# Appendix A

# Miscellaneous

In this chapter we collect a number of analytic tools that are used at some point in the monograph. These are all well-known, and we present them without proofs providing references to relevant sources when needed. Thus, here we make short expositions of topics including local hypoellipticity and solvability, operator semigroups, fractional powers of operators, singular integrals, almost orthogonality, and the analytic interpolation.

## A.1 General properties of hypoelliptic operators

In this section, we recall the definition and first properties of locally hypoelliptic operators. We will also point out the useful duality between local solvability and local hypoellipticity in Theorem A.1.3.

Roughly speaking, a differential operator $L$ is (locally) *hypoelliptic* if whenever $u$ and $f$ are distributions satisfying $Lu = f$, $u$ must be smooth where $f$ is smooth. Usually, we omit the word 'local' and just speak of hypoellipticity. More precisely:

**Definition A.1.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $L$ be a differential operator on $\Omega$ with smooth coefficients. Then $L$ is said to be *hypoelliptic* if, for any distribution $u \in \mathcal{D}'(\Omega)$ and any open subset $\Omega'$ of $\Omega$, the condition $Lu \in C^\infty(\Omega')$ implies that $u \in C^\infty(\Omega')$.

This definition extends to an open subset of a smooth manifold.

Of course elliptic operators such as Laplace operators are hypoelliptic. Less obvious examples are provided by the celebrated Hörmander's Theorem on sums of squares of vector fields [Hör67a] which we recall here even if we will not use it in this monograph:

**Theorem A.1.2** (Hörmander sum of squares)**.** *Let $X_o, X_1, \ldots, X_p$ be smooth real-valued vector fields on an open set $\Omega \subset \mathbb{R}^n$, and let $c_o \in C^\infty(\Omega)$. We assume*



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*that the vector fields $X_o, X_1, \ldots, X_p$ satisfy Hörmander's condition, that is, the Lie algebra generated by $\{X_o, X_1, \ldots, X_p\}$ is of dimension $n$ at every point of $\Omega$. Then the operator $X_1^2 + \ldots + X_p^2 + X_o + c$ is hypoelliptic on $\Omega$.*

*This extends to smooth manifolds.*

Consequently any sub-Laplacian (see Definition 4.1.6) on a stratified Lie group is hypoelliptic on the whole group since any basis of the first stratum satisfies Hörmander's condition.

Hörmander's condition in Theorem A.1.2 is sufficient but not necessary for the hypoellipticity of sums of squares, thus allowing for sharper versions, see e.g. [BM95].

In the following sense, local hypoellipticity is dual to local solvability:

**Theorem A.1.3.** *Let $L$ be hypoelliptic on $\Omega$. Then $L^t$ is locally solvable at every point of $\Omega$.*

Let us briefly recall the definitions of the local solvability and of transpose:

**Definition A.1.4.** Let $L$ be a linear differential operator with smooth coefficients on $\Omega$. We say that $L$ is *locally solvable* at $x \in \Omega$ if $x$ has an open neighbourhood $V$ in $\Omega$ such that, for every function $f \in \mathcal{D}(V)$ there is a distribution $u \in \mathcal{D}'(V)$ satisfying $Lu = f$ on $V$.

**Definition A.1.5.** The *transpose* of a differential operator $L$ with smooth coefficients on an open subset $\Omega$ of $\mathbb{R}^n$ is the operator, denoted by $L^t$, given by

$$\forall \phi, \psi \in \mathcal{D}(\Omega) \qquad \langle L\phi, \psi \rangle = \langle \phi, L^t \psi \rangle.$$

This extends to manifolds.

Note that if

$$Lf(x) = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha f(x),$$

then

$$L^t f(x) = \sum_{|\alpha| \le m} \partial^\alpha \big(a_\alpha(x) f(x)\big) = \sum_{|\alpha| \le m} b_\alpha(x) \partial^\alpha f(x),$$

where the $b_\alpha$'s are linear combinations of derivatives of the $a_\alpha$'s, in particular they are smooth functions.

We will need the following property:

**Theorem A.1.6** (Schwartz-Trèves)**.** *Let $L$ be a differential operator with smooth coefficients on an open subset $\Omega$ of $\mathbb{R}^n$. We assume that $L$ and $L^t$ are hypoelliptic on $\Omega \subset \mathbb{R}^n$. Then the $\mathcal{D}'(\Omega)$ and $C^\infty(\Omega)$ topologies agree on*

$$N_L(\Omega) = \{f \in \mathcal{D}'(\Omega) \ : \ Lf = 0\}.$$

For its proof, we refer to [Tre67, Corollary 1 in Ch. 52].

## A.2 Semi-groups of operators

In this section we discuss operator semi-groups and their infinitesimal generators.

**Definition A.2.1.** Suppose that for every $t \in (0,\infty)$, there is an associated bounded linear operator $Q(t)$ on a Banach space $\mathcal{X}$ in such a way that

$$\forall s,t>0 \qquad Q(s+t)=Q(s)Q(t).$$

Then the family $\{Q(t)\}_{t>0}$ is called a *semi-group* of operators on $\mathcal{X}$.

If we have for every $x \in \mathcal{X}$, that

$$\|Q(t)x-x\|_{\mathcal{X}} \underset{t\to 0}{\longrightarrow} 0,$$

then the semi-group is said to be *strongly continuous.*

If the operator norm of each $Q(t)$ is less or equal to one, $\|Q(t)\|_{\mathscr{L}(\mathcal{X})} \leq 1$, then the semi-group is called a *contraction* semi-group.

Let $\{Q(t)\}_{t>0}$ be a semi-group of operators on $\mathcal{X}$. If $x \in \mathcal{X}$ is such that $\frac{1}{\epsilon}\left(Q(\epsilon)x-x\right)$ converges in the norm topology of $\mathcal{X}$ as $\epsilon \to 0$, then we denote its limit by $Ax$ and we say that $x$ is in the domain $\mathrm{Dom}(A)$ of $A$. Clearly $\mathrm{Dom}(A)$ is a linear subspace of $\mathcal{X}$ and $A$ is a linear operator on $\mathrm{Dom}(A) \subset X$. This operator is essentially $A = Q'(0)$.

**Definition A.2.2.** The operator $A$ defined just above is called the *infinitesimal generator* of the semi-group $\{Q(t)\}_{t>0}$.

We now collect some properties of semi-groups and their generators.

**Proposition A.2.3.** *Let $\{Q(t)\}_{t>0}$ be a strongly continuous semi-group with infinitesimal generator $A$. We also set $Q(0) := \mathrm{I}$, the identity operator. Then*

1. *there are constants $C,\gamma$ such that for all $t \in [0,\infty)$,*

$$\|Q(t)\|_{\mathscr{L}(\mathcal{X})} \leq Ce^{\gamma t};$$

2. *for every $x \in \mathcal{X}$, the map $[0,\infty) \ni t \mapsto Q(t)x \in \mathcal{X}$ is continuous;*
3. *the operator $A$ is closed with dense domain;*
4. *the differential equation*

$$\partial_t Q(t)x = Q(t)A\,x = AQ(t)x,$$

*holds for every $x \in \mathrm{Dom}(A)$ and $t \geq 0$;*

5. *for every $x \in \mathcal{X}$ and $t>0$,*

$$Q(t)x = \lim_{\epsilon\to 0} \exp(tA_\epsilon)x,$$

*where*

$$A_\epsilon = \frac{1}{\epsilon}\,(Q(\epsilon) - \mathrm{I}) \quad \textit{and} \quad \exp(tA_\epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!}(tA_\epsilon)^k;$$

*furthermore the convergence is uniform on every compact subset of* $[0,\infty)$*;*

6. *if* $\lambda \in \mathbb{C}$ *and* $\operatorname{Re}\lambda > \gamma$ *(where* $\gamma$ *is any constant such that (1) holds), the integral*

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} Q(t)x\,dt,$$

*defines a bounded linear operator* $R(\lambda)$ *on* $\mathcal{X}$ *(* often called the resolvent of the semi-group $\{Q(t)\}$*) whose range is* $\operatorname{Dom}(A)$ *and which inverts* $\lambda\mathrm{I} - A$*. In particular, the spectrum of* $A$ *lies in the half plane* $\{\lambda : \operatorname{Re}\lambda \leq \gamma\}$*.*

For the proof, see e.g. Rudin [Rud91, §13.35].

**Theorem A.2.4** (Hille-Yosida)**.** *A densely defined operator* $A$ *on a Banach space* $\mathcal{X}$ *is the infinitesimal generator of a strongly continuous semi-group* $\{Q(t)\}_{t>0}$ *if and only if there are constants* $C, \gamma$ *such that*

$$\forall \lambda > \gamma,\ m \in \mathbb{N} \qquad \|(\lambda\mathrm{I} - A)^{-m}\| \leq C(\lambda - \gamma)^{-m}.$$

*The constant* $\gamma$ *can be taken as in Proposition A.2.3.*

For the proof of the Hille-Yosida Theorem, see e.g [Rud91, §13.37].

In this case the operators of the semi-group $\{Q(t)\}_{t>0}$ generated by $A$ are denoted by

$$Q(t) = e^{tA}.$$

**Theorem A.2.5** (Lumer-Phillips)**.** *A densely defined operator* $A$ *on a Banach space* $\mathcal{X}$ *is the infinitesimal generator of a strongly continuous contraction semi-group* $\{Q(t)\}_{t>0}$ *if and only if*

- *A is dissipative, i.e.*

$$\forall \lambda > 0,\ x \in \operatorname{Dom}(A) \qquad \|(\lambda\mathrm{I} - A)x\| \geq \lambda\|x\|;$$

- *there is at least one* $\lambda_o$ *such that* $A - \lambda_o\mathrm{I}$ *is surjective.*

For the proof of the Lumer-Phillips Theorem, see [LP61].

For this monograph, the facts given in this section will be enough. We refer for the general theory of semi-groups to the fundamental work of Hille and Phillips [HP57], or to later expositions e.g. by Davies [Dav80] or Pazy [Paz83].

## A.3 Fractional powers of operators

Here we summarise the definition of fractional powers for certain operators. We refer the interested reader to the monograph of Martinez and Sanz [MCSA01] and all the explanations and historical discussions therein.

Let $A : \mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ be a linear operator on a Banach space $\mathcal{X}$. In order to present only the part of the theory that we use in this monograph, we make the following assumptions

(i) The operator $A$ is closed and densely defined.

(ii) The operator $A$ is injective, that is, $A$ is one-to-one on its domain.

(iii) The operator $A$ is Komatsu-non-negative, that is, $(-\infty, 0)$ is included in the resolvent $\rho(A)$ of $A$ and

$$\exists M > 0 \quad \forall \lambda > 0 \quad \|(\lambda + A)^{-1}\| \leq M\lambda^{-1}.$$

*Remark* A.3.1. This implies (cf. [MCSA01, Proposition 1.1.3 (iii)]) that for all $n, m \in \mathbb{N}$, $\mathrm{Dom}(A^n)$ is dense in $\mathcal{X}$, and $\mathrm{Range}(A^m)$ as well as $\mathrm{Dom}(A^n) \cap \mathrm{Range}(A^m)$ are dense in the closure of $\mathrm{Range}(A)$.

The powers $A^n$, $n \in \mathbb{N}$, are defined using iteratively the following definition:

**Definition A.3.2.** The product of two (possibly) unbounded operators $A$ and $B$ acting on the same Banach space $\mathcal{X}$ is as follows. A vector $x$ is in the domain of the operator $AB$ whenever $x$ is in the domain of $B$ and $Bx$ is in the domain of $A$. In this case $(AB)(x) = A(Bx)$.

*Remark* A.3.3. Note that if an operator $A$ satisfies (i), (ii) and (iii), then it is also the case for $\mathrm{I} + A$.

Following Balakrishnan (cf. [MCSA01, Section 3.1]), the (Balakrishnan) operators $J^\alpha$, $\alpha \in \mathbb{C}_+ := \{z \in \mathbb{C},\ \mathrm{Re}\, z > 0\}$, are (densely) defined by the following:

- If $0 < \mathrm{Re}\, \alpha < 1$, $\mathrm{Dom}(J^\alpha) := \mathrm{Dom}(A)$ and for $\phi \in \mathrm{Dom}(A)$,

$$J^\alpha \phi := \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda \mathrm{I} + A)^{-1} A\phi \, d\lambda.$$

- If $\mathrm{Re}\, \alpha = 1$, $\mathrm{Dom}(J^\alpha) := \mathrm{Dom}(A^2)$ and for $\phi \in \mathrm{Dom}(A^2)$,

$$J^\alpha \phi := \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} \left[ (\lambda \mathrm{I} + A)^{-1} - \frac{\lambda}{\lambda^2 + 1} \right] A\phi \, d\lambda + \sin \frac{\alpha\pi}{2} A\phi.$$

- If $n < \mathrm{Re}\, \alpha < n + 1$, $n \in \mathbb{N}$, $\mathrm{Dom}(J^\alpha) := \mathrm{Dom}(A^{n+1})$ and for $\phi \in \mathrm{Dom}(A)$,

$$J^\alpha \phi := J^{\alpha - n} A^n \phi.$$

- If $\operatorname{Re}\alpha = n+1$, $n \in \mathbb{N}$, $\operatorname{Dom}(J^\alpha) := \operatorname{Dom}(A^{n+2})$ and for $\phi \in \operatorname{Dom}(A^{n+2})$,

$$J^\alpha \phi := J^{\alpha-n} A^n \phi.$$

We now define fractional powers distinguishing between three different cases:

**Case 0:** $A$ is bounded.

**Case I:** $A$ is unbounded and $0 \in \rho(A)$, that is, the resolvent of $A$ contains zero; in other words, $A^{-1}$ is bounded.

**Case II:** $A$ is unbounded and $0 \in \sigma(A)$, that is, the spectrum of $A$ contains zero.

The fractional powers $A^\alpha$, $\alpha \in \mathbb{C}_+$, are defined in the following way (cf. [MCSA01, Section 5.1]):

**Case 0:** $A$ being bounded, $J^\alpha$ is bounded and we define $A^\alpha := J^\alpha$, $\alpha \in \mathbb{C}_+$.

**Case I:** $A^{-1}$ being bounded, we can use Case 0 to define $(A^{-1})^\alpha$ which is injective; then we define

$$A^\alpha := \left[(A^{-1})^\alpha\right]^{-1} \quad (\alpha \in \mathbb{C}_+).$$

**Case II:** Using Case I for $A + \epsilon\mathrm{I}$, $\epsilon > 0$, we define

$$A^\alpha := \lim_{\epsilon \to 0} (A + \epsilon\mathrm{I})^\alpha \quad (\alpha \in \mathbb{C}_+);$$

that is, the domain of $A^\alpha$ is composed of all the elements $\phi \in \operatorname{Dom}\left[(A+\epsilon\mathrm{I})^\alpha\right]$, $\epsilon > 0$ close to zero, and such that $(A+\epsilon\mathrm{I})\phi$ is convergent for the norm topology of $\mathcal{X}$ as $\epsilon \to 0$; the limit defines $A^\alpha\phi$.

In all cases, $J^\alpha$ is closable and we have (cf. [MCSA01, Theorem 5.2.1]):

$$A^\alpha = (A+\lambda\mathrm{I})^n \overline{J^\alpha} (A+\lambda\mathrm{I})^{-n} \qquad (\alpha \in \mathbb{C}_+,\ \lambda \in \rho(-A),\ n \in \mathbb{N}).$$

Hence $A^\alpha$, $\alpha \in \mathbb{C}_+$, can be understood as the maximal domain operator which extends $J^\alpha$ and commutes with the resolvent of $A$ (in other words *commutes strongly with* $A$).

We can now define the powers for complex numbers also with non-positive real parts (cf. [MCSA01, Section 7.1]):

- Given $\alpha \in \mathbb{C}_+$, the operators $A^\alpha$, $\alpha \in \mathbb{C}_+$, are injective, and we can define

$$A^{-\alpha} := (A^\alpha)^{-1}.$$

- Given $\tau \in \mathbb{R}$, we define

$$A^{i\tau} := (A+\mathrm{I})^2 A^{-1} A^{1+i\tau} (A+\mathrm{I})^{-2}.$$

We now collect properties of fractional powers.

**Theorem A.3.4.** *Let $A : \mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ be a linear operator on a Banach space $\mathcal{X}$. Assume that the operator $A$ satisfies Properties (i), (ii) and (iii), and define its fractional powers $A^\alpha$ as above.*

1. *For every $\alpha \in \mathbb{C}$, the operator $A^\alpha$ is closed and injective with $(A^\alpha)^{-1} = A^{-\alpha}$. In particular, $A^0 = \mathrm{I}$.*
2. *For $\alpha \in \mathbb{C}_+$, the operator $A^\alpha$ coincides with the closure of $J^\alpha$.*
3. *If $A$ has dense range and for all $\tau \in \mathbb{R}$, $A^{i\tau}$ is bounded, then there exist $C > 0$ and $\theta \in (0, \pi)$ such that*
$$\forall \tau \in \mathbb{R} \qquad \|A^{i\tau}\|_{\mathscr{L}(\mathcal{X})} \leq Ce^{\theta\tau}.$$
*Given $\tau \in \mathbb{R}\backslash\{0\}$, if $A^{i\tau}$ is bounded then $\mathrm{Dom}(A^\alpha) \subset \mathrm{Dom}(A^{\alpha+i\tau})$ for all $\alpha \in \mathbb{R}$. Conversely, if $\mathrm{Dom}(A^\alpha) \subset \mathrm{Dom}(A^{\alpha+i\tau})$ for all $\alpha \in \mathbb{R}\backslash\{0\}$, then $A^{i\tau}$ is bounded.*
4. *For any $\alpha, \beta \in \mathbb{C}$, we have $A^\alpha A^\beta \subset A^{\alpha+\beta}$, and if $\mathrm{Range}(A)$ is dense in $\mathcal{X}$ then the closure of $A^\alpha A^\beta$ is $A^{\alpha+\beta}$.*
5. *Let $\alpha_o \in \mathbb{C}_+$.*
   - *If $\phi \in \mathrm{Range}(A^{\alpha_o})$ then $\phi \in \mathrm{Dom}(A^\alpha)$ for all $\alpha \in \mathbb{C}$ with $0 < -\mathrm{Re}\,\alpha < \mathrm{Re}\,\alpha_o$ and the function $\alpha \mapsto A^\alpha\phi$ is holomorphic in $\{\alpha \in \mathbb{C} : -\mathrm{Re}\,\alpha_o < \mathrm{Re}\,\alpha < 0\}$.*
   - *If $\phi \in \mathrm{Dom}(A^{\alpha_o})$ then $\phi \in \mathrm{Dom}(A^\alpha)$ for all $\alpha \in \mathbb{C}$ with $0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\alpha_o$ and the function $\alpha \mapsto A^\alpha\phi$ is holomorphic in $\{\alpha \in \mathbb{C} : 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\alpha_o\}$.*
   - *If $\phi \in \mathrm{Dom}(A^{\alpha_o}) \cap \mathrm{Range}(A^{\alpha_o})$ then $\phi \in \mathrm{Dom}(A^\alpha)$ for all $\alpha \in \mathbb{C}$ with $|\mathrm{Re}\,\alpha| < \mathrm{Re}\,\alpha_o$ and the function $\alpha \mapsto A^\alpha\phi$ is holomorphic in $\{\alpha \in \mathbb{C} : -\mathrm{Re}\,\alpha_o < \mathrm{Re}\,\alpha < \mathrm{Re}\,\alpha_o\}$.*
6. *If $\alpha, \beta \in \mathbb{C}_+$ with $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha$, then*
$$\exists C = C_{A,\alpha,\beta} > 0 \quad \forall \phi \in \mathrm{Dom}(A^\beta) \quad \|A^\alpha\phi\|_{\mathcal{X}} \leq C\|\phi\|_{\mathcal{X}}^{1-\frac{\mathrm{Re}\,\alpha}{\mathrm{Re}\,\beta}}\|A^\beta\phi\|_{\mathcal{X}}^{\frac{\mathrm{Re}\,\alpha}{\mathrm{Re}\,\beta}}.$$
7. *If $B^*$ denotes the dual of an operator $B$ on $\mathcal{X}$, then $(A^\alpha)^* = (A^*)^\alpha$.*
8. *For $\alpha \in \mathbb{C}_+$ and $\epsilon > 0$, $\mathrm{Dom}\left[(A + \epsilon\mathrm{I})^\alpha\right] = \mathrm{Dom}(A^\alpha)$.*
9. *Let $\tau \in \mathbb{R}$. Let $S_{i\tau}$ be the strong limit of $(A + \epsilon\mathrm{I})^{i\tau}$ as $\epsilon \to 0^+$, with domain $\mathrm{Dom}(S_{i\tau}) = \{\phi \in \mathrm{Dom}\left[(A + \epsilon)^{i\tau}\right] : \exists \lim_{\epsilon\to 0^+}(A + \epsilon)^{i\tau}\phi\}$. Then $S_{i\tau}$ is closable and the closure of (the graph of) $J^{i\tau}$ is included in the closure of (the graph of) $S_{i\tau}$ which is included in (the graph of) $A^{i\tau}$.*

   *In particular, if $A$ has dense domain and range, then the closure of $S_{i\tau}$ is $A^{i\tau}$.*

10. *Let us assume that $A$ generates an equibounded semi-group $\{e^{-tA}\}_{t>0}$ on $\mathcal{X}$, that is,*
$$\exists M \qquad \forall t>0 \qquad \|e^{-tA}\|_{\mathcal{X}} \le M. \tag{A.1}$$
*If $0<\operatorname{Re}\alpha<1$ and $\phi \in \operatorname{Range}(A)$ then*
$$A^{-\alpha}\phi = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-tA}\phi\,dt, \tag{A.2}$$
*in the sense that $\lim_{N\to\infty}\int_0^N$ converges in the $\mathcal{X}$-norm.*

*Moreover, if $\{e^{-tA}\}_{t>0}$ is exponentially stable, that is,*
$$\exists M,\mu>0 \quad \forall t>0 \qquad \|e^{-tA}\|_{\mathscr{L}(\mathcal{X})} \le Me^{-t\mu},$$
*then Formula (A.2) holds for all $\alpha \in \mathbb{C}_+$ and $\phi \in \mathcal{X}$, and the integral converges absolutely: $\int_0^\infty \|t^{\alpha-1}e^{-tA}\phi\|_{\mathcal{X}}dt < \infty$.*

References for these results are in [MCSA01] as follows:

(1) Corollary 5.2.4 and Section 7.1;

(2) Corollary 5.1.12;

(3) Proposition 8.1.1, Section 7.1 and Corollary 7.1.2;

(4) Theorem 7.1.1;

(5) Proposition 7.1.5 with its proof, and Corollary 5.1.13;

(6) Corollary 5.1.13;

(7) Corollary 5.2.4 for $\alpha \in \mathbb{C}_+$, consequently for any $\alpha \in \mathbb{C}$;

(8) Theorem 5.1.7;

(9) Theorem 7.4.6;

(10) Lemma 6.1.5.

In Theorem A.3.4 Part (10), $\Gamma$ denotes the Gamma function. Let us recall briefly its definition. For each $\alpha \in \mathbb{C}_+$, it is defined by the convergent integral
$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt.$$
A direct computation gives $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$ and an integration by parts yields the functional equation $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$. Hence the Gamma function coincides with the factorial in the sense that if $\alpha \in \mathbb{N}$, then the equality $\Gamma(\alpha) = (\alpha-1)!$ holds. It is easy to see that $\Gamma$ is analytic on the half plane $\{\operatorname{Re}\alpha > 0\}$. Because of the functional equation, it admits a unique analytic continuation to the whole complex plane except for non-positive integers where it has simple pole. We keep the same notation $\Gamma$ for its analytic continuation.

For $\operatorname{Re} z > 0$, we have the Sterling estimate

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(\frac{1}{z})). \tag{A.3}$$

Also, the following known relation will be of use to us,

$$\int_{t=0}^{1} t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re} x > 0, \ \operatorname{Re} y > 0. \tag{A.4}$$

We will use Part (6) also in the following form: let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\operatorname{Re}\alpha < \operatorname{Re}\beta$ and $\operatorname{Re}\alpha \leq \operatorname{Re}\gamma \leq \operatorname{Re}\beta$; then there exists $C = C_{\alpha,\beta,\gamma,A} > 0$ such that for any $f \in \operatorname{Dom}(A^\alpha)$ with $A^\alpha f \in \operatorname{Dom}(A^{\beta-\alpha})$, we have

$$\|A^\gamma f\|_{\mathcal{X}} \leq C \|A^\alpha f\|_{\mathcal{X}}^{1-\theta} \|A^\beta f\|_{\mathcal{X}}^{\theta} \qquad \text{where} \quad \theta := \frac{\operatorname{Re}(\gamma - \alpha)}{\operatorname{Re}(\beta - \alpha)}.$$

## A.4 Singular integrals (according to Coifman-Weiss)

The operators appearing ‘in practice’ in the theory of partial differential equations on $\mathbb{R}^n$ often have kernels $\kappa$ satisfying the following properties:

1. the restriction of $\kappa(x, y)$ to $(\mathbb{R}^n_x \times \mathbb{R}^n_y) \backslash \{x = y\}$ coincides with a smooth function $\kappa_o = \kappa_o(x, y) \in C^\infty((\mathbb{R}^n_x \times \mathbb{R}^n_y) \backslash \{x = y\})$;
2. away from the diagonal $x = y$, the function $\kappa_o$ decays rapidly;
3. at the diagonal, $\kappa_o$ is singular but not completely wild: $\kappa_o$ and some of its first derivatives admit a control of the form $|\kappa_o(x, y)| \leq C_x |x - y|^k$ for some power $k \in (-\infty, \infty)$ with $C_x$ varying slowly in $x$.

These types of operators include all the (Hörmander, Shubin, semi-classical, ...) pseudo-differential operators, and these types of operators appear when looking for fundamental solutions or parametrices of differential operators.

In general, we want our operator $T$ to map continuously some well-known functional space to another. For example, we are looking for conditions to ensure that our operator extends to a bounded operator from $L^p$ to $L^q$. This is the subject of the theory of singular integrals on $\mathbb{R}^n$, especially when the power $k$ above equals $-n$. In the classical Euclidean case, we refer to the monograph [Ste93] by Stein for a detailed presentation of this theory.

Here, let us present the main lines of the generalisation of the theory of singular integrals to the setting of ‘spaces of homogeneous type’ where there is no (apparent) trace of a group structure. This generalisation is relevant for us since examples of such spaces are compact manifolds and homogeneous nilpotent Lie groups. We omit the proofs, referring to [CW71a, Chapitre III] for details.

**Definition A.4.1.** A *quasi-distance* on a set $X$ is a function $d : X \times X \to [0, \infty)$ such that

1. $d(x, y) > 0$ if and only if $x \neq y$;
2. $d(x, y) = d(y, x)$;
3. there exists a constant $K > 0$ such that

$$\forall x, y, z \in X \qquad d(x, z) \leq K \left( d(x, y) + d(y, z) \right).$$

We call

$$B(x, r) := \{y \in G \ : \ d(x, y) < r\},$$

the *quasi-ball of radius* $r$ *around* $x$.

**Definition A.4.2.** A *space of homogeneous type* is a topological space $X$ endowed with a quasi-distance $d$ such that

1. The quasi-balls $B(x, r)$ form a basis of open neighbourhood at $x$;
2. homogeneity property

   there exists $N \in \mathbb{N}$ such that for every $x \in X$ and every $r > 0$ the ball $B(x, r)$ contains at most $N$ points $x_i$ such that $d(x_i, x_j) > r/2$.

The constants $K$ in Definition A.4.1 and $N$ in Definition A.4.2 are called the constants of the space of homogeneous type $X$.

Some authors (like in the original text of [CW71a]) prefer using the vocabulary pseudo-norms, pseudo-distance, etc. instead of quasi-norms, quasi-distance, etc. In this monograph, following e.g. both Stein [Ste93] and Wikipedia, we choose the perhaps more widely adapted convention of the term quasi-norm.

*Examples of spaces of homogeneous type:*

1. A homogeneous Lie group endowed with the quasi-distance associated to any homogeneous quasi-norm (see Lemma 3.2.12).
2. The unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with the quasi-distance

$$d(x, y) = |1 - x \cdot y|^{\alpha},$$

   where $\alpha > 0$ and $x \cdot y = \sum_{j=1}^{n} x_j y_j$ is the real scalar product of $x, y \in \mathbb{R}^n$.
3. The unit sphere $\mathbb{S}^{2n-1}$ embedded in $\mathbb{C}^n$ with the quasi-distance

$$d(z, w) = |1 - (z, w)|^{\alpha},$$

   where $\alpha > 0$ and $(z, w) = \sum_{j=1}^{n} z_j \bar{w}_j$.
4. Any compact Riemannian manifold.

The proof that these spaces are effectively of homogeneous type comes easily from the following lemma:

**Lemma A.4.3.** *Let $X$ be a topological set endowed with a quasi-distance $d$ satisfying (1) of Definition A.4.2.*

*Assume that there exist a Borel measure $\mu$ on $X$ satisfying*

$$0 < \mu\left(B(x,r)\right) \leq C\mu\left(B(x,\frac{r}{2})\right) < \infty. \tag{A.5}$$

*Then $X$ is a space of homogeneous type.*

The condition (A.5) is called the *doubling condition.* For instance, the Riemannian measure of a Riemannian compact manifold or the Haar measure of a homogeneous Lie group satisfy the doubling condition; we omit the proof of these facts, as well as the proof of Lemma A.4.3.

Let $(X,d)$ be a space of homogeneous type. The hypotheses are 'just right' to obtain a covering lemma. We assume now that $X$ is also equipped with a measure $\mu$ satisfying the doubling condition (A.5). A maximal function with respect to the quasi-balls may be defined. Then given a level, any function $f$ can be decomposed 'in the usual way' into good and bad functions $f = g + \sum_j b_j$. The Euclidean proof of the Singular Integral Theorem can be adapted to obtain

**Theorem A.4.4** (Singular integrals)**.** *Let $(X,d)$ be a space of homogeneous type equipped with a measure $\mu$ satisfying the doubling condition given in (A.5).*

*Let $T$ be an operator which is bounded on $L^2(X)$:*

$$\exists C_o \qquad \forall f \in L^2 \quad \|Tf\|_2 \leq C_o\|f\|_2. \tag{A.6}$$

*We assume that there exists a locally integrable function $\kappa$ on $(X \times X)\setminus \{(x,y) \in X \times X : x = y\}$ such that for any compactly supported function $f \in L^2(X)$, we have*

$$\forall x \notin \operatorname{supp} f \qquad Tf(x) = \int_X \kappa(x,y) f(y) d\mu(y).$$

*We also assume that there exist $C_1, C_2 > 0$ such that*

$$\forall y, y_o \in X \qquad \int_{d(x,y_o) > C_1 d(y,y_o)} |\kappa(x,y) - \kappa(x,y_o)| d\mu(x) \leq C_2. \tag{A.7}$$

*Then for all $p$, $1 < p \leq 2$, $T$ extends to a bounded operator on $L^p$ because*

$$\exists A_p \qquad \forall f \in L^2 \cap L^p \quad \|Tf\|_p \leq A_p \|f\|_p;$$

*for $p = 1$, the operator $T$ extends to a weak-type (1,1) operator since*

$$\exists A_1 \qquad \forall f \in L^2 \cap L^1 \quad \mu\{x \ : \ |Tf(x)| > \alpha\} \leq A_1 \frac{\|f\|_1}{\alpha};$$

*the constants $A_p$, $1 \leq p \leq 2$, depend only on $C_o$, $C_1$ and $C_2$.*

*Remark* A.4.5. 1. In the statement of the fundamental theorem of singular integrals on spaces of homogeneous types, cf. [CW71a, Théorème 2.4 Chapitre III], the kernel $\kappa$ is assumed to be square integrable in $L^2(X \times X)$. However, the proof requires only that the kernel $\kappa$ is locally integrable away from the diagonal, beside the $L^2$-boundedness of the operator $T$. We have therefore chosen to state it in the form given above.

2. Following the constants in the proof of [CW71a, Théorème 2.4 Chapitre III], we find

$$A_2 = C_1 \qquad \text{and} \qquad A_1 = C(C_1^2 + C_3),$$

where $C$ is a constant which depends only on the constants of the space of homogeneous type. The constants $A_p$ for $p \in (1,2)$ are obtained via the constants appearing in the Marcinkiewicz interpolation theorem (see e.g. [DiB02, Theorem 9.1]):

$$A_p = \frac{2p}{(2-p)(1-p)} A_1^\delta A_2^{1-\delta} \quad \text{with } \delta = 2(\frac{1}{p} - \frac{1}{2}).$$

Let us discuss the two main hypotheses of Theorem A.4.4.

*About Condition (A.7) in the Euclidean case.* As explained at the beginning of this section, we are interested in 'nice' kernels $\kappa_o(x,y)$ with a control of the form $|\kappa_o(x,y)| \le C_x |x-y|^k$ with a particular interest for $k = -n$, and similar estimates for their derivatives with power $-n-1$. Hence they should satisfy Condition (A.7). They are called Calderón-Zygmund kernels, which we now briefly recall:

### Calderón-Zygmund kernels on $\mathbb{R}^n$

A *Calderón-Zygmund kernel* on $\mathbb{R}^n$ is a measurable function $\kappa_o$ defined on $(\mathbb{R}^n_x \times \mathbb{R}^n_y)\backslash\{x = y\}$ satisfying for some $\gamma$, $0 < \gamma \le 1$, the inequalities

$$\begin{aligned} |\kappa_o(x,y)| &\le A|x-y|^{-n}, \\ |\kappa_o(x,y) - \kappa_o(x',y)| &\le A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \qquad \text{if } |x-x'| \le \frac{|x-y|}{2}, \\ |\kappa_o(x,y) - \kappa_o(x,y')| &\le A\frac{|y-y'|^\gamma}{|x-y|^{n+\gamma}} \qquad \text{if } |y-y'| \le \frac{|x-y|}{2}. \end{aligned}$$

Sometimes the condition of Calderón-Zygmund kernels refers to a smooth function $\kappa_o$ defined on $(\mathbb{R}^n_x \times \mathbb{R}^n_y)\backslash\{x = y\}$ satisfying

$$\forall \alpha, \beta \quad \exists C_{\alpha,\beta} \qquad \left|\partial_x^\alpha \partial_y^\beta \kappa_o(x,y)\right| \le C_{\alpha,\beta}|x-y|^{-n-\alpha-\beta}.$$

For a detailed discussion, the reader is directed to [Ste93, ch.VII].

A *Calderón-Zygmund operator* on $\mathbb{R}^n$ is an operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ such that the restriction of its kernel $\kappa$ to $(\mathbb{R}^n_x \times \mathbb{R}^n_y)\backslash\{x = y\}$ is a Calderón-Zygmund kernel $\kappa_o$. In other words, $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a Calderón-Zygmund operator if there exists a Calderón-Zygmund kernel $\kappa_o$ satisfying

$$Tf(x) = \int_{\mathbb{R}^n} \kappa_o(x,y)f(y)dy,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ with compact support and $x \in \mathbb{R}^n$ outside the support of $f$.

The Calderón-Zygmund conditions imply Condition (A.7) for the operator $T$ and its formal adjoint $T^*$ but they are not sufficient to imply the $L^2$-boundedness for which some additional 'cancellation' conditions are needed.

*About Condition (A.6).* The difficulty with applying the main theorem of singular integrals (i.e. Theorem A.4.4) is often to know that the operator is $L^2$-bounded. The next section explains the Cotlar-Stein lemma which may help to prove the $L^2$-boundedness in many cases.

## A.5 Almost orthogonality

On $\mathbb{R}^n$, a convolution operator (for the usual convolution) is bounded on $L^2(\mathbb{R}^n)$ if and only if the Fourier transform of its kernel is bounded. Similar result is valid on compact Lie groups, see (2.23), and more generally on any Hausdorff locally compact separable group, see the decomposition of group von Neumann algebras in the abstract Plancherel theorem in Theorem B.2.32. For operators on spaces without readily available Fourier transform or with no control on the Fourier transform of its kernel, or for non-convolution operators this becomes more complicated (however, see Theorem 2.2.5 for the case of non-invariant operators on compact Lie groups).

Fortunately, the space $L^2$ is a Hilbert space and to prove that an operator is bounded on $L^2$, it suffices to do the same for $TT^*$ (or $T^*T$). The reason that this observation is useful in practice is that if $T$ is formally representable by a kernel $\kappa$ (see Schwartz kernel theorem, Theorem 1.4.1), then $T^*T$ is representable by the kernel

$$\int \overline{\kappa(z,x)}\kappa(z,y)\, dz;$$

the latter kernel is often better than $\kappa$ because the integration can have a smoothing effect and/or can take into account the cancellation properties of $\kappa$. This remark alone does not always suffice to prove the $L^2$-boundedness. Sometimes some 'smart' decomposition $T = \sum_k T_k$ of the operator is needed and again the properties of a Hilbert space may help.

The next statement is an easy case of 'exact' orthogonality:

**Proposition A.5.1.** *Let $\mathcal{H}$ be a Hilbert space and let $\{T_k,\ k \in \mathbb{Z}\}$ be a sequence of linear operators on $\mathcal{H}$. We assume that the operators $\{T_k\}$ are uniformly bounded:*

$$\exists C > 0 \qquad \forall k \in \mathbb{Z} \qquad \|T_k\|_{\mathscr{L}(\mathcal{H})} \leq C,$$

*and that*

$$\forall j \neq k \qquad T_j^* T_k = 0 \quad \text{and} \quad T_j T_k^* = 0. \tag{A.8}$$

*Then the series $\sum_{k\in\mathbb{Z}} T_k$ converges in the strong operator norm topology to an operator $S$ satisfying $\|S\|_{\mathscr{L}(\mathcal{H})} \leq C$.*

Note that (A.8) is equivalent to

$$\forall j \neq k \qquad (\ker T_j)^\perp \perp (\ker T_k)^\perp \quad \text{and} \quad \operatorname{Im} T_j \perp \operatorname{Im} T_k.$$

*Proof.* Let $v \in \mathcal{H}$ and $N \in \mathbb{N}$. Since the images of the $T_j$'s are orthogonal, the Pythagoras equality implies

$$\| \sum_{|j|\leq N} T_j v\|^2 = \sum_{|j|\leq N} \|T_j v\|^2.$$

Denoting by $P_j$ the orthogonal projection onto $(\ker T_j)^\perp$, we have

$$\|T_j v\| = \|T_j P_j v\| \leq C\|P_j v\|,$$

since $\|T_j\|_{\mathscr{L}(\mathcal{H})} \leq C$. Thus

$$\| \sum_{|j|\leq N} T_j v\|^2 \leq C^2 \sum_{|j|\leq N} \|P_j v\|^2.$$

As the kernels of the $T_j$'s are mutually orthogonal, we have

$$\sum_{|j|\leq N} \|P_j v\|^2 \leq \|v\|^2.$$

We have obtained that

$$\| \sum_{|j|\leq N} T_j v\|^2 \leq C^2 \|v\|^2,$$

for any $N \in \mathbb{N}$ and $v \in \mathcal{H}$. The constant $C$ here is the uniform bound of the operator norms of the $T_j$'s and is independent of $v$ or $N$. The same proof shows that the sequence $(\sum_{|j|\leq N} T_j v)_{N\in\mathbb{N}}$ is Cauchy when $v$ is in a finite number of $(\ker T_j)^\perp$. This allows us to define the operator $S$ on the dense subspace $\sum_j (\ker T_j)^\perp$. The conclusion follows. $\square$

In practice, the orthogonality assumption above is rather demanding, and is often substituted by a condition of 'almost' orthogonality:

**Theorem A.5.2** (Cotlar-Stein lemma). *Let $\mathcal{H}$ be a Hilbert space and $\{T_k,\ k \in \mathbb{Z}\}$ be a sequence of linear operators on $\mathcal{H}$. We assume that we are given a sequence of positive constants $\{\gamma_j\}_{j=-\infty}^{\infty}$ with*

$$A = \sum_{j=-\infty}^{\infty} \gamma_j < \infty.$$

*If for any $i, j \in \mathbb{Z}$,*

$$\max\left(\|T_i^* T_j\|_{\mathscr{L}(\mathcal{H})}, \|T_i T_j^*\|_{\mathscr{L}(\mathcal{H})}\right) \leq \gamma_{i-j}^2,$$

*then the series $\sum_{k\in\mathbb{Z}} T_k$ converges in the strong operator topology to an operator $S$ satisfying $\|S\|_{\mathscr{L}(\mathcal{H})} \leq A$.*

For the proof of the Cotlar-Stein lemma, see e.g. [Ste93, Ch. VII §2], and for its history see Knapp and Stein [KS69].

When working on groups, one sometimes has to deal with operators mapping the $L^2$-space on the group to the $L^2$-space on its unitary dual. This requires one to use the version of Cotlar's lemma for operators mapping between two different Hilbert spaces. In this case, the statement of Theorem A.5.2 still holds, for an operator $T : \mathcal{H} \to \mathcal{G}$, provided we take the operator norms $T_i^* T_j$ and $T_i T_j^*$ in appropriate spaces. For details, we refer to [RT10a, Theorem 4.14.1].

The following crude version of the Cotlar lemma will be also useful to us:

**Proposition A.5.3** (Cotlar-Stein lemma; crude version). *Let $\mathcal{H}$ be a Hilbert space and $\{T_k,\ k \in \mathbb{Z}\}$ be a sequence of linear operators on $\mathcal{H}$. We assume that*

$$T_i T_j^* = 0 \qquad \text{if } i \neq j. \tag{A.9}$$

*We also assume that the operators $T_k$, $k \in \mathbb{Z}$, are uniformly bounded,*

$$\text{i.e.} \quad \sup_{k\in\mathbb{Z}} \|T_k\|_{\mathscr{L}(\mathcal{H})} < \infty, \tag{A.10}$$

*and that the following sum is finite*

$$\sum_{i\neq j} \|T_i^* T_j\|_{\mathscr{L}(\mathcal{H})} < \infty. \tag{A.11}$$

*Then the series $\sum_{k\in\mathbb{Z}} T_k$ converges in the strong operator topology to an operator $S$ satisfying*

$$\|S\|_{\mathscr{L}(\mathcal{H})}^2 \leq 2\max\left(\sup_{k\in\mathbb{Z}} \|T_k\|_{\mathscr{L}(\mathcal{H})}^2, \sum_{i\neq j} \|T_i^* T_j\|_{\mathscr{L}(\mathcal{H})}\right).$$

For the proof of this statement, see [Ste93, Ch. VII §2.3].

*Remark* A.5.4. The condition (A.9) can can be relaxed slightly with the following modifications.

For instance, (A.9) can be replaced with

$$T_i^* T_j = 0 \quad \text{if} \quad i \neq j \text{ have the same parity.}$$

(This condition appears often when considering dyadic decomposition.) Indeed, applying Proposition A.5.3 to $\{T_{2k+1}\}_{k\in\mathbb{Z}}$ and to $\{T_{2k}\}_{k\in\mathbb{Z}}$, we obtain that the series $\sum_k T_k = \sum_k T_{2k} + \sum_k T_{2k+1}$ converges in the strong operator norm topology to an operator $S$ satisfying

$$\|S\|_{\mathscr{L}(\mathcal{H})} \leq 2^{1/2} \times 2 \times \max\left( \sup_{k\in\mathbb{Z}} \|T_k\|_{\mathscr{L}(\mathcal{H})}, \big(2 \sum_{i-j\in 2\mathbb{N}} \|T_i^* T_j\|_{\mathscr{L}(\mathcal{H})}\big)^{1/2} \right).$$

More generally, (A.9) can be replaced with

$$T_i^* T_j = 0 \quad \text{for} \quad |i-j| > a,$$

where $a \in \mathbb{N}$ is a fixed positive integer. It suffices to apply Proposition A.5.3 to each $\{T_{ak+b}\}_{k\in\mathbb{Z}}$ for $b = 0, \ldots, a-1$. Then the series $\sum T_k = \sum_{0\leq b<a} T_{ak+b}$ converges in the strong operator norm topology to an operator $S$ satisfying

$$\|S\|_{\mathscr{L}(\mathcal{H})} \leq 2^{1/2} \times a \times \max\left( \sup_{k} \|T_k\|_{\mathscr{L}(\mathcal{H})}, \big(2 \sum_{i-j>a} \|T_i^* T_j\|_{\mathscr{L}(\mathcal{H})}\big)^{1/2} \right).$$

## A.6 Interpolation of analytic families of operators

Let $(M, \mathcal{M}, \mu)$ and $(N, \mathcal{N}, \nu)$ be measure spaces. We suppose that to each $z \in \mathbb{C}$ in the strip

$$S := \{z \in \mathbb{C} \ : \ 0 \leq \operatorname{Re} z \leq 1\},$$

there corresponds a linear operator $T_z$ from the space of simple functions in $L^1(M)$ to measurable functions on $N$, in such a way that $(T_z f)g$ is integrable on $N$ whenever $f$ is a simple function in $L^1(M)$ and $g$ is a simple function in $L^1(N)$. (Recall that a simple function is a measurable function which takes only a finite number of values.)

We assume that the family $\{T_z\}_{z\in S}$ is admissible in the sense that the mapping

$$z \mapsto \int_N (T_z f) g \, d\nu$$

is analytic in the interior of $S$, continuous on $S$, and there exists a constant $a < \pi$ such that

$$e^{-a|\operatorname{Im} z|} \ln \left| \int_N (T_z f) g \, d\nu \right|,$$

is uniformly bounded from above in the strip $S$.

**Theorem A.6.1.** *Let $\{T_z\}_{z\in S}$ be an admissible family as above. We assume that*

$$\|T_{iy}f\|_{q_0} \leq M_0(y)\|f\|_{p_0} \quad \textit{and} \quad \|T_{1+iy}f\|_{q_1} \leq M_1(y)\|f\|_{p_1},$$

*for all simple functions in $L^1(M)$ where $1 \leq p_j, q_j \leq \infty$, and functions $M_j(y)$, $j = 1, 2$ are independent of $f$ and satisfy*

$$\sup_{y\in\mathbb{R}} e^{-b|y|} \ln M_j(y) < \infty,$$

*for some $b < \pi$. Then if $0 \leq t \leq 1$, there exists a constant $M_t$ such that*

$$\|T_t f\|_{q_t} \leq M_t \|f\|_{p_t},$$

*for all simple functions $f$ in $L^1(M)$, provided that*

$$\frac{1}{p_t} = (1-t)\frac{1}{p_0} + t\frac{1}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = (1-t)\frac{1}{q_0} + t\frac{1}{q_1}.$$

For the proof of this theorem, we refer e.g. to [SW71, ch. V §4].

*Remark* A.6.2. The following remarks are useful.

- The constant $M_t$ depends only on $t$ and on $a, b, M_0(y), M_1(y)$, but not on $T$.
- From the proof, it appears that, if $N = M = \mathbb{R}^n$ is endowed with the usual Borel structure and the Lebesgue measures, one can require the assumptions and the conclusion to be on simple functions $f$ with compact support.

We also refer to Definition 6.4.17 for the notion of the complex interpolation (which requires stronger estimates).

# Appendix B

# Group $C^*$ and von Neumann algebras

In this chapter we make a short review of the machinery related to group von Neumann algebras that will be useful for setting up the Fourier analysis in other parts of book, in particular in Section 1.8.2. We try to make a short and concise presentation of notions and ideas without proofs trying to make the presentation as informal as possible. All the material presented in this chapter is well known but is often scattered over the literature in different languages and with different notation. Here we collect what is necessary for us giving references along the exposition. The final aim of this chapter is to introduce the notion of the von Neumann algebra of the group (or the group von Neumann algebra) and describe its main properties.

## B.1 Direct integral of Hilbert spaces

We start by describing direct integrals of Hilbert spaces. For more details and overall proofs we can refer to more classical literature such as Bruhat [Bru68] or to more modern exposition of Folland [Fol95, p. 219].

### B.1.1 Convention: Hilbert spaces are assumed separable

All the Hilbert spaces considered in this chapter are separable, unless stated otherwise. Let us recall the definition and some properties of separable spaces.

**Definition B.1.1.** A topological space is *separable* if its topology admits a countable basis of neighbourhoods.

When a topological space is metrisable, being separable is equivalent to having a (countable) sequence which is dense in the space.



V. Fischer, M. Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics 314,
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Moreover, a separable Hilbert space of infinite dimension is unitarily equivalent to the Hilbert space of square integrable complex sequences: that is, to

$$\ell^2(\mathbb{N}_0) = \{(x_j)_{j\in\mathbb{N}_0}, \sum_{j=0}^{\infty} |x_j|^2 < \infty\}.$$

Naturally a separable Hilbert space of finite dimension $n$ is unitarily equivalent to $\mathbb{C}^n$.

We can refer e.g. to Rudin [Rud91] for different topological implications of the separability.

### B.1.2 Measurable fields of vectors

Here we recall the definitions of measurable fields of Hilbert spaces, of vectors and of operators.

**Definition B.1.2.** Let $Z$ be a set and let $(\mathcal{H}_\zeta)_{\zeta\in Z}$ is a family of vector spaces (on the same field) indexed by $Z$. Then $\prod_{\zeta\in Z}\mathcal{H}_\zeta$ denotes the *direct product* of $(\mathcal{H}_\zeta)_{\zeta\in Z}$, that is, the set of all tuples $v = (v(\zeta))_{\zeta\in Z}$ with $v(\zeta) \in \mathcal{H}_\zeta$ for each $\zeta \in Z$. It is naturally endowed with a structure of a vector space with addition and scalar multiplication being performed componentwise.

An element of $\prod_{\zeta\in Z}\mathcal{H}_\zeta$, that is, a tuple $v = (v(\zeta))_{\zeta\in Z}$, may be called a field of vectors parametrised by $Z$, or, when no confusion is possible, a *vector field*.

We will use this definition for a measurable space $Z$. In practice, for the set $\Gamma$ in the following definition, we may also choose $\Gamma \subset \prod_{\zeta\in Z}\mathcal{H}_\zeta^\infty$ in view of Gårding's theorem (see Proposition 1.7.7).

**Definition B.1.3.** Let $Z$ be a measurable space and $\mu$ a positive sigma-finite measure on $Z$. A *$\mu$-measurable field of Hilbert spaces* over $Z$ is a pair $\mathcal{E} = \big((\mathcal{H}_\zeta)_{\zeta\in Z}, \Gamma\big)$ where $(\mathcal{H}_\zeta)_{\zeta\in Z}$ is a family of (separable) Hilbert spaces indexed by $Z$ and where $\Gamma \subset \prod_{\zeta\in Z}\mathcal{H}_\zeta$ satisfies the following conditions:

(i) $\Gamma$ is a vector subspace of $\prod_{\zeta\in Z}\mathcal{H}_\zeta$;

(ii) there exists a sequence $(x_\ell)_{\ell\in\mathbb{N}}$ of elements of $\Gamma$ such that for every $\zeta \in Z$, the sequence $(x_\ell(\zeta))_{\ell\in\mathbb{N}}$ spans $\mathcal{H}_\zeta$ (in the sense that the subspace formed by the finite linear combination of the $x_\ell(\zeta), \ell \in \mathbb{N}$, is dense in $\mathcal{H}_\zeta$);

(iii) for every $x \in \Gamma$, the function $\zeta \mapsto \|x(\zeta)\|_{\mathcal{H}_\zeta}$ is $\mu$-measurable;

(iv) if $x \in \prod_{\zeta\in Z}\mathcal{H}_\zeta$ is such that for every $y \in \Gamma$, the function

$$Z \ni \zeta \mapsto (x(\zeta), y(\zeta))_{\mathcal{H}_\zeta}$$

is measurable, then $x \in \Gamma$.

Under these conditions, the elements of $\Gamma$ are called the *measurable vector fields* of $\mathcal{E}$. We always identify two vector fields which are equal almost everywhere. This means that we identify two elements $x$ and $x'$ of $\Gamma$ when, for every $y \in \Gamma$, the two mappings

$$Z \ni \zeta \mapsto (x(\zeta), y(\zeta))_{\mathcal{H}_\zeta} \quad \text{and} \quad Z \ni \zeta \mapsto (x'(\zeta), y(\zeta))_{\mathcal{H}_\zeta},$$

can be identified as measurable functions.

A vector field $x$ is *square integrable* if $x \in \Gamma$ and $\int_Z \|x(\zeta)\|^2_{\mathcal{H}_\zeta} d\mu(\zeta) < \infty$. One may write then

$$x = \int_Z^{\oplus} x(\zeta) d\mu(\zeta).$$

The set of square integrable vector fields form a (possibly non-separable) Hilbert space denoted by

$$\mathcal{H} := \int_Z^{\oplus} \mathcal{H}_\zeta d\mu(\zeta),$$

and called the *direct integral* of the $\mathcal{H}_\zeta$. The inner product is given via

$$(x|y)_{\mathcal{H}} = \int_Z^{\oplus} (x(\zeta)|y(\zeta)_{\mathcal{H}_\zeta} d\mu(\zeta), \quad x, y \in \mathcal{H}.$$

### B.1.3 Direct integral of tensor products of Hilbert spaces

After a brief recollection of the definitions of tensor products, we will be able to analyse the direct integral of tensor products of Hilbert spaces, as well as their decomposable operators.

#### Definition of tensor products

Here we define firstly the algebraic tensor product of two vector spaces, and secondly the tensor products of Hilbert spaces.

**Definition B.1.4.** Let $V$ and $W$ be two complex vector spaces.

The free space generated by $V$ and $W$ is the vector space $\mathbb{F}(V \times W)$ linearly spanned by $V \times W$, that is, the space of finite $\mathbb{C}$-linear combinations of elements of $V \times W$.

The *algebraic tensor product* of $V$ and $W$ is the quotient of $\mathbb{F}(V \times W)$ by its subspace generated by the following elements

$$\begin{aligned} (v_1, w) + (v_2, w) - (v_1 + v_2, w), &\qquad (v, w_1) + (v, w_2) - (v, w_1 + w_2), \\ c(v, w) - (cv, w), &\qquad c(v, w) - (v, cw), \end{aligned}$$

where $v, v_1, v_2$ are arbitrary elements of $V$, $w, w_1, w_2$ are arbitrary elements of $W$, and $c$ is an arbitrary complex number.

The equivalence class of an element $(v, w) \in V \times W \subset \mathbb{F}(V \times W)$ is denoted $v \otimes w$.

The algebraic tensor product of $V$ and $W$ is naturally a complex vector space which we will denote in this monograph by

$$V \overset{alg}{\otimes} W.$$

The algebraic tensor product has the following universal property (which may be given as an alternate definition):

**Proposition B.1.5** (Universal property)**.** *Let $V$, $W$ and $X$ be (complex) vector spaces and let $\Psi : V \times W \to X$ be a bilinear mapping. Then there exists a unique map $\tilde{\Psi} : V \overset{alg}{\otimes} W \to X$ such that*

$$\Psi = \tilde{\Psi} \circ \pi$$

*where $\pi : V \times W \to V \overset{alg}{\otimes} W$ is the map defined by $\pi(v, w) = v \otimes w$.*

More can be said when the complex vector spaces are also Hilbert spaces. Indeed one checks easily:

**Lemma B.1.6.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then the mapping defined on $\mathcal{H}_1 \overset{alg}{\otimes} \mathcal{H}_2$ via*

$$(u_1 \otimes v_1, u_2 \otimes v_2) := (u_1, u_2)(v_1, v_2), \quad u_1, u_2 \in \mathcal{H}_1,\ v_1, v_2 \in \mathcal{H}_2,$$

*is a complex inner product on $\mathcal{H}_1 \overset{alg}{\otimes} \mathcal{H}_2$.*

This shows that $\mathcal{H}_1 \overset{alg}{\otimes} \mathcal{H}_2$ is a pre-Hilbert space.

**Definition B.1.7.** The *tensor product of the Hilbert spaces* $\mathcal{H}_1$ and $\mathcal{H}_2$ is the completion of $\mathcal{H}_1 \overset{alg}{\otimes} \mathcal{H}_2$ for the natural sesquilinear form from Lemma B.1.6. It is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Naturally we have the universal property of tensor products of Hilbert spaces:

**Proposition B.1.8** (Universal property)**.** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$ be Hilbert spaces and let $\Psi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$ be a continuous bilinear mapping. Then there exists a unique continuous map $\tilde{\Psi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}$ such that*

$$\Psi = \tilde{\Psi} \circ \pi$$

*where $\pi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ is the map defined by $\pi(v, w) = v \otimes w$.*

### Tensor products of Hilbert spaces as Hilbert-Schmidt spaces

The tensor product of two Hilbert spaces may be identified with a space of Hilbert Schmidt operators in the following way. To any vector $w \in \mathcal{H}_2$, we associate the continuous linear form on $\mathcal{H}_2$

$$w^* : v \longmapsto (v, w)_{\mathcal{H}_2}.$$

Conversely any element of $\mathcal{H}_2^*$, that is, any continuous linear form on $\mathcal{H}_2$, is of this form. To any $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$, we associate the rank-one operator

$$\Psi_{u,v} : \begin{cases} \mathcal{H}_2^* & \longrightarrow & \mathcal{H}_1 \\ w^* & \longmapsto & w^*(v)u \end{cases}$$

**Lemma B.1.9.** *With the notation above, the continuous bilinear mapping*

$$\Psi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)$$

*extends to an isometric isomorphism of Hilbert spaces*

$$\tilde{\Psi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1).$$

*Moreover, if $T_1 \in \mathscr{L}(\mathcal{H}_1)$ and $T_2 \in \mathscr{L}(\mathcal{H}_2)$, then the operator $T_1 \otimes T_2$ defined via*

$$(T_1 \otimes T_2)(v_1 \otimes v_2) := (T_1 v_1) \otimes (T_2 v_2), \quad v_1 \in \mathcal{H}_1, v_2 \in \mathcal{H}_2,$$

*is in $\mathscr{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and corresponds to the bounded operator*

$$\tilde{\Psi}(T_1 \otimes T_2)\tilde{\Psi}^{-1} : \begin{cases} \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1) & \longrightarrow & \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1) \\ A & \longmapsto & T_1 A T_2 \end{cases}.$$

Recall that the scalar product of $\mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)$ is given by

$$(T_1, T_2)_{\mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)} = \sum_j (T_1 f_j^*, T_2 f_j^*)_{\mathcal{H}_1}.$$

where $(f_j^*)_{j \in \mathbb{N}}$ is any orthonormal basis of $\mathcal{H}_2^*$.

*Proof.* By Proposition B.1.8, $\Psi$ leads to a continuous linear mapping $\tilde{\Psi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)$. The image of $\tilde{\Psi}$ contains the rank-one operators, thus all the finite ranked operators which form a dense subset of $\mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)$. Thus $\tilde{\Psi}$ is surjective.

If $(f_j^*)_{j \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_2^*$, we can compute easily the scalar product between $\Psi_{u_1,v_1}$ and $\Psi_{u_2,v_2}$:

$$\begin{aligned}
(\Psi_{u_1,v_1}, \Psi_{u_2,v_2})_{\mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)} &= \sum_j (\Psi_{u_1,v_1} f_j^*, \Psi_{u_2,v_2} f_j^*)_{\mathcal{H}_1} \\
&= \sum_j (f_j^*(v_1) u_1, f_j^*(v_2) u_2)_{\mathcal{H}_1} = (u_1, u_2)_{\mathcal{H}_1} \sum_j f_j^*(v_1) \overline{f_j^*(v_2)} \\
&= (u_1, u_2)_{\mathcal{H}_1} \sum_j (v_1, f_j) \overline{(v_2, f_j)} = (u_1, u_2)_{\mathcal{H}_1} (v_1, v_2)_{\mathcal{H}_2}.
\end{aligned}$$

This implies that the mapping $\tilde{\Psi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathtt{HS}(\mathcal{H}_2^*, \mathcal{H}_1)$ is an isometry.

For the last part of the statement, one checks easily that

$$(T_1 \Psi_{u,v} T_2)(w^*) = w^*(T_2 v)\ T_1 u,$$

concluding the proof. $\square$

Let us apply this to $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H}^*$.

**Corollary B.1.10.** *Let $\mathcal{H}$ be a Hilbert space. The Hilbert space given by the tensor product $\mathcal{H} \otimes \mathcal{H}^*$ of Hilbert spaces is isomorphic to $\mathtt{HS}(\mathcal{H})$ via*

$$u \otimes v^* \longleftrightarrow \Psi_{u,v}, \quad \Psi_{u,v}(w) = (w, v)_{\mathcal{H}} u.$$

*Via this isomorphism, the bounded operator $T_1 \otimes T_2^*$ where $T_1, T_2 \in \mathscr{L}(\mathcal{H})$, corresponds to the bounded operator*

$$\tilde{\Psi}(T_1 \otimes T_2)\tilde{\Psi}^{-1} : \left\{ \begin{array}{rcl} \mathtt{HS}(\mathcal{H}) & \longrightarrow & \mathtt{HS}(\mathcal{H}) \\ A & \longmapsto & T_1 A T_2^* \end{array} \right. .$$

**Direct integral of tensor products of Hilbert spaces**

Let $\mu$ be a positive sigma-finite measure on a measurable space $Z$ and $\mathcal{E} = \big((\mathcal{H}_\zeta)_{\zeta \in Z}, \Gamma\big)$ a *$\mu$-measurable field of Hilbert spaces* over $Z$. Then

$$\mathcal{E}^{\otimes} := \big((\mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^*)_{\zeta \in Z}, \Gamma \otimes \Gamma^*\big)$$

is a *$\mu$-measurable field of Hilbert spaces* over $Z$.

Identifying each tensor product $\mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^*$ with $\mathtt{HS}(\mathcal{H}_\zeta)$, see Corollary B.1.10, we may write

$$\int_Z^{\oplus} \mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^* d\mu(\zeta) \equiv \int_Z^{\oplus} \mathtt{HS}(\mathcal{H}_\zeta) d\mu(\zeta).$$

Furthermore if $x \in \int_Z^{\oplus} \mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^* d\mu(\zeta)$ then

$$\|x\|^2 = \int_Z \|x(\zeta)\|^2_{\mathtt{HS}(\mathcal{H}_\zeta)} d\mu(\zeta).$$

## B.1.4 Separability of a direct integral of Hilbert spaces

In this chapter, we are always concerned with separable Hilbert spaces (see Section B.1.1). A sufficient condition to ensure the separability of a direct integral is that the measured space is standard (the definition of this notion is recalled below):

**Proposition B.1.11.** *Keeping the setting of Definition B.1.3, if $(Z, \mu)$ is a standard space, then $\int_Z^{\oplus} \mathcal{H}_\zeta d\mu(\zeta)$ is a separable Hilbert space.*

For the proof we refer to Dixmier [Dix96, §II.1.6].

**Definition B.1.12.** A measurable space $Z$ is a *standard Borel space* if $Z$ is a Polish space (i.e. a separable complete metrisable topological space) and the considered sigma-algebra is the Borel sigma-algebra of $Z$ (i.e. the smallest sigma-algebra containing the open sets of $Z$).

These Borel spaces have a simple classification: they are isomorphic (as Borel spaces) either to a (finite or infinite) countable set, or to $[0, 1]$. For these and other details see, for instance, Kechris [Kec95, Chapter II, Theorem 15.6] and its proof.

**Definition B.1.13.** A positive measure $\mu$ on a measure space $Z$ is a *standard measure* if $\mu$ is sigma-finite, (i.e. there exists a sequence of mutually disjoint measurable sets $Y_1, Y_2, \ldots$ such that $\mu(Y_j) < \infty$ and $Z = Y_1 \cup Y_2 \cup \ldots$) and there exists a null set $E$ such that $Z \backslash E$ is a standard Borel space.

In this monograph, we consider only the setting described in Proposition B.2.24 which is standard.

### B.1.5 Measurable fields of operators

Let $Z$ be a measurable space and $\mu$ a positive sigma-finite measure on $Z$. The main application for our analysis of these constructions will be in Section 1.8.3 dealing with measurable fields of operators over $\widehat{G}$.

**Definition B.1.14.** Let $\mathcal{E} = \big((\mathcal{H}_\zeta)_{\zeta \in Z}, \Gamma\big)$ be a $\mu$-measurable field of Hilbert spaces over $Z$. A *$\mu$-measurable field of operators* over $Z$ is a collection of operators $(T(\zeta))_{\zeta \in Z}$ such that $T(\zeta) \in \mathscr{L}(\mathcal{H}_\zeta)$ and for any $x \in \Gamma$, the field $(T(\zeta)x(\zeta))_{\zeta \in Z}$ is measurable. If furthermore the function $\zeta \mapsto \|T(\zeta)\|_{\mathscr{L}(\mathcal{H}_\zeta)}$ is $\mu$-essentially bounded, then the field of operators $(T(\zeta))_{\zeta \in Z}$ is *essentially bounded.*

Let us continue with the notation of Definition B.1.14. Let $(T(\zeta))_{\zeta \in Z}$ be an essentially bounded field of operators. Then we can define the operator $T$ on the Hilbert space $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}_\zeta d\mu(\zeta)$ via $(Tx)(\zeta) := T(\zeta)x(\zeta)$. Clearly the operator $T$ is linear and bounded. It is often denoted by

$$T := \int_Z^{\oplus} T(\zeta) d\mu(\zeta).$$

Naturally two fields of operators which are equal up to a $\mu$-negligible set yield the same operator on $\mathcal{H}$ and may be identified. Furthermore the operator norm of $T \in \mathscr{L}(\mathcal{H})$ is

$$\|T\|_{\mathscr{L}(\mathcal{H})} = \sup_{\zeta \in Z} \|T(\zeta)\|_{\mathscr{L}(\mathcal{H}_\zeta)},$$

where sup denotes here the essential supremum with respect to $\mu$.

**Definition B.1.15.** An operator on $\mathcal{H}$ as above, that is, obtained via

$$T := \int_Z^{\oplus} T(\zeta) d\mu(\zeta)$$

where $(T(\zeta))_{\zeta \in Z}$ is an essentially bounded field of operators, is said to be *decomposable.*

The set of decomposable operators form a subspace of $\mathscr{L}(\mathcal{H})$ stable by composition and taking the adjoint.

### B.1.6 Integral of representations

In the following definition, $\mu$ is a positive sigma-finite measure on a measurable space $Z$, $\mathcal{A}$ is a separable $C^*$-algebra, and $G$ is a (Hausdorff) locally compact separable group. For further details on the constructions of this section we refer to Dixmier [Dix77, §8]. For the definition of representations of $C^*$-algebras see Definition B.2.16.

**Definition B.1.16.** Let $\mathcal{E} = \big((\mathcal{H}_\zeta)_{\zeta\in Z}, \Gamma\big)$ be a $\mu$-measurable field of Hilbert spaces over $Z$. A *$\mu$-measurable field of representations* of $\mathcal{A}$, resp. $G$, is a $\mu$-measurable field of operator $(T(\zeta))_{\zeta\in Z}$ (see Definition B.1.14) such that for each $\zeta \in Z$, $T(\zeta) = \pi_\zeta$ is a representation of $\mathcal{A}$, resp. a unitary continuous representation of $G$, in $\mathcal{H}_\zeta$.

In this case, for each $x \in G$, we can define the operator

$$\pi(x) := \int_Z^\oplus \pi_\zeta(x) d\mu(\zeta) \quad \text{acting on } \mathcal{H} := \int_Z^\oplus \mathcal{H}_\zeta d\mu(\zeta).$$

One checks easily that this yields a representation $\pi$ of $\mathcal{A}$, resp. a unitary continuous representation of $G$, on $\mathcal{H}$ denoted by

$$\pi := \int_Z^\oplus \pi_\zeta d\mu(\zeta),$$

often called *the integral of the representations* $(\pi_\zeta)_{\zeta\in Z}$.

The following technical properties give sufficient conditions for two integrals of representations to yield equivalent representations. Again $\mathcal{A}$ is a separable $C^*$-algebra and $G$ a (Hausdorff) locally compact separable group.

**Proposition B.1.17.** *Let $\mu_1$ and $\mu_2$ be two positive sigma-finite measures on measurable spaces $Z_1$ and $Z_2$ respectively. For $j = 1, 2$, let $\mathcal{E}_j = \big((\mathcal{H}^{(j)}_{\zeta_j})_{\zeta_j\in Z_j}, \Gamma_j\big)$ be a $\mu_j$-measurable field of Hilbert spaces over $Z_j$ and let $(\pi^{(j)}_{\zeta_j})$ be a measurable field of representations of $\mathcal{A}$, resp. of unitary continuous representations of $G$.*

*We assume that $\mu_1$ and $\mu_2$ are standard. We also assume that there exist a Borel $\mu_1$-negligible part $E_1 \subset Z_1$, a Borel $\mu_2$-negligible part $E_2 \subset Z_2$ and a Borel isomorphism $\eta : Z_1\backslash E_1 \to Z_2\backslash E_2$ which transforms $\mu_1$ to $\mu_2$ and such that $\pi^{(1)}_{\zeta_1}$ and $\pi^{(2)}_{\eta(\zeta_1)}$ are equivalent for any $\zeta_1 \in Z_1\backslash E_1$. Then there exists a unitary mapping from $\mathcal{H}^{(1)} := \int_{Z_1}^\oplus \mathcal{H}^{(1)}_{\zeta_1} d\mu_1(\zeta_1)$ onto $\mathcal{H}^{(2)} := \int_{Z_2}^\oplus \mathcal{H}^{(2)}_{\zeta_2} d\mu_2(\zeta_2)$ which intertwines the representations of $\mathcal{A}$, resp. the unitary continuous representations of $G$,*

$$\pi^{(1)} := \int_{Z_1}^\oplus \pi^{(1)}_{\zeta_1} d\mu_1(\zeta_1) \quad \textit{and} \quad \pi^{(2)} := \int_{Z_2}^\oplus \pi^{(2)}_{\zeta_2} d\mu_2(\zeta_2).$$

# B.2 $C^*$- and von Neumann algebras

The main reference for this section are Dixmier's books [Dix81, Dix77], Arveson [Arv76] or Blackadar [Bla06]. For a more basic introduction to $C^*$-algebras and elements of the Gelfand theory see also Ruzhansky and Turunen [RT10a, Chapter D].

## B.2.1 Generalities on algebras

Here we recall the definitions of an algebra, together with its possible additional structures (involution, norm) and sets usually associated with it (spectrum, bi-commutant).

### Algebra

Let us start with the definition of an algebra over a field.

Let $\mathcal{A}$ be a vector space over a field $\mathbb{K}$ equipped with an additional binary operation

$$\begin{array}{rcl} \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A}, \\ (x, y) & \longmapsto & x \cdot y. \end{array}$$

It is an *algebra* over $\mathbb{K}$ when the binary operation (then often called the product) satisfies:

- left distributivity: $(x + y) \cdot z = x \cdot z + y \cdot z$ for any $x, y, z \in \mathcal{A}$,
- right distributivity: $z \cdot (x + y) = z \cdot x + z \cdot y$ for any $x, y, z \in \mathcal{A}$,
- compatibility with scalars: $(ax) \cdot (by) = (ab)(x \cdot y)$ for any $x, y \in \mathcal{A}$ and $a, b \in \mathbb{K}$.

The algebra $\mathcal{A}$ is said to be *unital* when there exists a unit, that is, an element $1 \in \mathcal{A}$ such that $x \cdot 1 = 1 \cdot x = x$ for every $x \in \mathcal{A}$.

A subspace $\mathcal{Y} \subset \mathcal{A}$ is a *sub-algebra* of $\mathcal{A}$ whenever $y_1 \cdot y_2 \in \mathcal{Y}$ for any $y_1, y_2 \in \mathcal{Y}$.

### Commutant and bi-commutant

We will need the notion of commutant:

**Definition B.2.1.** Let $\mathcal{M}$ be a subset of the algebra $\mathcal{A}$. The *commutant* of $\mathcal{M}$ is the set denoted by $\mathcal{M}'$ of the elements which commute with all the elements of $\mathcal{M}$, that is,

$$\mathcal{M}' := \{x \in \mathcal{A} \ : \ xm = mx \ \text{ forall } \ m \in \mathcal{M}\}.$$

The *bi-commutant* of $\mathcal{M}$ is the commutant of the commutant of $\mathcal{M}$, that is,

$$\mathcal{M}'' := (\mathcal{M}')'.$$

Keeping the notation of Definition B.2.1, one checks easily that a commutant $\mathcal{M}'$ is a sub-algebra of $\mathcal{A}$. It contains the unit if $\mathcal{A}$ is unital. Furthermore, in any case, $\mathcal{M} \subset \mathcal{M}''$.

### Involution and norms

We consider now algebras endowed with an involution:

**Definition B.2.2.** Let $\mathcal{A}$ be an algebra over the complex numbers $\mathbb{C}$. It is called an *involutive algebra* or a $*$-*algebra* when there exists a map $* : \mathcal{A} \to \mathcal{A}$ which is

- sesquilinear (that is, $(ax + by)^* = \bar{a}x^* + \bar{b}y^*$ for every $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$),
- involutive (that is, $(x^*)^* = x$ for every $x \in \mathcal{A}$).

In this case, $x^*$ may be called the *adjoint* of $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is *hermitian* if $x^* = x$. An element $x \in \mathcal{A}$ is *unitary* if $xx^* = x^*x = 1$.

*Example* B.2.3. Let $\mathcal{A}$ be a $*$-algebra. If $\mathcal{M}$ is a subset of $\mathcal{A}$ stable under the involution (that is, $m^* \in \mathcal{M}$ for every $m \in \mathcal{M}$), then its commutant $\mathcal{M}'$ is a $*$-subalgebra of $\mathcal{A}$.

**Definition B.2.4.** A *normed involutive algebra* is an involutive algebra $\mathcal{A}$ endowed with a norm $\|\cdot\|$ such that

$$\|x^*\| = \|x\|$$

for each $x \in \mathcal{A}$. If, in addition, $\mathcal{A}$ is $\|\cdot\|$-complete, then $\mathcal{A}$ is called an *involutive Banach algebra.*

The notions of (involutive, normed involutive / involutive Banach) sub-algebra and morphism between (involutive / normed involutive / involutive Banach) algebras follow naturally. Furthermore if $\mathcal{A}$ is a (involutive / normed involutive / involutive Banach) non unital algebra, then there exists a unique (involutive / normed involutive / involutive Banach) unital algebra $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}1$, up to isomorphism, which contains $\mathcal{A}$ as a (involutive, normed involutive / involutive Banach) sub-algebra.

### Examples

*Example* B.2.5. The complex field $\mathcal{A} = \mathbb{C}$ is naturally a unital commutative involutive Banach algebra.

*Example* B.2.6. Let $X$ be a locally compact space and let $\mathcal{A} = C_o(X)$ be the space of continuous functions $f : X \to \mathbb{C}$ vanishing at infinity, that is, for every $\epsilon > 0$, there exists a compact neighbourhood out of which $|f| < \epsilon$. Then $\mathcal{A}$ is a commutative involutive Banach algebra when endowed with pointwise multiplication and involution $f \mapsto \bar{f}$. When $X$ is a singleton, this reduces to Example B.2.5.

*Example* B.2.7. If $\eta$ is a positive measure on a measurable space $X$ and if $\mathcal{A}$ is the space of $\eta$-essentially bounded functions $f : X \to \mathbb{C}$, that is, $\mathcal{A} = L^\infty(X, \eta)$, then $\mathcal{A}$ is a unital commutative involutive Banach algebra when endowed with pointwise multiplication and involution $f \mapsto \bar{f}$. When $X$ is a singleton, this reduces to Example B.2.5.

Recall that all the Hilbert spaces we consider are separable.

*Example* B.2.8. The space $\mathscr{L}(\mathcal{H})$ of continuous linear operators on a Hilbert space $\mathcal{H}$ is naturally a unital involutive Banach algebra for the usual structure. This means that the product is given by the composition of operators $(A, B) \mapsto AB$, the involution by the adjoint and the norm by the operator norm. The unit is the identity mapping $\mathrm{I}_{\mathcal{H}} = \mathrm{I} : v \mapsto v$.

*Example* B.2.9. If $G$ is a locally compact (Hausdorff) group which is unimodular, then $L^1(G)$ is naturally an involutive Banach algebra where the product is given by the convolution and the involution $f \mapsto f^*$ by $f^*(x) = \bar{f}(x^{-1})$. If $G$ is separable then $L^1(G)$ is separable.

Example B.2.9 can be generalised to locally compact groups which are not necessarily unimodular. First, let us recall the following definitions:

**Definition B.2.10.** Let $G$ be a locally compact (Hausdorff) group. Let us fix a left Haar measure $dx$. We also denote by $|E|$ the volume of a Borel set for this measure. Then there exists a unique function $\Delta$ such that

$$|Ex| = \Delta(x)|E|$$

for any Borel set $E$ and $x \in G$. It is called the *modular* function of $G$ and is independent of the chosen left Haar measure. It is a group homomorphism $G \to (\mathbb{R}^+, \times)$.

If the modular function is constant then $\Delta \equiv 1$ and $G$ is said to be *unimodular.*

*Remark* B.2.11. Any Lie group is a separable locally compact (Hausdorff) group. Any compact (Hausdorff) group is necessarily a locally compact (Hausdorff) group and it is also unimodular. Any abelian locally compact (Hausdorff) group is unimodular. Any nilpotent or semi-simple Lie group is unimodular.

*Example* B.2.12. If $G$ is a locally compact (Hausdorff) group then $L^1(G)$ is naturally an involutive Banach algebra often called the *group algebra.* The product is given by the convolution and the involution $f \mapsto f^*$ by

$$f^*(x) = \bar{f}(x^{-1})\Delta(x)^{-1},$$

where $\Delta$ is the modular function (see Definition B.2.10).

The space $M(G)$ of complex measures on $G$ is also naturally an involutive Banach algebra and $L^1(G)$ may be viewed as a closed involutive sub-algebra. The algebra $M(G)$ always admits the Dirac measure $\delta_e$ at the neutral element of the group as unit.

Note that $L^1(G)$ is unital if and only if $G$ is discrete and in this case $L^1(G) = M(G)$.

### B.2.2 $C^*$-algebras

In this subsection we briefly review the notion of $C^*$-algebra and its main properties. We can refer to Ruzhansky and Turunen [RT10a, Chapter D] for a longer exposition.

**Definition B.2.13.** A $C^*$*-algebra* is an involutive Banach algebra $\mathcal{A}$ such that

$$\|x\|^2 = \|x^* x\|$$

for every $x \in \mathcal{A}$.

*Example* B.2.14. Examples B.2.5, B.2.6, B.2.7, and B.2.8 are $C^*$-algebras.

*Remark* B.2.15.

1. If we choose a Hilbert space $\mathcal{H}$ of finite dimension $n$ in Example B.2.8, the Banach algebra $\mathscr{L}(\mathcal{H}) \sim \mathscr{L}(\mathbb{C}^n) \sim \mathbb{C}^{n\times n}$ is a $C^*$-algebra if endowed with the operator norm, but is not a $C^*$-algebra when equipped with the Euclidean norm of $\mathbb{C}^{n^2}$ for instance.

2. Example B.2.6 is fundamental in the sense that one can show that any commutative $C^*$-algebra $\mathcal{A}$ is isomorphic to $C_o(X)$, where $X$ is the spectrum of $\mathcal{A}$, that is, the set of non-zero complex homomorphisms with its usual topology. Moreover the isomorphism often called the Gelfand-Fourier transform is $*$-isometric. For further details see e.g. Rudin [Rud91] but with a different vocabulary.

3. In the non-commutative setting, the previous point may be generalised via the Gelfand-Naimark theorem: this theorem states that any $C^*$-algebra is $*$-isometric to a closed sub-*-algebra of $\mathscr{L}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. Note that Example B.2.8 give the precise structure of $\mathscr{L}(\mathcal{H})$ and shows that a closed sub-*-algebra of $\mathscr{L}(\mathcal{H})$ is indeed a $C^*$-algebra. The proof is based on the Gelfand-Naimark-Segal construction, see e.g. Arveson [Arv76] for more precise statements.

The general definition of the spectrum of a (not necessarily commutative) $C^*$ algebra is more involved than in the commutative case (Remark B.2.15 (2)):

**Definition B.2.16** (Representations of $C^*$-algebras)**.** Let $\mathcal{A}$ be a $C^*$-algebra.

A *representation* of $\mathcal{A}$ is a continuous mapping $\mathcal{A} \to \mathscr{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, this mapping being a homomorphism of involutive algebras. Two representations $\pi_j : \mathcal{A} \to \mathscr{L}(\mathcal{H}_j)$, $j = 1, 2$, of $\mathcal{A}$, are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(x) = \pi_2(x)U$ for every $x \in \mathcal{A}$. A representation $\pi : \mathcal{A} \to \mathscr{L}(\mathcal{H})$ is *irreducible* if the only subspaces of $\mathcal{H}$ which are invariant under $\pi$, that is, under every $\pi(x)$, $x \in \mathcal{A}$, are trivial: $\{0\}$ and $\mathcal{H}$.

The *dual* (or spectrum) of $\mathcal{A}$ is the set of unitary irreducible representations of $\mathcal{A}$ modulo unitary equivalence. It is denoted by $\widehat{\mathcal{A}}$.

*Remark* B.2.17. The dual of a $C^*$-algebra is equipped with the hull-kernel topology due to Jacobson, and, if it is separable, with a structure of measurable space due to Mackey, see Dixmier [Dix77, §3].

### B.2.3 Group $C^*$-algebras

In general, the group algebra of a locally compact (Hausdorff) group $G$, that is, the involutive Banach algebra $L^1(G)$ in Example B.2.12, is not a $C^*$ algebra (see Remark B.2.26 below). The group $C^*$ algebra is the $C^*$-enveloping algebra of $L^1(G)$, meaning that it is a 'small' $C^*$ algebra containing $L^1(G)$ and built in the following way.

First, let us mention that many authors, for instance Jacques Dixmier, prefer to use for the Fourier transform

$$\pi_{\mathscr{D}}(f) := \int_G \pi(x) f(x) dx, \quad f \in L^1(G), \tag{B.1}$$

instead of $\pi(f)$ defined via

$$\pi(f) = \int_G \pi(x)^* f(x) dx, \quad f \in L^1(G), \tag{B.2}$$

which we adopt in this monograph, starting from (1.2), see Remark 1.1.4 for the explanation of this choice.

An advantage of using $\pi_{\mathscr{D}}$ would be that it yields a morphism of involutive Banach algebras from $L^1(G)$ to $\mathscr{L}(\mathcal{H}_\pi)$ as one checks readily:

**Lemma B.2.18.** *Let $\pi$ be a unitary continuous representation of $G$. Then $\pi_{\mathscr{D}}$ is a (non-degenerate) representation of the involutive Banach algebra $L^1(G)$:*

$$\forall f, g \in L^1(G) \qquad \pi_{\mathscr{D}}(f * g) = \pi_{\mathscr{D}}(f)\pi_{\mathscr{D}}(g), \quad \pi_{\mathscr{D}}(f)^* = \pi_{\mathscr{D}}(f^*),$$

*and*

$$\|\pi_{\mathscr{D}}(f)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq \|f\|_{L^1(G)}.$$

For the proof, see Dixmier [Dix77, Proposition 13.3.1].

The choice of the Fourier transform in (B.2) made throughout this monograph, yields in contrast

$$\forall f, g \in L^1(G) \qquad \pi(f * g) = \pi(g)\pi(f)$$

and still

$$\pi(f)^* = \pi(f^*), \quad \|\pi(f)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le \|f\|_{L^1(G)}.$$

The main advantage of our choice of Fourier transform is the fact that the Fourier transform of left-invariant operators will act on the left, as is customary in harmonic analysis, see our presentation of the abstract Plancherel theorem in Section 1.8.2.

**Definition B.2.19.** On $L^1(G)$, we can define $\|\cdot\|_*$ via

$$\|f\|_* := \sup_\pi \|\pi_{\mathcal{D}}(f)\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad f \in L^1(G),$$

where the supremum runs over all continuous unitary irreducible representations $\pi$ of the group $G$.

One checks easily that $\|\cdot\|_*$ is a seminorm on $L^1(G)$ which satisfies

$$\|f\|_* \le \|f\|_{L^1} < \infty.$$

One can show that it is in fact also a norm on $L^1(G)$, see Dixmier [Dix77, §13.9.1].

**Definition B.2.20.** The *group $C^*$-algebra* is the Banach space obtained by completion of $L^1(G)$ for the norm $\|\cdot\|_*$. It is often denoted by $C^*(G)$.

*Remark* B.2.21. Choosing the definition of $\|\cdot\|_*$ using $\pi_{\mathcal{D}}$ as above or using our usual Fourier transform leads to the same $C^*$-algebra of the group. Indeed one checks easily that the adjoint of the operator $\pi(f)$ acting on $\mathcal{H}_\pi$ is $\pi_{\mathscr{D}}(\bar{f})$:

$$\pi(f) = \pi_{\mathscr{D}}(\bar{f})^* = \pi_{\mathscr{D}}(\bar{f}^*) \quad \text{and} \quad \|\pi(f)\|_{\mathscr{L}(\mathcal{H}_\pi)} = \|\pi_{\mathscr{D}}(\bar{f})\|_{\mathscr{L}(\mathcal{H}_\pi)}, \tag{B.3}$$

for all $f \in L^1(G)$.

Naturally $C^*(G)$ is a $C^*$-algebra and there are natural one-to-one correspondences between the representation theories of the group $G$, of the involutive Banach algebra $L^1(G)$, and of the $C^*$-algebra $C^*(G)$ in the following sense:

**Lemma B.2.22.** *If $\pi$ is a continuous unitary representation of $G$, then $f \mapsto \pi_{\mathscr{D}}(f)$ defined via (B.1) is a non-degenerate $*$-representation of $L^1(G)$ which extends naturally to $C^*(G)$. Conversely any non-degenerate $*$-representation of $L^1(G)$ or $C^*(G)$ arise in this way.*

*Hence*

$$\|f\|_* = \sup_\pi \|\pi(f)\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad f \in L^1(G),$$

*where the supremum runs over all representations $\pi$ of the involutive Banach algebra $L^1(G)$ or over all representations $\pi$ of the $C^*$-algebra $C^*(G)$.*

For the proof see Dixmier [Dix77, §13.3.5 and §13.9.1].

**Definition B.2.23.** The dual of the group $G$ is the set $\widehat{G}$ of (continuous) irreducible unitary representations of $G$ modulo equivalence, see (1.1).

Given the correspondence explained in Lemma B.2.22, $\widehat{G}$ can be identified with the dual of $C^*(G)$ and inherit the structure that may occur on $\widehat{C^*(G)}$, see Remark B.2.17.

In particular, $\widehat{G}$ inherits a topology, called the *Fell topology*, corresponding to the hull-kernel (Jacobson) topology on $C^*(G)$, see e.g. Folland [Fol95, §7.2], Dixmier [Dix77, §18.1 and §3]. If $G$ is separable, then $C^*(G)$ is separable, see [Dix77, §13.9.2], and $\widehat{G}$ also inherits the Mackey structure of measurable space.

**Proposition B.2.24.** *Let $G$ be a separable locally compact group of type I. Then its dual $\widehat{G}$ is a standard Borel space. Moreover the Mackey structure coincides with the sigma-algebra associated with the Fell topology.*

For the definition of groups of type I, see Dixmier [Dix77, §13.9.4] or Folland [Fol95, §7.2]. See also hypothesis (H) in Section 1.8.2 for a relevant discussion. For the definition of the Plancherel measure, see (1.28), as well as Dixmier [Dix77, Definition 8.8.3] or Folland [Fol95, §7.5].

*References for the proof of Proposition B.2.24.* As $G$ is of type I and separable, its group $C^*$-algebra $C^*(G)$ is of type I, postliminar and separable, see Dixmier [Dix77, §13.9]. Hence the Mackey Borel structure on the spectrum of this $C^*$-algebra (cf. [Dix77, §3.8]) is a standard Borel space by Dixmier [Dix77, Proposition 4.6.1]. □

### Reduced group $C^*$-algebra

Although we do not use the following in this monograph, let us mention that one can also define another 'small' $C^*$ algebra which contains $L^1(G)$.

Let us recall that the left regular representation $\pi_L$ is defined on the group via

$$\pi_L(x)\phi(y) := \phi(x^{-1}y), \quad x, y \in G, \ \phi \in L^2(G). \tag{B.4}$$

This leads to the representation of $L^1(G)$ given by

$$(\pi_L)_{\mathscr{D}}(f)\phi = \int_G f(x)\pi_L(x)\phi \, dx = \int_G f(x)\phi(x^{-1}\cdot) \, dx = f * \phi, \tag{B.5}$$

which may be extended onto the closure $\overline{(\pi_L)_{\mathcal{D}}(L^1(G))}$ of $(\pi_L)_{\mathcal{D}}(L^1(G))$ for the operator norm, see Lemma B.2.22. This closure is naturally a $C^*$-algebra, often called the *reduced* $C^*$-algebra of the group and denoted by $C_r^*(G)$. Equivalently, $C_r^*(G)$ may be realised as the closure of $L^1(G)$ for the norm given by

$$\|f\|_{C_r^*} = \|(\pi_L)_{\mathscr{D}}(f)\|_{\mathscr{L}(L^2(G))} = \{\|f * \phi\|_{L^2}, \phi \in L^2(G) \text{ with } \|\phi\|_{L^2} = 1\}.$$

The 'full' and reduced $C^*$ algebras of a group may be different. When they are equal, that is, $C_r^*(G) = C^*(G)$, then the group $G$ is said to be *amenable*. Amenability can be described in many other ways. The advantage of considering the 'full'

$C^*$-algebra of a group is the one-to-one correspondence between the representations theories of $G$, $L^1(G)$, and $C^*(G)$.

The groups considered in this monograph, that is, compact groups and nilpotent Lie groups, are amenable.

**Pontryagin duality**

Although we do not use it in this monograph, let us recall briefly the Pontryagin duality, as this may be viewed as one of the historical motivation to develop the theory of (noncommutative) $C^*$-algebras.

The case of a locally compact (Hausdorff) abelian ($\equiv$ commutative) group $G$ is described by the Pontryagin duality, see Section 1.1. In this case, the group algebra $L^1(G)$ (see Example B.2.9) is an abelian involutive Banach algebra. Its spectrum $\widehat{G}$ may be identified with the set of the continuous characters of $G$ and is naturally equipped with the structure of a locally compact (Hausdorff) abelian group. The group $G$ is amenable, that is, the full and reduced group $C^*$-algebras coincide: $C^*(G) = C_r^*(G)$. Moreover, the Fourier-Gelfand transform (see Remark B.2.15 (2)) extends into an isometry of $C^*$-algebra from $C^*(G)$ onto $C_o(\widehat{G})$.

*Example* B.2.25. In the particular example of the abelian group $G = \mathbb{R}^n$, the dual $\widehat{G}$ may also be identified with $\mathbb{R}^n$ and the Fourier-Gelfand transform in this case is the (usual) Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$.

The group $C^*$-algebra $C^*(\mathbb{R}^n) = C_r^*(\mathbb{R}^n)$ may be viewed as a subspace of $\mathcal{S}'(\mathbb{R}^n)$ which contains $L^1(\mathbb{R}^n)$. Recall that, by the Riemann-Lebesgue Theorem (see e.g. [RT10a, Theorem 1.1.8]), the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ maps $L^1(\mathbb{R}^n)$ to $C_o(\mathbb{R}^n)$, and one can show that

$$C^*(\mathbb{R}^n) = \mathcal{F}_{\mathbb{R}^n}^{-1} C_o(\mathbb{R}^n).$$

*Remark* B.2.26. Note that the inclusion $\mathcal{F}_{\mathbb{R}^n}(L^1(\mathbb{R}^n)) \subset C_o(\mathbb{R}^n)$ is strict. Indeed for $n > 1$, the kernel of the Bochner Riesz means $\mathcal{F}_{\mathbb{R}^n}^{-1}\{\sqrt{1-|\xi|^2}1_{|\xi|\leq 1}\}$ is not in $L^1(\mathbb{R}^n)$ but its Fourier transform is in $C_o(\mathbb{R}^n)$. For $n = 1$, see e.g. Stein and Weiss [SW71, Ch 1, §4.1].

## B.2.4 Von Neumann algebras

Let us recall the von Neumann bi-commutant theorem:

**Theorem B.2.27.** *Let $\mathscr{L}(\mathcal{H})$ be the space of continuous linear operators on a Hilbert space $\mathcal{H}$ with its natural structure (see Example B.2.8). Let $\mathcal{M}$ be a $*$-subalgebra of $\mathscr{L}(\mathcal{H})$ containing the identity mapping* I. *Then the following are equivalent:*

*(i) $\mathcal{M}$ is equal to its bi-commutant (in the sense of Definition B.2.1):*

$$\mathcal{M} = \mathcal{M}''.$$

*(ii)* *$\mathcal{M}$ is closed in the weak-operator topology, i.e. the topology given by the family of seminorms $\{T \mapsto (Tv, w)_{\mathcal{H}},\ v, w \in \mathcal{H}\}$.*

*(iii)* *$\mathcal{M}$ is closed in the strong-operator topology, i.e. the topology on $\mathscr{L}(\mathcal{H})$ given by the family of seminorms $\{T \mapsto \|Tv\|_{\mathcal{H}},\ v \in \mathcal{H}\}$.*

This leads to the notion of a von Neumann algebra where we take the above equivalent properties as its definition:

**Definition B.2.28.** We keep the notation of Theorem B.2.27. A *von Neumann algebra in $\mathcal{H}$ is a $*$-subalgebra $\mathcal{M}$ of $\mathscr{L}(\mathcal{H})$ which satisfies any of the equivalent properties* $(i)$, $(ii)$, *or* $(iii)$ *in Theorem B.2.27.*

Note that the operator-norm topology on $\mathscr{L}(\mathcal{H})$ is stronger than the strong-operator topology, which in turn is stronger than the weak-operator topology. Thus a von Neumann algebra in $\mathcal{H}$ is a $*$-subalgebra of $\mathscr{L}(\mathcal{H})$ closed for the operator-norm topology, hence is a $C^*$-subalgebra of $\mathscr{L}(\mathcal{H})$ and a $C^*$-algebra itself. Among $C^*$-algebras, the von Neumann algebras are the $C^*$-algebras which are realised as a closed $*$-subalgebra of $\mathscr{L}(\mathcal{H})$ and furthermore satisfy any of the equivalent properties $(i)$, $(ii)$, or $(iii)$ in Theorem B.2.27.

It is also possible to define the von Neumann algebras abstractly as the $C^*$-algebras having a predual, see e.g. Sakai [Sak98].

*Example* B.2.29. Naturally $\mathscr{L}(\mathcal{H})$ and $\mathbb{C}\mathrm{I}_{\mathcal{H}}$ are von Neumann algebras in $\mathcal{H}$.

*Example* B.2.30. If $\eta$ is a positive and sigma-finite measure on a locally compact space $X$, then $\mathcal{A} = L^\infty(X, \eta)$ is a commutative unital $C^*$-algebra (see Example B.2.7). The operator of pointwise multiplication

$$L^\infty(X, \eta) \ni f \mapsto T_f \in \mathscr{L}(L^2(X, \mu)), \quad T_f(\phi) = f\phi,$$

is an isometric ($*$-algebra) morphism. This yields a $C^*$-algebra isomorphism from $\mathcal{A} = L^\infty(X, \eta)$ onto an abelian von Neumann algebra acting on the separable Hilbert space $L^2(X, \mu)$.

Conversely any abelian von Neumann algebra on a separable Hilbert space may be realised in the way described in Example B.2.30, see Dixmier [Dix96, §I.7.3].

The main example of von Neumann algebras of interest for us is the one associated with a group. This is explained in the next subsection.

### B.2.5 Group von Neumann algebra

In this section we follow Dixmier [Dix77, §13]. The main application of these constructions are in Section 1.8.2, see Definition 1.8.7 and the subsequent discussion.

Now, first let us define the (isomorphic) left and right von Neumann algebras of a (Hausdorff) locally compact group $G$.

The left, resp. right, von Neumann algebra of $G$ is the von Neumann algebra $\mathrm{VN}_L(G)$, resp. $\mathrm{VN}_R(G)$, in $L^2(G)$ generated by the left, resp. right, regular representation. This means that $\mathrm{VN}_L(G)$ *is the smallest von Neumann algebra containing all the operators* $\pi_L(x)$, $x \in G$, where $\pi_L$ is defined in (B.4), i.e.

$$\pi_L(x)\phi(y) := \phi(x^{-1}y), \quad x, y \in G, \ \phi \in L^2(G).$$

Let us recall that the right regular representation $\pi_R$ is given by

$$\pi_R(x)\phi(y) = \Delta(x)^{\frac{1}{2}}\phi(yx).$$

Here $\Delta$ denotes the modular function (see Definition B.2.10).

One checks easily that the isomorphism $U$ of $L^2(G)$ given by

$$U\phi(y) = \Delta(y)^{\frac{1}{2}}\phi(y^{-1}), \quad \phi \in L^2(G), \ y \in G,$$

intertwines $\pi_L$ and $\pi_R$:

$$\forall x \in G \qquad U\pi_L(x) = \pi_R(x)U.$$

Thus one is sometimes allowed to speak of 'the regular representation' and 'the group von Neumann algebra'. However, in this subsection, we will keep making the distinction between left and right regular representations.

Let us assume that the group $G$ is also separable. In this case, the group von Neumann algebra can be described further.

Clearly $\mathrm{VN}_L(G)$, resp. $\mathrm{VN}_R(G)$, is the smallest von Neumann algebra containing all the operators $(\pi_L)_{\mathscr{D}}(f)$, $f \in C_c(G)$, resp. $(\pi_R)_{\mathscr{D}}(f)$, $f \in C_c(G)$, see [Dix77, §13.10.2]. Here $C_c(G)$ denotes the space of continuous functions with compact support on $G$. For the definitions of $(\pi_L)_{\mathscr{D}}(f)$ and $(\pi_R)_{\mathscr{D}}$, see (B.5) and (B.1). This easily implies that $\mathrm{VN}_L(G)$, resp. $\mathrm{VN}_R(G)$, is the smallest von Neumann algebra containing all the operators $(\pi_L)_{\mathscr{D}}(f)$, resp. $(\pi_R)_{\mathscr{D}}(f)$, where $f$ runs over $L^1(G)$ or $C^*(G)$.

Applying the commutation theorem (cf. Dixmier [Dix96, Ch 1, §5.2]) to the quasi-Hilbertian algebra $C_c(G)$ ([Eym72, p. 210]) we see that

$$\mathrm{VN}_L(G) = (\mathrm{VN}_R(G))' \quad \text{and} \quad \mathrm{VN}_R(G) = (\mathrm{VN}_L(G))'.$$

See Definition B.2.1 for the definition of the commutant. This implies

**Proposition B.2.31.** *The group von Neumann algebra coincides with the invariant bounded operators in the following sense:*

- $\mathrm{VN}_L(G)$ *is the space* $\mathscr{L}_R(L^2(G))$ *of operators in* $\mathscr{L}(L^2(G))$ *which commute with* $\pi_R(x)$*, for all* $x \in G$*,*
- $\mathrm{VN}_R(G)$ *is the space* $\mathscr{L}_L(L^2(G))$ *of operators in* $\mathscr{L}(L^2(G))$ *which commute with* $\pi_L(x)$*, for all* $x \in G$*:*

$$\mathrm{VN}_L(G) = \mathscr{L}_R(L^2(G)) \quad \textit{and} \quad \mathrm{VN}_R(G) = \mathscr{L}_L(L^2(G)).$$

*Denoting by $J$ the involutive anti-automorphism on $L^2(G)$ given by*

$$J(\phi)(x) := \bar{\phi}(x^{-1})\Delta(x)^{-\frac{1}{2}}, \quad \phi \in L^2(G),\ x \in G,$$

*we also have*

$$J\ \mathrm{VN}_L(G)\ J = \mathrm{VN}_R(G) \quad \textit{and} \quad J\ \mathrm{VN}_R(G)\ J = \mathrm{VN}_L(G).$$

Under our hypotheses, it is possible to describe the group von Neumann algebra as a space of convolution operators, see Eymard [Eym72, Theorem 3.10 and Proposition 3.27]. In the special case of Lie groups, this is a consequence of the Schwartz kernel theorem, see Corollary 3.2.1 and its right-invariant version.

### B.2.6 Decomposition of group von Neumann algebras and abstract Plancherel theorem

The full abstract version of the Plancherel theorem allows us to decompose not only the Hilbert space $L^2(G)$ (thus obtaining the Plancherel formula) but also the operators in $\mathrm{VN}_R(G)$ and $\mathrm{VN}_L(G)$:

**Theorem B.2.32** (Plancherel theorem)**.** *We assume that the (Hausdorff locally compact separable) group $G$ is also unimodular and of type I and that a (left) Haar measure has been fixed.*

*Then there exist*

- *a positive sigma-finite measure $\mu$ on $\widehat{G}$,*
- *a $\mu$-measurable field of unitary continuous representations $(\pi_\zeta)_{\zeta\in\widehat{G}}$ of $G$ on the $\mu$-measurable field of Hilbert spaces $(\mathcal{H}_\zeta)_{\zeta\in\widehat{G}}$,*
- *and a unitary map $W$ from $L^2(G)$ onto*

$$\int_{\widehat{G}}^{\oplus} (\mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^*)\ d\mu(\zeta) \equiv \int_{\widehat{G}}^{\oplus} \mathtt{HS}(\mathcal{H}_\zeta)\ d\mu(\zeta),$$

*(see Subsection B.1.3)*

*such that $W$ satisfies the following properties:*

1. *If $\phi \in L^2(G)$, then $W\phi = \int_{\widehat{G}}^{\oplus} v_\zeta d\mu(\zeta)$ where each $v_\zeta$ is a Hilbert-Schmidt operator on $\mathcal{H}_\zeta$ and we have*

$$WJ\phi = \int_{\widehat{G}}^{\oplus} v_\zeta^*\ d\mu(\zeta), \quad \textit{where} \quad (J\phi)(x) = \bar{\phi}(x^{-1}).$$

2. *For any $f \in L^1(G)$ (or $C^*(G)$), the operators $(\pi_R)_{\mathcal{D}}(f)$ and $(\pi_L)_{\mathcal{D}}(f)$ acting on $L^2(G)$ are transformed via $W$ into the decomposable operators (in the sense of Definition B.1.15) on $\int_{\widehat{G}}^{\oplus}(\mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^*)d\mu(\zeta)$,*

$$W\left\{(\pi_L)_{\mathcal{D}}(f)\right\}W^{-1} = \int_{\widehat{G}}^{\oplus} (\pi_\zeta)_{\mathcal{D}}(f) \otimes \mathrm{I}_{\mathcal{H}_\zeta^*}\ d\mu(\zeta),$$

*and*

$$W\left\{(\pi_R)_{\mathcal{D}}(f)\right\}W^{-1} = \int_{\widehat{G}}^{\oplus} \mathrm{I}_{\mathcal{H}_\zeta} \otimes (\pi_\zeta^{dual})_{\mathcal{D}}(f)\ d\mu(\zeta).$$

*See (B.1) for the notation $(\pi)_{\mathcal{D}}$, and here $\pi_\zeta^{dual}$ denotes the dual representation to $\pi_\zeta$ which acts on $\mathcal{H}_\zeta^*$ via*

$$(\pi_\zeta^{dual}(x))v^* : w \mapsto (\pi_\zeta(x^{-1})w, v)_{\mathcal{H}_\zeta}.$$

3. *If $T$ is a bounded operator on $L^2(G)$ which commutes with $\pi_L(x)$, for all $x \in G$, that is, $T \in \mathrm{VN}_R(G) = \mathscr{L}_L(L^2(G))$, then $T$ is transformed via $W$ into a decomposable operator (in the sense of Definition B.1.15) on the Hilbert space $\int_{\widehat{G}}^{\oplus}(\mathcal{H}_\zeta \otimes \mathcal{H}_\zeta^*)d\mu(\zeta)$ of the form*

$$WTW^{-1} = \int_{\widehat{G}}^{\oplus} T_\zeta \otimes \mathrm{I}_{\mathcal{H}_\zeta^*}\ d\mu(\zeta).$$

*Conversely any decomposable operator of this type yields an operator in $\mathscr{L}_L(L^2(G))$. Hence we may summarise this by writing*

$$\mathrm{VN}_R(G) = \mathscr{L}_L(L^2(G)) = W^{-1}\int_{\widehat{G}}^{\oplus} \mathscr{L}(\mathcal{H}_\zeta) \otimes \mathbb{C}\ d\mu(\zeta)\ W.$$

*Similarly*

$$\mathrm{VN}_L(G) = \mathscr{L}_R(L^2(G)) = W^{-1}\int_{\widehat{G}}^{\oplus} \mathbb{C} \otimes \mathscr{L}(\mathcal{H}_\zeta^*)\ d\mu(\zeta)\ W.$$

*A consequence of Points 1. and 2. is that if $f \in L^1(G) \cap L^2(G)$, then $(\pi_\zeta)_{\mathcal{D}}(f) \in \mathtt{HS}(\mathcal{H}_\zeta)$ for almost every $\zeta \in \widehat{G}$ and*

$$Wf = \int_{\widehat{G}}^{\oplus} (\pi_\zeta)_{\mathcal{D}}(f) d\mu(\zeta) \quad \textit{thus} \quad \|f\|_{L^2(G)}^2 = \int_{\widehat{G}} \|(\pi_\zeta)_{\mathcal{D}}(f)\|_{\mathtt{HS}(\mathcal{H}_\zeta)}^2 d\mu(\zeta).$$

*The measure $\mu$ is standard (in the sense of Definition B.1.13, see also Proposition B.2.24) and unique modulo equivalence (see Proposition B.1.17).*

*Reference for the proof of Theorem B.2.32.* For the Plancherel measure being standard, see Dixmier [Dix77, Proposition 18.7.7 and Theorems 8.8.1 and 8.8.2]. For the Plancherel theorem expressed in terms of the canonical fields, see [Dix77, 18.8.1 and 18.8.2]. □

The main application of the above theorem for us is Theorem 1.8.11.

**Definition B.2.33.** The measure $\mu$ is called the *Plancherel measure* (associated to the fixed Haar measure).

A different choice of the Haar measure would lead to a different Plancherel measure. Up to this choice, the Plancherel measure is unique. Proposition B.1.17 then implies that we do not need to specify the choice of a measurable field of continuous representations.

In our monograph, our group Fourier transform and Dixmier's defined in (B.2) and (B.1) respectively, are related via (B.3). This implies that the statement of Theorem B.2.32 remains valid if we replace firstly $(\pi)_{\mathcal{D}}$ with our definition of the group Fourier transform and, secondly, $W$ with the isometric isomorphism

$$\tilde{W} : L^2(G) \to \int_{\widehat{G}}^{\oplus} \mathtt{HS}(\mathcal{H}_\zeta) d\mu(\zeta)$$

given by

$$\tilde{W}\phi := W(\phi \circ \mathrm{inv}) \quad \text{where} \quad \mathrm{inv}(x) = x^{-1}.$$

In particular, if $\phi \in L^2(G)$ then

$$\tilde{W}\phi = \int_{\widehat{G}}^{\oplus} \phi_\zeta d\mu(\zeta), \tag{B.6}$$

and we understand $(\phi_\zeta)_{\zeta\in\widehat{G}}$ as the group Fourier transform of $\phi$. If $T \in \mathscr{L}_L(L^2(G))$ then it may be decomposed by

$$\tilde{W}T\tilde{W}^{-1} = \int_{\widehat{G}}^{\oplus} T_\zeta \otimes \mathrm{I}_{\mathcal{H}_\zeta^*} \, d\mu(\zeta),$$

which means that if $\phi \in L^2(G)$ with (B.6), then

$$\tilde{W}(T\phi) = \int_{\widehat{G}}^{\oplus} T_\zeta \phi_\zeta d\mu(\zeta).$$

Theorem B.2.32 is reformulated in Theorem 1.8.11 with our choice of group Fourier transform.

We end this appendix with the following observation. Comparing closely the contents of Chapter 1 and Chapter B, there is a small discrepancy about the separability of Hilbert spaces. Indeed, in Chapter B, all the Hilbert spaces on which the representations act are assumed separable, see Section B.1.1, whereas the separability of the Hilbert spaces is not mentioned in Chapter 1. This leeds however to no contradiction when considering a continuous irreducible unitary representation $\pi$ of a Hausdorff locally compact separable group $G$ on a Hilbert space $\mathcal{H}_\pi$. Indeed, in this case, this yields a continuous non-degenerate representation of $L^1(G)$ on $\mathcal{H}_\pi$ as in Lemma B.2.18. As $L^1(G)$ is separable [Dix77, §13.2.4] and $\pi$ is irreducible, one can easily adapt the arguments in [Dix77, §2.3.3] to show that $\mathcal{H}_\pi$ is separable. Consequently, the dual $\widehat{G}$ of a Hausdorff locally compact separable group $G$ may be defined as in Section 1.1 as the equivalence classes of the continuous unitary representations, without stating the hypothesis of separability on the representation spaces.

## Schrödinger representations and Weyl quantization

Here we summarise the choices of normalisations and give some relations between the Schrödinger representations $\pi_\lambda$, $\lambda \in \mathbb{R}\backslash 0$, of the Heisenberg group $\mathbb{H}_n$ and the Weyl quantization on $L^2(\mathbb{R}^n)$. Detailed justifications and some proofs are given in Section 6.2.

Euclidean Fourier transform (for $f \in \mathcal{S}(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$)

$$\boxed{\mathcal{F}_{\mathbb{R}^N} f(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x) e^{-ix\xi} dx}$$

Weyl quantization (for $f \in \mathcal{S}(\mathbb{R}^N)$ and $u \in \mathbb{R}^N$)

$$\boxed{\mathrm{Op}^W(a) f(u) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i(u-v)\xi} a(\xi, \frac{u+v}{2}) f(v) dv d\xi}$$

The useful convention for abbreviating the expressions below is

$$\sqrt{\lambda} := \mathrm{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0. \end{cases} \tag{B.7}$$

Schrödinger representations (for $(x, y, t) \in \mathbb{H}_n$, $h \in L^2(\mathbb{R}^n)$, and $u \in \mathbb{R}^n$)

$$\boxed{\pi_\lambda(x, y, t) h(u) = e^{i\lambda(t + \frac{1}{2}xy)} e^{i\sqrt{\lambda} yu} h(u + \sqrt{|\lambda|} x)}$$

Notation for the group Fourier transform

$$\boxed{\pi_\lambda(\kappa) \equiv \widehat{\kappa}(\pi_\lambda) = \int_{\mathbb{H}_n} \kappa(x, y, t)\ \pi_\lambda(x, y, t)^*\ dx dy dt}$$

Relation between Schrödinger representation and Weyl quantization

$$\boxed{\pi_\lambda(\kappa) = (2\pi)^{\frac{2n+1}{2}} \mathrm{Op}^W \left[ \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\, \cdot, \sqrt{\lambda}\, \cdot, \lambda) \right]}$$

or, with more details,

$$\begin{aligned} \pi_\lambda(\kappa) h(u) &= \int_{\mathbb{H}_n} \kappa(x, y, t)\ \pi_\lambda(x, y, t)^* h(u)\ dx dy dt \\ &= \int_{\mathbb{R}^{2n+1}} \kappa(x, y, t) e^{i\lambda(-t + \frac{1}{2}xy)} e^{-i\sqrt{\lambda} yu} h(u - \sqrt{|\lambda|} x) dx dy dt \\ &= (2\pi)^{\frac{2n+1}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) h(v) dv d\xi. \end{aligned}$$

Plancherel formula

$$\boxed{\int_{\mathbb{H}_n} |f(x, y, t)|^2 dx dy dt = c_n \int_{\lambda \in \mathbb{R}\backslash\{0\}} \|\widehat{f}(\pi_\lambda)\|^2_{\mathtt{HS}(L^2(\mathbb{R}^n))} |\lambda|^n d\lambda}$$

# Explicit symbolic calculus on the Heisenberg group

Here we give a summary of some explicit formulae for symbolic analysis of concrete operators on the Heisenberg group $\mathbb{H}_n$. We refer to Section 6.3.3 for more details. We always employ the convention in (B.7) for $\sqrt{\lambda}$.

Symbols of left-invariant vector fields and the sub-Laplacian

$$\boxed{\begin{array}{rcccl} \pi_\lambda(X_j) &=& \sqrt{|\lambda|}\partial_{u_j} &=& \mathrm{Op}^W\left(i\sqrt{|\lambda|}\xi_j\right) \\ \pi_\lambda(Y_j) &=& i\sqrt{\lambda}u_j &=& \mathrm{Op}^W\left(i\sqrt{\lambda}u_j\right) \\ \pi_\lambda(T) &=& i\lambda \mathrm{I} &=& \mathrm{Op}^W(i\lambda) \\ \pi_\lambda(\mathcal{L}) &=& |\lambda|\sum_j(\partial_{u_j}^2 - u_j^2) &=& \mathrm{Op}^W\left(|\lambda|\sum_j(-\xi_j^2 - u_j^2)\right) \end{array}}$$

Difference operators

$$\boxed{\begin{array}{rcl} \Delta_{x_j}|_{\pi_\lambda} &=& \frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(Y_j)) \quad = \quad \frac{1}{\sqrt{|\lambda|}}\mathrm{ad}u_j \\ \Delta_{y_j}|_{\pi_\lambda} &=& -\frac{1}{i\lambda}\mathrm{ad}\,(\pi_\lambda(X_j)) \quad = \quad -\frac{1}{i\sqrt{\lambda}}\mathrm{ad}\partial_{u_j} \\ \Delta_t|_{\pi_\lambda} &=& i\partial_\lambda + \frac{1}{2}\sum_{j=1}^n \Delta_{x_j}\Delta_{y_j}|_{\pi_\lambda} + \frac{i}{2\lambda}\sum_{j=1}^n\left\{\pi_\lambda(Y_j)|_{\pi_\lambda}\Delta_{y_j} + \Delta_{x_j}|_{\pi_\lambda}\pi_\lambda(X_j)\right\} \end{array}}$$

Difference operators acting on symbols of left-invariant vector fields

| | $\pi_\lambda(X_k)$ | $\pi_\lambda(Y_k)$ | $\pi_\lambda(T)$ | $\pi_\lambda(\mathcal{L})$ |
|---|---|---|---|---|
| $\Delta_{x_j}$ | $-\delta_{j=k}$ | 0 | 0 | $-2\pi_\lambda(X_j)$ |
| $\Delta_{y_j}$ | 0 | $-\delta_{j=k}$ | 0 | $-2\pi_\lambda(Y_j)$ |
| $\Delta_t$ | 0 | 0 | $-\mathrm{I}$ | 0 |

Relation between the group Fourier transform and the $\lambda$-symbols

$$\boxed{\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda) = \mathrm{Op}^W(\tilde{a}_\lambda(\sqrt{|\lambda|}\cdot, \sqrt{\lambda}\cdot))}$$

$$\text{with } \boxed{\begin{array}{rcl} a_\lambda &=& \{a_\lambda(\xi,u) = \sqrt{2\pi}\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u, \lambda)\} \\ \tilde{a}_\lambda &=& \{\tilde{a}_\lambda(\xi,u) = \sqrt{2\pi}\mathcal{F}_{\mathbb{R}^{2n+1}}(\kappa)(\xi, u, \lambda)\} \end{array}}$$

Difference operators in terms of the Weyl quantization of $\lambda$-symbols

$$\boxed{\begin{array}{rcccl} \Delta_{x_j}\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right) &=& i\mathrm{Op}^W\left(\partial_{\xi_j}\tilde{a}_\lambda\right) \\ \Delta_{y_j}\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\frac{1}{\sqrt{\lambda}}\partial_{u_j}a_\lambda\right) &=& i\mathrm{Op}^W\left(\partial_{u_j}\tilde{a}_\lambda\right) \\ \Delta_t\pi_\lambda(\kappa) &=& i\mathrm{Op}^W\left(\tilde{\partial}_{\lambda,\xi,u}a_\lambda\right) &=& i\mathrm{Op}^W\left(\partial_\lambda\tilde{a}_\lambda\right) \end{array}}$$

$$\left(\text{with } \tilde{\partial}_{\lambda,\xi,u} = \partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^n\{u_j\partial_{u_j} + \xi_j\partial_{\xi_j}\}\right)$$

# List of quantizations

We refer to Sections 2.2, 5.1.3 and 6.5.1 for the cases of compact, graded, and Heisenberg groups, respectively.

Quantization on compact Lie groups (for $\varphi \in C^\infty(G)$ and $x \in G$)

$$\boxed{A\varphi(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\ \sigma_A(x,\pi)\ \widehat{\varphi}(\pi)\right)}$$

with the formula for the symbol

$$\boxed{\sigma_A(x,\pi) = \pi(x)^*(A\pi)(x)}$$

Quantization on general graded Lie groups (for $\varphi \in \mathcal{S}(G)$ and $x \in G$)

$$\boxed{A\varphi(x) = \int_{\widehat{G}} \operatorname{Tr}\left(\pi(x)\ \sigma_A(x,\pi)\ \widehat{\varphi}(\pi)\right) d\mu(\pi)}$$

Symbols of vector fields $\sigma_X(\pi) \equiv d\pi(X) = X\pi(e)$, see (1.22)

In the compact and graded cases, relation with the right-convolution kernel

$$\boxed{A\varphi(x) = \varphi * \kappa_x(x) = \int_G \varphi(y)\kappa_x(y^{-1}x)dy \quad \text{with} \quad \widehat{\kappa_x}(\pi) = \sigma_A(x,\pi)}$$

Quantization on the Heisenberg group (for $\varphi \in \mathcal{S}(\mathbb{H}_n)$ and $g = (x,y,t) \in \mathbb{H}_n$)

$$\boxed{A\varphi(g) = c_n \int_{\mathbb{R}\backslash\{0\}} \operatorname{Tr}\left(\pi_\lambda(g)\ \sigma_A(g,\lambda)\ \widehat{\varphi}(\pi_\lambda)\right) |\lambda|^n d\lambda}$$

and in terms of $\lambda$-symbols $a_{g,\lambda} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$\boxed{\sigma_A(g,\lambda) = \operatorname{Op}^W(a_{g,\lambda})} \quad (g \in \mathbb{H}_n,\ \lambda \in \mathbb{R}\backslash\{0\})$$

$$\begin{aligned}
&A\varphi(g) \\
&= c_n' \int_{\mathbb{R}\backslash\{0\}} \operatorname{Tr}\left(\pi_\lambda(g)\operatorname{Op}^W(a_{g,\lambda})\ \operatorname{Op}^W\left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\varphi)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda)\right]\right) |\lambda|^n d\lambda \\
&= c_n' \int_{\mathbb{R}\backslash\{0\}} \operatorname{Tr}\left(\operatorname{Op}^W(a_{g,\lambda})\ \operatorname{Op}^W\left[\mathcal{F}_{\mathbb{R}^{2n+1}}(\varphi(g\,\cdot))(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\,\cdot, \lambda)\right]\right) |\lambda|^n d\lambda
\end{aligned}$$