds\right) \tag{3}$$

Note that within this setting there is no “last default”. The intensity is defined for the whole marked point process and not just for a concrete single default time, so it does not go to zero after default realizations. This combined with the fact that intensity is continuous makes the market filtration $G_t$ behave like a background filtration in the pricing formulas. So we can avoid using the generalized Duffie, Schroder, and Skiadas [7] formula. Furthermore, we can denote $q_e(t) = \int_E q\,(\omega;t,x)\,F_t(dx)$ to be the expected loss. So we have that the pricing formula is dependent on the generalized intensity $h(t)q_e(t)$. Due to the multiplicative nature of the last expression, only from market information, as discussed in Schönbucher (2003), we cannot distinguish between the pure intensity effect $h(t)$ and the recovery induced one $q_e(t)$.

Under a recovery of par (RP) setting, in case of default, the recovery is a separate fixed or random quantity independent of the default indicator and the risk-free rate. So we have $E = \{0, 1 - R(\omega)\}$ and $\upsilon(\omega; dt, dx) = h(\omega; t)(1 - R_e)dt$ with $R_e = E^Q(R(\omega) \mid G_t)$. Since we have just one jump, we can write:

$$\mu(\omega, dt, dx) = 1_{\{\Delta N(\omega,t)\neq 0\}}\delta_{(t,\Delta N(\omega,t))}(dt, dx)$$

The bond price is:

$$\begin{aligned} P^{*RP}(t,T) &= E^Q\left(\exp\left(-\int_t^T r(s)ds\right)\left(R(\omega)\,1_{\{\tau\leq T\}} + 1_{\{\tau>T\}}\right)|G_t\right) \\ &= 1_{\{\tau>t\}}\exp\left(-\int_t^T f^{*RP}(t,s)ds\right) \end{aligned} \tag{4}$$

In contrast to RMV, here, as discussed in Schönbucher (2003), he can distinguish between the pure intensity and recovery induced effects.

## 2.2 *Model Formulation*

In this section we develop our HJM model for pricing of local and foreign currency bonds of a risky country. However, before this being done formally, it is essential to elaborate on the nature of the problem. Although we do not put here explicitly macrofinancial structure, but just proxy it by jumps and correlations, it, by all means, stays in the background and must be conceptually considered.

### 2.2.1 General Notes

A risky emerging market country can have bonds denominated both in local and foreign currency that give rise to two risky yield curves and risky spreads—credit and currency. Generally, the latter arise due to the possibility of the respective credit events to occur and their severity. To investigate them, formal assumptions are needed both on their characteristics and interdependence.

We will consider that the two types of debt have different priorities. The country is first engaged to meeting the foreign debt obligation from its limited international reserves. The impossibility of this being done leads to default or restructuring. In both cases, we have a credit event according to the ISDA classification. The foreign debt has a senior status. The spread that arises reflects the credit risk of the country. It is a function of: (1) the probability of the credit event to occur; (2) the expected loss given default; (3) the risk aversion of the market participants to the credit event.

The domestic debt economically stands differently. It reflects the priority of the payments in hard currency and it incurs instantly the losses in case of default of the country. So this debt is the first to be affected by a default and is subordinated. Technically, the credit event can be avoided under a flexible exchange rate regime because the country can always make a debt monetization and pay the amounts due in local currency taking advantage of the fact that there is no resource constraint on it. However, the price for this is inflation pick-up and exchange rate depreciation. This leads to real devaluation of the domestic debt. It is exactly the seigniorage and the dilution effect that cause the loss in the value.[2] This resembles the case of a firm issuing more equity to avoid default. The spread of the domestic debt over the foreign one forms the currency spread. Its nature is very broad, and it is not only due to the currency mismatch. Namely, it is a function of: (1) the probability of the credit event to occur and the need for monetization; (2) the negative side effect of the credit event on the exchange rate by a sudden depreciation of the latter; (3) the volatility of the exchange rate; (4) the expected depreciation of the exchange rate without taking into consideration the monetization; (5) the risk aversion of market participants to the credit event and the need for monetization, the sudden exchange rate depreciation and its size; (6) the risk aversion of the market participants to the volatility of the exchange rate. All these effects are captured by our model.

### 2.2.2 Multi-currency Risky Bonds Model

We use the setting of Sect. 2.1 modified to a multi-currency debt. Firstly, we consider the case of no monetization and then analyze the case with monetization. Secondly, to avoid using an additional marked point process, and thus a second intensity, the default on the foreign debt is modeled indirectly. Namely, we assume that default on domestic debt leads to default on foreign debt, but due to the different priority of the two, we have just different losses incurred, respectively recoveries. This means that by controlling recoveries we control default and the inherent subordination without imposing too much structure. If the default on the domestic debt is so strong that it leads to a default on the foreign debt as well, we incur zero recovery on the domestic debt and some positive one on the foreign debt. If the insolvency is mild, we have a loss only on the domestic debt, so we incur some positive recovery on the domestic debt and a full recovery on the foreign debt. Thirdly, for notational purposes, we take as a benchmark Germany and EUR as the base hard currency. Lastly, we employ the recovery of market value assumption. The reason for this is twofold. On one hand, in that way, we are consistent with the HJM methodology of Schönbucher [12] for a single risky curve under RMV and produce parsimonious no-arbitrage conditions for the extension to a multi-curve environment. On the other

[2]This pattern can be observed historically for almost all EM countries resorting to a galloping inflation to avoid a nominal domestic debt default. The Russian default of 1998 somehow seems to be a partial notable exception where there was along with the inflation surge an actual default on certain ruble ($RUR$) bonds—GKOs and OFZs.

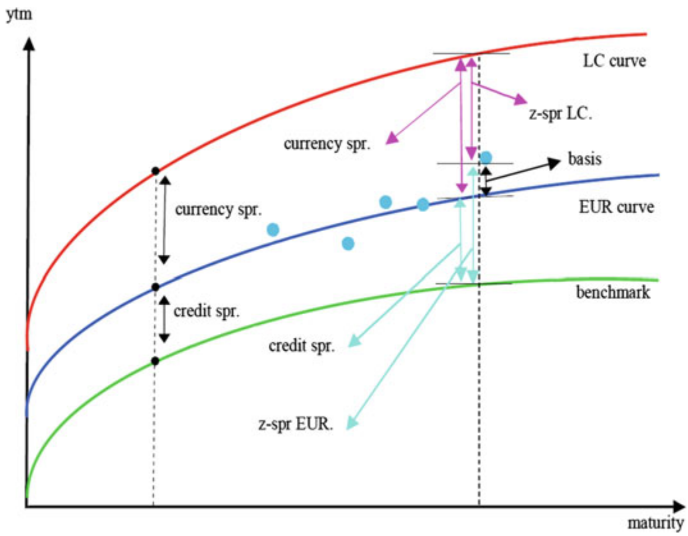


**Fig. 1** Risky spreads

hand, as pointed out in Bonnaud et al. [5], for bonds denominated in a different currency than the numerator employed in discounting, the RMV assumption should be the working engine. Their argument is exactly as ours above, in case of default, the sovereign would rather dilute by depreciating the exchange rate and thus the remaining cash flows of the bond produce in essence the RMV structure. Moreover, rather than using EUR denominated bonds, we could take advantage of the CDS quotes and produce synthetic bonds having an RMV recovery structure. Using them is actually preferable for empirical work since major academic studies argue that it is the CDS market that first captures the market information about the credit risk stance of the risky sovereign. Furthermore, with a few exceptions, if the EM sovereigns have in most cases both well developed local currency treasury markets and are subject to CDS quotation, they do have only few Eurobonds outstanding. Figure 1 represents the typical situation the risky sovereign faces.

*Mathematical formulation* We continue with the model setup. Firstly, we give the suitable notation and assumptions. Then we move to the derivation of the no-arbitrage conditions and the pricing.

- **Notation**
  $f_{EUR}(t, T)$—nominal forward rate, EUR, Ger.
  $f^*_{EUR}(t, T)$—nominal forward rate, EUR, EM
  $f^*_{LC}(t, T)$—nominal forward rate in LC, EM
  $r_{EUR}(t)$—nominal short rate, EUR, Ger.
  $r^*_{EUR}(t)$—nominal short rate, EUR, EM
  $r^*_{LC}(t)$—nominal short rate in LC, EM
  $h^*_{EUR}(t, T) = f^*_{EUR}(t, T) - f_{EUR}(t, T)$—credit spr., EM

$h^*_{LC,EUR}(t,T) = f^*_{LC}(t,T) - f^*_{EUR}(t,T)$—currency spr., EM
$h^*_{LC}(t,T) = f^*_{LC}(t,T) - f_{EUR}(t,T)$—general currency spr., EM
$P_{EUR}(t,T) = \exp(-\int_t^T f_{EUR}(t,s)ds)$—bond, EUR, Ger.
$P^*_{f,EUR}(t,T) = R_{f,EUR}(t)\exp(-\int_t^T f^*_{EUR}(t,s)ds)$—for. bond price., EUR, EM
$P^*_{d,LC}(t,T) = R_{d,LC}(t)\exp(-\int_t^T f^*_{LC}(t,s)ds)$—dom. bond price., LC, EM
$B_{EUR}(t) = \exp(\int_0^t r_{EUR}(s)ds)$—bank account, EUR, Ger.
$B^*_{f,EUR}(t) = R_{f,EUR}(t)\exp(\int_0^t r^*_{EUR}(s)ds)$—for. bank account, EUR, EM
$B^*_{d,LC}(t) = R_{d,LC}(t)\exp(\int_0^t r^*_{LC}(s)ds)$—dom. bank account, LC, EM
$X(t)$—exchange rate, EUR for 1 LC, $\widetilde{X}(t)$—exchange rate, LC for 1 EUR
$R_{f,EUR}(t)$—bond recovery, EUR, EM, $R_{d,LC}(t)$—bond recovery, LC, EM

We use the asterisk to denote risk, the first letter ($d$ or $f$) to denote domestic or foreign debt, and finally the currency of denomination is shown as $EUR$ or LC.[3]

- **Currency denominations**
$P^*_{d,EUR}(t,T) = X(t)P^*_{d,LC}(t,T)$—dom. bond, EUR
$P^*_{f,LC}(t,T) = \widetilde{X}(t)P^*_{f,EUR}(t,T)$—for. bond, LC
$B^*_{d,EUR}(t) = X(t)B^*_{d,LC}(t)$—dom. bank account, EUR
$B^*_{f,LC}(t) = \widetilde{X}(t)B^*_{f,EUR}(t)$—for. bank account, LC
- **Intensities**

Foreign debt, EUR:
Intensity: $h_{EUR}(t) = h(t)$
Compensator: $h_{EUR}(t)q_{e,EUR}(t) = h(t)\int_E q_{f,EUR}(\omega;t,x)\,F_t(dx)$
Domestic debt, LC:
Intensity: $h_{LC}(t) = h(t)$
Compensator: $h_{LC}(t)q_{e,LC}(t) = h(t)\int_E q_{d,LC}(\omega;t,x)\,F_t(dx)$

The compensator (generalized intensity) characterizes default. Controlling in a suitable way the recovery, we can control the compensator and thus the default event. We turn attention now to the dynamics of the instruments under consideration.

- **Forward rates**
$df_{EUR}(t,T) = \alpha_{EUR}(t,T)dt + \sum_{i=1}^n \sigma_{EUR,i}(t,T)dW_i^P(t)$
$df^*_{EUR}(t,T) = \alpha^*_{EUR}(t,T)dt + \sum_{i=1}^n \sigma^*_{EUR,i}(t,T)dW_i^P(t)$
$+\int_E \delta^*_{EUR}(x,t,T)\mu(dx,dt)$

[3]It must be further noted that we actually used standard definitions for the risky forward rates as in Schönbucher [12]. Namely, $f^*_{EUR/LC}(t,T) = -\frac{\partial}{\partial T}\log P^*_{f,EUR/d,LC}(t,T)$ with terminal conditions $P^*_{f,EUR/d,LC}(T,T) = R_{f,EUR/d,LC}(T)$. The risky bank accounts economically just represent a unit of currency invested at the respective short rates and continuously rolled over accounting for any default losses. However, since the forward rates, resp. the bonds, are our basic modeling object, it would be more precise to consider the bank accounts derived quantities from them similar to Björk et al. [4] without going here deeper into the modified technical details.

$df^*_{LC}(t,T) = \alpha^*_{LC}(t,T)dt + \sum_{i=1}^n \sigma^*_{LC,i}(t,T)dW_i^P(t)$
$+\int_E \delta^*_{LC}(x,t,T)\mu(dx,dt)$
We assume that in case of default there is a market turmoil leading to a jump in both curves. At maturity $T$, the $EUR$ curve jumps by a size of $\int_E \delta^*_{EUR}(x,t,T)\mu(dx,dt)$, and that of the local currency by $\int_E \delta^*_{LC}(x,t,T)\mu(dx,dt)$. The terms $\delta^*_{EUR}(x,t,T)$ and $\delta^*_{LC}(x,t,T)$ show the jump sizes of the respective curves for every maturity. As indicated at the beginning of the section, everywhere we will work under the market filtration $G_t$ so both the Brownian motions and the point process are adapted to it.

- **Recoveries**

  $\frac{dR_{f,EUR}(t)}{R_{f,EUR}(t)} = -\int_E q_{f,EUR}(x,t)\mu(dx,dt)$

  $\frac{dR_{d,LC}(t)}{R_{d,LC}(t)} = -\int_E q_{d,LC}(x,t)\mu(dx,dt)$

  After each default we have a devaluation of the respective bond by an expected value of $\int_E q_{f/d}(x,t)\mu(dx,dt)$. The stochasticity of the loss is captured by the random jump size $q(.,.)$ as elaborated in Sect. 2.1.

- **Bank accounts**

  $\frac{dB_{EUR}(t)}{B_{EUR}(t)} = r_{EUR}(t)dt$

  $\frac{dB^*_{f,EUR}(t)}{B^*_{f,EUR}(t)} = r^*_{EUR}(t)dt - \int_E q_{f,EUR}(x,t)\mu(dx,dt)$

  $\frac{dB^*_{d,LC}(t)}{B^*_{d,LC}(t)} = r^*_{LC}(t)dt - \int_E q_{d,LC}(x,t)\mu(dx,dt)$

- **Exchange rate**

  $\frac{dX(t)}{X(t)} = \alpha_X(t)dt + \sum_{i=1}^n \sigma_{X,i}(t)dW_i^P(t) - \int_E \delta_X(x,t)\mu(dx,dt)$
  We assume that in case of default the market turmoil causes an exchange rate devaluation by $\int_E \delta_X(x,t)\mu(dx,dt)$.

- **Bond prices**

  $P_{EUR}(t,T) = \exp(-\int_t^T f_{EUR}(t,s)ds) = E^{Q^f}(\exp(-\int_t^T r_{EUR}(s)ds)|G_t)$

  $P^*_{f,EUR}(t,T) = R_{f,EUR}(t)\exp(-\int_t^T f^*_{EUR}(t,s)ds)$

  $= E^{Q^f}(\exp(-\int_t^T r_{EUR}(s)ds)R_{f,EUR}(T)|G_t)$

  $P^*_{d,EUR}(t,T) = P^*_{d,LC}(t,T)X(t) = R_{d,LC}(t)X(t)\exp(-\int_t^T f^*_{LC}(t,s)ds)$

  $= E^{Q^f}(\exp(-\int_t^T r_{EUR}(s)ds)R_{d,LC}(T)X(T)|G_t)$

  It must be emphasized that the effects of exchange rate, recovery, and the expected devaluation sizes are incorporated in the respective forward rates of the bonds. Furthermore, the expectations are taken under $Q^f$, the foreign risk-neutral measure.

- **Arbitrage**

  Under standard regularity conditions, for the system to be free of arbitrage, all traded assets denominated in euro must have a rate of return $r_{EUR}$ under $Q^f$. This means that the processes:

$$\frac{P_{EUR}(t,T)}{B_{EUR}(t)}, \frac{B^*_{f,EUR}(t)}{B_{EUR}(t)}, \frac{P^*_{f,EUR}(t,T)}{B_{EUR}(t)}, \frac{B^*_{d,LC}(t)X(t)}{B_{EUR}(t)}, \frac{P^*_{d,LC}(t,T)X(t)}{B_{EUR}(t)}$$

  must be local martingales under $Q^f$. For our purposes being martingales would be enough.
  Taking the stochastic differentials of the upper expressions, omitting the technicalities to the appendix, we can get the respective no-arbitrage conditions.

- Spreads:

$$r^*_{EUR}(t) - r_{EUR}(t) = h(t)\varphi_{q_{f,EUR}}(t) \tag{5.1}$$

$$\begin{aligned} r^*_{LC}(t) - r^*_{EUR}(t) &= -\alpha_X(t) - \phi(t)\sigma_X(t) \\ &+h(t)(\varphi_{\delta_X}(t) - \varphi_{q_{d,LC},\delta_X}(t) + \varphi_{q_{d,LC}}(t) - \varphi_{q_{f,EUR}}(t)) \end{aligned} \tag{5.2}$$

- Drifts:

  $\alpha_{EUR}(t,T) = \sigma_{EUR}(t,T)\int_t^T \sigma_{EUR}(t,v)dv - \sigma_{EUR}(t,T)\phi(t)$

  $\alpha^*_{EUR}(t,T) = \sigma^*_{EUR}(t,T)\int_t^T \sigma^*_{EUR}(t,v)dv - \sigma^*_{EUR}(t,T)\phi(t)$

  $+h_{EUR}(t)\varphi_{\theta^*_{EUR}}^{q_{f,EUR},\delta_X}(t)$

  $\alpha^*_{LC}(t,T) = \sigma^*_{LC}(t,T)\int_t^T \sigma^*_{LC}(t,v)dv - \sigma^*_{LC}(t,T)\phi(t) - \sigma^*_{LC}(t,T)\sigma_X(t,T)$

  $+h_{LC}(t)\varphi_{\theta^*_{LC}}^{q_{d,LC},\delta_X}(t)$,

  where we have used the notation:

  $\theta^*_{EUR} = \exp(-\int_t^T \delta^*_{EUR}(x,t,s)ds)$ , $\theta^*_{LC} = \exp(-\int_t^T \delta^*_{LC}(x,t,s)ds)$

  $\varphi_{a.b,\ldots}^{x,y,\ldots}(t) = \int_E (ab\ldots)((1-x)(1-y)\ldots)\Phi(t,x)F_t(dx)$

  and used vector notation and scalar products where necessary for simplicity.

  By $\Phi(t,x)$ and $\phi(t)$ we denoted the Girsanov's kernels of the counting process and the Brownian motion respectively when changing the probability measure from $P$ to $Q^f$. The term $\varphi(t)$ represents the scaled expected jump sizes of the counting process. We can give the interpretation that $\phi(t)$ is the market price of diffusion risk and $\varphi(t)$ is the market price of jump risk. Parametrizing the volatilities and the market prices of risk, as well as imposing suitable dynamics for $h(t)$, we give a full characterization of our system. Furthermore, the intensity could be a function of the underlying processes of the rates, so we could get correlation between the intensity, the interest rates, and the exchange rate.

*Spreads diagnostics from a reduced form point of view* It is important to give a deeper interpretation of the no-arbitrage conditions and see which factors drive the credit and currency spreads. Despite the heavy notation, the analysis actually goes fluently. The drift equations give the modified HJM drift restrictions. The slight change from the classical riskless case is due to the jumps that arise. Equation (5.1) shows that the credit risk is proportional to the intensity of default and the scaled expected LGD by the coefficient controlling the risk aversion. The higher they are, the higher the spread is. Equation (5.2) gives the currency spread. It arises due to two main reasons. Firstly, the intensity of default and the difference between the two *LGDs* in local currency and euro, scaled by the coefficient for the risk aversion, act as in the previous case. They also make explicit the subordination. Secondly, the expected local currency depreciation, its volatility, and the risk aversion to diffusion risk act similarly to the standard uncovered interest parity (UIP) relationship. The higher they are, the higher the spread is. It is both important and interesting to note that inflation does not appear directly and it influences the spreads, as the next section shows, only through a secondary channel.

*Monetization* The analysis so far considered a loss of $1 - R_{d,LC}(T)$ on default of the domestic debt. However, if a full monetization is applied, then we would have $R_{d,LC}(T) = 1$ and thus $\varphi_{q_{d,LC}}(t) = 0$ and $\varphi_{q_{d,LC},\delta_X}(t) = 0$. If such a monetary injection is neutral to nominal values, it is certainly not to real ones. Devaluation arises due to the negative market sentiment following the default and the higher amount of money in circulation. Its effect can be measured differently based on what we take as a base—the price index or the exchange rate. Most naturally, we can expect both of them to depreciate due to the structural macrolinks that exist between these variables. For quantifying the amount we would need a macromodel which is beyond the scope of the reduced form model presented. The latter only shows what characteristics the market prices in general without imposing concrete macrolinks among them. Depending on what the base is, we would have a direct estimation of certain type of indicators and an indirect one of the rest up to their structural influence on the former. If the inflation is taken as a base, then we would have the comparison of inflation indexed bonds to the non-indexed ones. The spread between them would give an estimate for the expected inflation. Unfortunately, such an analysis is unrealistic due to the fact that such bonds are issued very rarely by emerging market countries. If the exchange rate is taken as a base, then we would have the comparison of domestic debt bonds to foreign debt bonds. The spread between them would give an estimate for the currency risk and the devaluation effect. The estimate for the inflation would be indirect and based on hypothetical structural links.

Whether the country would monetize or declare a formal default is based on strategic considerations. It is a matter of structural analysis which option it would take. By all means, its decision is priced. In case of default, the pricing formula is Eq. (5.2). In case of monetization, we would have a jump in the exchange rate. Let us denote its size by $\widehat{\delta}_X$. It will be different from the no-monetization one, $\delta_X$, due to the different regimes that are followed, and we would thus get:

$$r^*_{LC}(t) - r^*_{EUR}(t) = h(t)(\varphi_{\widehat{\delta}_X}(t) - \varphi_{q_{f,EUR}}(t)) - \alpha_X(t) - \phi(t)\sigma_X(t) \quad (6)$$

There is no a priori no-arbitrage argument that $\varphi_{\widehat{\delta}_X}(t) = \varphi_{\delta_X}(t) - \varphi_{q_{d,LC},\delta_X}(t) + \varphi_{q_{d,LC}}(t)$ must hold so that the two scenarios are equivalent.[4] The only information we get from the market is an estimate for the generalized intensity being $h(t)\varphi_{\widehat{\delta}_X}(t)$ or $h(t)(\varphi_{\delta_X}(t) - \varphi_{q_{d,LC},\delta_X}(t) + \varphi_{q_{d,LC}}(t))$ but not knowing which possible scenario will be realized.

# 3 CDS-Bond Basis

## *3.1 General Notes*

The setting we built gives us an alternative for evaluating the CDS-Bond basis. This is represented in Fig. 1. There the LC zero-coupon yield curve is built by employing local currency treasuries and an appropriate smoothing method. The EUR zero-coupon yield curve is built by employing CDS quotes with the maths represented in the sequel. Along with the curves, there are few Eurobonds represented in light blue colored dots. Both credit and currency spreads can be computed for them employing a standard Z-spread methodology. Despite its various shortcomings, as discussed in Berd et al. [2] and Elizalde et al. [10], it allows us to have a certain measure for the spreads and it is widely accepted by practitioners. Subtracting from the yield curves' implied credit and currency spreads the bond implied spreads, we get two alternative specifications for the CDS-Bonds basis. Several things need a comment.

Firstly, the two basis measures are not equal by default. The one representing the credit spread is subject to Z-spread measurement based on a parallel shift of the benchmark curve. So it depends on the whole benchmark curve and has nothing to do with the LC one. Vice versa, the basis implied by the LC curve is subject to Z-spread measurement based on a parallel shift of the LC curve. So it depends on the whole $LC$ curve, but has nothing to do with the benchmark one. This provides intuition how the introduction of the LC curve brings additional information in the picture and provides more market completeness that must be utilized in relative value trades.

Secondly, as mentioned above, the EUR curve is built by utilizing CDS quotes. As shown below, in the procedure employed, an assumption is needed for the recovery scheme. What it should be depends on our purposes. On one hand, if we would like to just extract the credit and currency spreads from the yield curves and calibrate a reduced form model,[5] it would be convenient to employ the setting from Sect. 2. So

[4] This is a delicate issue. As indicated, a further structural analysis is needed for a complete answer. The crucial point is that the two scenarios affect in a different way the monetary base. It will have a neutral effect on the macro variables in general and the risky spreads in particular only in case the economy is at the macro potential. Exactly when that is not the case, we can expect that the two scenarios will not be equivalent. A further elaboration on these issues from a structural point of view could be found in Yordanov [15, 16].

[5] We postpone the factors to build realization of the model from Sect. 2 so that it becomes operative for calibration and consequent further analysis to the forthcoming follow-up paper of Yordanov [17].

an RMV assumption for the EUR curve is the most appropriate one since the same assumption is imposed also for the LC curve and when subtracting the corresponding zero yields, we subtract apples from apples. On the other hand, if we like to extract the basis, we must be careful since the Eurobonds are priced under a firmly established RP assumption. So for a standard calculation via a Z-spread based on the benchmark curve we need an RP built EUR curve to be consistent. With the many problems of the Z-spread, it would be definitely bad to add further ones coming from a recovery assumption inconsistency which would only further contribute to an imprecise basis measurement. For a calculation via a Z-spread based on the LC curve, we should not use the RMV LC curve but a modified one. From the RMV LC curve we need to build an RP one and then compute the Z-spread and the basis to be consistent.

## 3.2 Technical Notes

Here we provide the technical notes regarding the above discussion.

- **EUR curve**
  Using OIS differential discounting as in Doctor and Goulden [6], we could modify[6] the standard CDS bootstrap procedure of ISDA and extract at time $t$ the $T-$maturity default probabilities $p^R_{EUR}(t,T)$ under a recovery assumption of $R$. Then we would get in a straightforward way the $EUR$ zero coupon yields ($ytm$) and credit spreads ($spr$) under RMV and RP:
  - **RMV**:
    $\text{spr}^{RMV,R}_{EUR}(t,T) = -\frac{(1-R)\log(1-p^R_{EUR}(t,T))}{T-t}$
    $\text{ytm}^{RMV,R}_{EUR}(t,T) = \text{spr}^{RMV,R}_{EUR}(t,T) + \exp(-y_{EUR}(t,T)(T-t))$
  - **RP**:
    $\text{spr}^{RP,R}_{EUR}(t,T) = -\frac{\log(Rp^R_{EUR}(t,T)+1-p^R_{EUR}(t,T))}{T-t}$
    $\text{ytm}^{RP,R}_{EUR}(t,T) = \text{spr}^{RP,R}_{EUR}(t,T) + \exp(-y_{EUR}(t,T)(T-t))$,
    where $y_{EUR}(t,T)$ is the $T-$maturity zero yield of the riskless benchmark curve (e.g. German bunds).
- **LC curve**
  - **RMV**:
    $\text{ytm}^{RMV,R}_{LC}(t,T)$–observed from the market

[6]The OIS discounting should be given a special comment since there is still no consensus on how to bootstrap OIS swaps to form the discount factors for the CDS swap bootstrap. The problem comes from the presence of gaps for certain maturities. A possible specification is given in West [14].

$$\text{spr}^{RMV,R}_{LC,EUR}(t,T) = \text{ytm}^{RMV,R}_{LC}(t,T) - \text{ytm}^{RMV,R}_{EUR}(t,T)$$

$$p^{R}_{LC}(t,T) = 1 - \exp\left(-\frac{\text{spr}^{RMV,R}_{LC,EUR}(t,T)}{1-R}(T-t)\right)$$

– **RP**:

$$\text{spr}^{RP,R}_{LC,EUR}(t,T) = -\frac{\log(R p^{R}_{LC}(t,T) + 1 - p^{R}_{LC}(t,T))}{T-t}$$

$$\text{ytm}^{RP,R}_{LC}(t,T) = \text{spr}^{RP,R}_{LC,EUR}(t,T) + \text{ytm}^{RP,R}_{EUR}(t,T)$$

Note that similarly to the EUR curve procedure, the LC curve one relies on the premise that both the RMV and RP cases must share the same $p^{R}_{LC}(t,T)$, which stands for the probability of default on the LC debt. However, according to the analysis we had in Sect. 2 on the no-arbitrage conditions, due to the monetization, such probability actually does not formally exist. Here it is only a derived quantity since although we assume the same point process as a driver of default on both the LC and EUR debt, we can control the compensator by changing the recoveries. However, we could just take the formulas above for the RP spread as definitions. Taking the limit case of zero EUR debt, they would be entirely consistent to the RP in case of EUR debt, thus providing a justification for our method.

### *3.3 CDS-Bond Basis Empirics*

For illustration we provide visualization of the Z-spread measured basis according to the two alternative ways for a set of European EM countries. They are chosen so that they have both Eurobonds outstanding in EUR and a liquid LC curve. The data sources are: Bloomberg, Datastream, and CBonds. We build the LC curves by employing the Bloomberg BFV curves. Since they are par curves, see Lee [11], we transform them to zero-coupon yield ones. For spreads extraction we use both EUR and USD denominated CDS. We give preference to the former, but in case of missing quotes we use USD quotes instead by making a quanto adjustment using cross currency basis swaps. The countries under focus are: Bulgaria (BGN), Czech Rep. (CZK), Hungary (HUF), Lithuania (LTL), Poland (PLN), Romania (RON), and Slovakia (SKK).

Since there are plenty of bonds outstanding, aggregate measures are presented based on duration weighting. The events: 1—GM turmoil of May 09, 2005, 2—Liquidity crisis of August 09, 2007, 3—Bear Sterns default of March 14, 2008, 4—Lehman default of September 15, 2008, 5—Greek turmoil of April 23, 2010, 6—August 5, 2011—the US rating downgrade, 7—06 May, 2012—ECB refi-rate woes are marked by the vertical dashed lines.

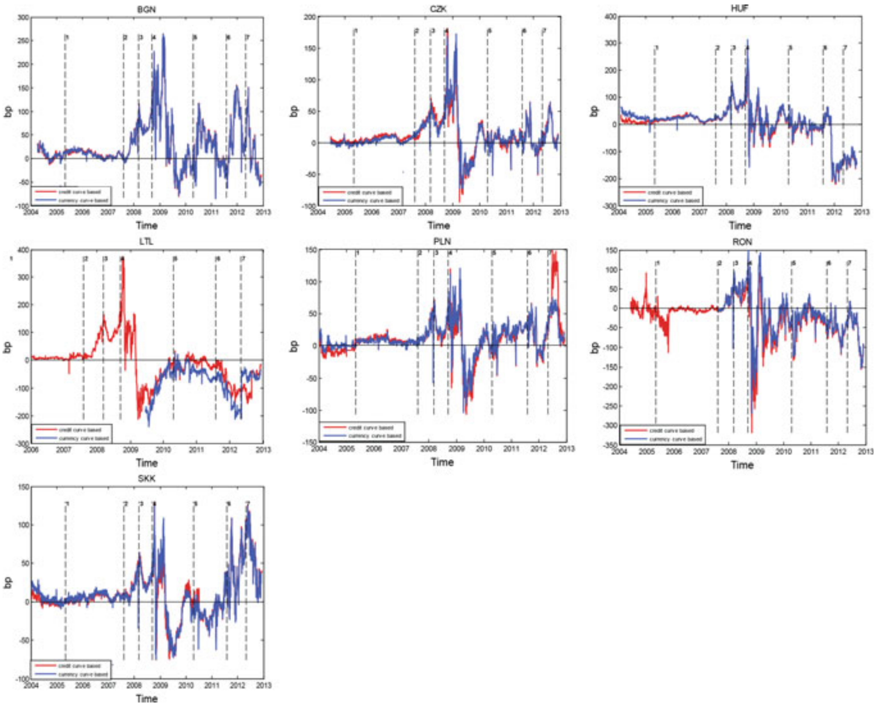


**Fig. 2** CDS-Bond basis across countries

The short conclusion from the patterns in Fig. 2 is that the bonds provide important input for extracting the credit and currency spreads. The two alternative basis formulations preserve general shape similarity, but still give different results that should not be underestimated. This is not surprising since the outcome is driven by the difference in shapes between the benchmark and the LC curves. Market strategists and arbitrage traders have a large scope for interpretations and trades design.

## 4 Conclusion

The paper considers the credit and currency spreads of a risky EM country. The necessary no-arbitrage conditions are derived and their informational content is analyzed. An application of the setting is made to proper building of the foreign and local currency yield curves of a sovereign as well as to providing ideas for relative value diagnostics in a multi-currency framework. In that direction, an alternative measure for the CDS-Bond basis is discussed when the local currency curve is employed as a pillar. The aim of the paper is both to point out the rich opportunities the setting gives for market-related research that could be of use to strategists and policy officers and to make the first several steps toward investigating such opportunities.

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## Appendix

Here we briefly elaborate on the derivation of Eqs. (5.1) and (5.2). Applying the Girsanov's theorem and the Ito's Lemma for jump diffusions to Eq. (2), we get the dynamics:

$$\begin{aligned}\frac{dP^*_{f,EUR}(t,T)}{P^*_{f,EUR}(t,T)} &= \left(-\int_t^T \alpha^*_{EUR}(t,s)ds + r^*_{EUR}(t) + \frac{1}{2}||\int_t^T \sigma^*_{EUR}(t,s)ds||^2\right)dt \\ &-\left(\int_t^T \sigma^*_{EUR}(t,s)ds\right)dW^P(t) \\ &+\int_E (1-q_{f,EUR}(x,t))\left(\exp\left(-\int_t^T \delta^*_{EUR}(x,t,s)ds\right)-1\right)\mu(dx,dt) \\ &-\int_E q_{f,EUR}(x,t)\mu(dx,dt)\end{aligned}$$

$$\begin{aligned}\frac{dP^*_{d,LC}(t,T)}{P^*_{d,LC}(t,T)} &= \left(-\int_t^T \alpha^*_{LC}(t,s)ds + r^*_{LC}(t) + \frac{1}{2}||\int_t^T \sigma^*_{LC}(t,s)ds||^2\right)dt \\ &-\left(\int_t^T \sigma^*_{LC}(t,s)ds\right)dW^P(t) \\ &+\int_E (1-q_{d,LC}(x,t))\left(\exp\left(-\int_t^T \delta^*_{LC}(x,t,s)ds\right)-1\right)\mu(dx,dt) \\ &-\int_E q_{d,LC}(x,t)\mu(dx,dt)\end{aligned}$$

Furthermore, we have the dynamics of the exchange rate:

$$\frac{dX(t)}{X(t)} = \alpha_X(t)dt + \sum_{i=1}^n \sigma_{X,i}(t)dW_i^P(t) - \int_E \delta_X(x,t)\mu(dx,dt)$$

So using the no-arbitrage conditions and equating the expected local drifts to the risk-free rate, we get the results shown.

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