Introduction

Nilpotent Lie groups appear naturally in the analysis of manifolds and provide an abstract setting for many notions of Euclidean analysis. As is generally the case when studying analysis on nilpotent Lie groups, we restrict ourselves to the very large subclass of homogeneous (nilpotent) Lie groups, that is, Lie groups equipped with a family of dilations compatible with the group structure. They are the groups appearing ‘in practice’ in the applications (some of them are described below). From the point of view of general harmonic analysis, working in this setting also leads to the distillation of the results of the Euclidean harmonic analysis depending only on the group and dilation structures.

In order to motivate the work presented in this monograph, we focus our attention in this introduction on three aspects of the analysis on nilpotent Lie groups: the use of nilpotent Lie groups as local models for manifolds, questions regarding hypoellipticity of differential operators, and the development of pseudo-differential operators in this setting. We only outline the historical developments of ideas and results related to these topics, and on a number of occasions we refer to other sources for more complete descriptions. We end this introduction with the main topic of this monograph: the development of a pseudo-differential calculus on homogeneous Lie groups.

Nilpotent Lie groups by themselves and as local models

It has been realised for a long time that the analysis on nilpotent Lie groups can be effectively used to prove subelliptic estimates for operators such as ‘sums of squares’ of vector fields on manifolds. Such ideas started coming to light in the works on the construction of parametrices for the Kohn-Laplacian $\Box_b$ (the Laplacian associated to the tangential CR complex on the boundary X of a strictly pseudoconvex domain), which was shown earlier by J. J. Kohn to be hypoelliptic (see e.g. an exposition by Kohn [Koh73] on the analytic and smooth hypoellipticities). Thus, the corresponding parametrices and subsequent subelliptic estimates have been obtained by Folland and Stein in [FS74] by first establishing a version of the results for a family of sub-Laplacians on the Heisenberg group, and then for the Kohn-Laplacian $\Box_b$ by replacing X locally by the Heisenberg group. These ideas soon led to powerful generalisations. The general techniques for approximating vector fields on a manifold by left-invariant operators on a nilpotent Lie group have been developed by Rothschild and Stein in [RS76]. Here the dimension of the nilpotent Lie group is normally larger than that of the manifold, and a first step of such a construction is to perform the 'lifting' of vector fields to the group. Consequently, this approach allowed one to produce parametrices for the original differential operator on the manifold by using the analysis on homogeneous Lie groups. A more geometric version of these constructions has been carried out by Folland in [Fol77b], see also Goodman [Goo76] for the presentation of nilpotent Lie algebras as tangent spaces (of sub-Riemannian manifolds). The functional analytic background for the analysis in the stratified setting was laid down by Folland in [Fol75]. A general approach to studying geometries appearing from systems of vector fields has been developed by Nigel, Stein and Wainger [NSW85].

Thus, one of the motivations for carrying out the analysis and the calculus of operators on nilpotent Lie groups comes from the study of differential operators on CR (Cauchy-Riemann) or contact manifolds, modelling locally the operators there on homogeneous invariant convolution operators on nilpotent groups. In 'practice' and from this motivation, only nilpotent Lie groups endowed with some compatible structure of dilations, i.e. homogeneous Lie groups, are considered. This will be also the setting of our present exposition.

The simplest example (apart from $\mathbb{R}^n$) of a nilpotent Lie group is the Heisenberg group, and the harmonic analysis there is a very well researched topic. We do not intend to make an overview of the subject here, but we refer to the books of Stein [Ste93] and Thangavelu [Tha98] for an introduction to the harmonic analysis on the Heisenberg group and for the historic development of the area. Elements of the harmonic analysis on different groups can be also found in Taylor's book [Tay86]. The Heisenberg group enters many applied areas, including various aspects of quantum mechanics, signal analysis, optics, thermodynamics; we refer to the recent book of Binz and Pods [BP08] for an overview of this subject. We mention another recent book by Calin, Chang and Greiner [CCG07] containing many explicit calculations related to the Heisenberg group and its sub-Riemannian geometry, as well as a sub-Riemannian treatment in Capogna, Danielli, Pauls and Tyson [CDPT07]. As such, in this monograph we will deal with the Heisenberg group almost exclusively in the context of pseudo-differential operators, and we refer to excellent surveys of Folland [Fol77a] and Howe [How80] on the role played by the Heisenberg group in the theory of partial differential equations and in harmonic analysis, as well as to Folland's book [Fol89] for its relation to the theory of pseudo-differential operators on $\mathbb{R}^n$ through the Weyl quantization. See also a more recent short survey by Semmes [Sem03] and a book by Krantz [Kra09].

Well-posedness questions for hyperbolic partial differential equations on the Heisenberg group have been considered parallel to their Euclidean counterparts. For example, the conditions for the well-posedness of the wave equation for the Laplacian associated to the $\bar{\partial}_b$ complex have been found by Nachman [Nac82], the L^p-estimates for the wave equation for the sub-Laplacian have been established by Müller and Stein [MS99], the smoothness of the Schrödinger kernel has been analysed by Sikora and Zienkiewicz [SZ02], a space-time estimate for the Schrödinger equation has been obtained by Zienkiewicz [Zie04], etc. Nonlinear wave and Schrödinger equations and Strichartz estimates have been analysed on the Heisenberg group as well, see e.g. Zuily [Zui93], Bahouri, Gérard and Xu [BGX00] and Furioli, Melzi and Veneruso [FMV07], as well as other equations, e.g. the Ginzburg-Landau equation by Birindelli and Valdinoci [BV08], quasilinear equations by Capogna [Cap99], etc.

The Hardy spaces on homogeneous Lie groups and the surrounding harmonic analysis have been investigated by Folland and Stein in their monograph [FS82]. In general, there are different machineries available depending on a degree of generality: the stratified Lie groups enjoy additional hypoellipticity techniques going back to Hörmander's celebrated sum of the squares theorem, while on the Heisenberg group explicit expressions from its representation theory can be used.

A typical example of such different degrees of generality within homogeneous Lie groups is, for instance, a problem of characterising the Hardy space H^1 in L^1 by families of singular integrals. Thus, in [CG84], Christ and Geller presented sufficient conditions for general homogeneous Lie groups, gave explicit examples of (generalised) Riesz transforms for such a family of integral operators on stratified Lie groups, and derived further necessary and sufficient conditions on the Heisenberg group in terms of its representation theory (see also further work by Christ [Chr84]).

A related aspect of harmonic analysis, the Calderón-Zygmund theory on homogeneous Lie groups, has a long history as well. Again, this started with the analysis of convolution operators (with earlier works e.g. by Korányi and Vági [KV71] in the nilpotent direction), but in this book we will adopt an utilitarian approach, and the setting of Coifman and Weiss [CW71a] of spaces of homogeneous type will be sufficient for our purposes (see Section 3.2.3 and Section A.4).

Proceeding with this part of the introduction on general homogeneous Lie groups, let us follow Folland and Stein [FS82] and mention another important occurrence of homogeneous Lie groups. If G is a non-compact real connected semi-simple Lie group, its Iwasawa decomposition $G = KAN$ contains the homogeneous Lie group N whose family of dilations comes from an appropriate one-parameter subgroup of the abelian group A (more precisely, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$, the decomposition $G = KAN$ corresponds to the Iwasawa decomposition of the Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{a}$ is the maximal abelian subalgebra of $\mathfrak{k}^\perp$, and the nilpotent Lie algebra $\mathfrak{n}$ is the sum of the positive root spaces corresponding to eigenvalues of $\mathfrak{a}$ acting on $\mathfrak{g}$). This decomposition generalises the decomposition of a real matrix as a product of an orthogonal, diagonal, and an upper triangular with 1 at the diagonal matrix. Furthermore, the the symmetric space G/K has the homogeneous nilpotent Lie group N as its 'boundary' in the sense that N may be identified with a dense subset of the maximal boundary of G/K. As we show in Section 6.1.1 for $n_o = 1$, if $G = \mathrm{SU}(n_o + 1, 1)$, G/K may be identified with the unit ball in $\mathbb{C}^{n_o+1}$ and the Heisenberg group $\mathbb{H}_{n_o}$ acts simply transitively on the complex sphere of $\mathbb{C}^{n_o+1}$

where one point has been excluded. This provides a link between the Heisenberg group $\mathbb{H}_{n_o}$, the analysis of the complex spheres, and the group $\mathrm{SU}(n_o + 1, 1)$ or, more generally, between general semi-simple Lie groups and homogeneous Lie groups as boundaries of their symmetric spaces. For example, harmonic functions on the symmetric space G/K can be represented by convolution operators on N (see e.g. the survey of Koranyi [Kor72]).

Our setting contains the realm of Carnot groups as this class of groups consists of the stratified Lie groups equipped with a specified metrics on the first layer, see e.g. Gromov [Gro96] for a survey on geometric analysis of Carnot groups. Our setting includes any class of stratified Lie groups, for instance H-groups, Heisenberg-Kaplan groups, Métivier-type groups [Mét80], filiform groups, as well as Kolmogorov-type groups appearing in the study of hypoelliptic ultra-parabolic operators including the Kolmogorov-Fokker-Planck operator (see Kolmogorov [Kol34], Lanconelli and Polidoro [LP94]). We refer to the book [BLU07] by Bonfiglioli, Lanconelli and Uguzonni for a detailed consideration of these groups and of their sub-Laplacians as well as related operators.

Hypoellipticity and Rockland operators

On compact Lie groups, the Fourier analysis and the symbolic calculus developed in [RT10a] are based on the Laplacian and on the growth rate of its eigenvalues. While on compact Lie groups the Laplacians (or the Casimir element) are operators naturally associated to the group, it is no longer the case in the nilpotent setting. Thus, on nilpotent Lie groups it is natural to work with operators associated with the group through its Lie algebra structure. On stratified Lie groups these are the sub-Laplacians, and such operators are not elliptic but hypoelliptic. More generally, on graded Lie groups invariant hypoelliptic differential operators are the so-called Rockland operators.

Indeed, in [Roc78], Rockland showed that if T is a homogeneous left-invariant differential operators on the Heisenberg group, then the hypoellipticity of T and T^t is equivalent to a condition now called the Rockland condition (see Definition 4.1.1). He also asked whether this equivalence would be true for more general homogeneous Lie groups. Soon after, Beals showed in [Bea77b] that the hypoellipticity of a homogeneous left-invariant differential operator on any homogeneous Lie group implies the Rockland condition. In the same paper he also showed that the converse holds in some step-two cases. Eventually in [HN79], Helffer and Nourrigat settled what has become known as Rockland's conjecture by proving that the hypoellipticity is equivalent to the Rockland condition (see Section 4.1.3). At the same time, it was shown by Miller [Mil80] that in the setting of homogeneous Lie groups, the existence of an operator satisfying the Rockland condition (hence of an invariant hypoelliptic differential operator in view of Helffer and Nourrigat's result), implies that the group is graded, see also Section 4.1.1. This means, altogether, that the setting of graded Lie groups is the right generality for marrying the harmonic analysis techniques with those coming from the theory of partial differential equations.

A number of well-known functional inequalities can be extended to the graded setting, for example, see Bahouri, Fermanian-Kammerer and Gallagher [BFKG12b]. Also, there are many contributions to questions of solvability related to the hypoellipticity problem: for a good introduction to local and non-local solvability questions on nilpotent Lie groups see Corwin and Rothschild [CR81] and, missing to mention many contributions, for a more recent discussion of the topic see Müller, Peloso and Ricci [MPR99].

The hypoellipticity of second order operators is a very well researched subject. Its beginning may be traced to the 19th century with the diffusion problems in probability arising in Kolmogorov's work [Kol34]. Hörmander made a major contribution [Hör67b] to the subject which then developed rapidly after that (see e.g. the book of Oleinik and Radkevich [OR73]) until nowadays. We will not be concerned much with these nor with the solvability problems in this book, since one of topics of importance to us will be Rockland operators of an arbitrary degree, and we will be giving more relevant references as we go along.

Here we want to mention that the question of the analytic hypoellipticity turns out to be more involved than that in the smooth setting. In general, if a graded Lie group is not stratified, there are no homogeneous analytic hypoelliptic left-invariant differential operators, a result by Helffer [Hel82]. For stratified Lie groups, the situation is roughly as follows: for H-type groups the analytic hypoellipticity is equivalent to the smooth hypoellipticity, while for step ≥ 3 (and an additional assumption that the second stratum is one-dimensional) the sub-Laplacians are not analytic hypoelliptic, see Métivier [Mét80] and Helffer [Hel82], respectively, and the discussions therein. For the Kohn-Laplacian $\Box_b$ in the $\bar{\partial}$-Neumann problem as well as for higher order operators in this setting the analytic hypoellipticity was shown earlier by Tartakoff [Tar78, Tar80]. Below we will mention a few more facts concerning the analytic hypoellipticity in the framework of the analytic calculus of pseudo-differential operators.

Pseudo-differential operators

Several versions of the smooth calculi of pseudo-differential operators on the Heisenberg group have been considered over the years. An earlier attempt yielding the calculus of invariant operators with symbols on the dual $\mathfrak{g}'$ of the Lie algebra of the group was made by Strichartz [Str72]. A calculus for (right-invariant) operators has been also constructed by Melin [Mel81] yielding parametrices for operators elliptic in the so-called generating directions. In particular, the symbolic calculus for invariant operators on stratified and graded Lie groups developed by Melin further in [Mel83] provided a simpler proof of many of Helffer and Nourrigat's arguments.

The question of a general symbolic calculus for convolution operators on nilpotent Lie groups was raised by Howe in [How84], who also tackled questions related to the Calderón-Vaillancourt theorem. A more recent development of the calculus for invariant operators on homogeneous Lie groups and applications to the corresponding symbolic conditions for the L^2-boundedness of convolution operators was given by Głowacki in [Gło04] and [Gło07]. All this analysis applies to invariant operators and employs the Euclidean Fourier transform yielding a symbol on the dual $\mathfrak{g}'$. The symbol classes of such operators on the group are defined as coming from the usual Hörmander classes on the (Euclidean) vector space $\mathfrak{g}'$. They satisfy the spectral invariance properties and yield further useful generalisations of parametrix constructions, see Głowacki [Gło12]. An approach to Melin's operators on nilpotent Lie groups from the point of view of the Weyl calculus was done by Manchon, with further applications to the Weyl spectral asymptotics for the infinitesimal representations of elliptic operators in his calculus, see [Man91]. There exists also a calculus of left-invariant integral operators on the Heisenberg group, using Laguerre polynomials for its Fourier analysis, see Beals, Gaveau, Greiner and Vauthier [BGGV86], or using Leray's quadratic Fourier transform by Gaveau, Greiner and Vauthier [GGV86].

While these are mostly the calculi of invariant operators, the geometric considerations require one to also understand operators in the non-invariant setting. However, here the amount of knowledge is more limited and most of the symbolic calculus is restricted to the Heisenberg group. Dynin's construction of certain operators on the Heisenberg group in [Dyn76] (see also [Dyn78]), was also developed by Folland into considering meta-Heisenberg groups in [Fol94]. Beside this, a non-invariant pseudo-differential calculus on any homogeneous Lie group was developed by Christ, Geller, Głowacki and Polin in [CGGP92] but this is not symbolic since the operator classes are defined via properties of the kernel. In the revised version of [Tay84], Taylor described several (non-invariant) operator calculi and, in a different direction, he also noted a way to develop symbolic calculi: using the representations of the group, he defines a general quantization and symbols on any unimodular type I group (by quantization, we mean a procedure which associates an operator with a symbol). He illustrated this on the Heisenberg group and obtained there several important applications for, e.g., the study of hypoellipticity. He used the fact that, because of the properties of the Schrödinger representations of the Heisenberg group, a symbol is a family of operators in the Euclidean space, themselves given by symbols via the Weyl quantization. Recently, the definition of suitable classes of Shubin type for these Weyl-symbols led to another version of the calculus on the Heisenberg group by Bahouri, Fermanian-Kammerer and Gallagher [BFKG12a].

A calculus of pseudo-differential operators on the Heisenberg group in the analytic setting was developed by Geller [Gel90], with applications to the analytic hypoellipticity and further extensions of the calculus to real analytic CR manifolds. In particular, this implies the analytic hypoellipticity of the Kohn-Laplacian on q-forms on pseudoconvex real analytic manifolds. It also implies that the Szegö projection preserves analyticity, recovering earlier results on the relations between Szegö projections and $\bar{\partial}_b^*$-operator by Greiner, Kohn and Stein [GKS75], in turn related to the solvability of the Lewy equation. The analytic hypoellipticity of the complex boundary Kohn-Laplacian on the (p, q)-forms arising in the $\bar{\partial}$-Neumann problem was proved earlier by Tartakoff [Tar78] using L^2-methods and by Trèves [Trè78] using the calculus. Here we note that a corresponding Euclidean symbolic calculus with applications to the propagation of analytic singularities and corresponding version of Fourier integral operators has been developed by Sjöstrand [Sjö82], and the propagation of the analytic wave front set for the sub-Laplacian was studied by Grigis and Sjöstrand [GS85].

The analysis of the nilpotent setting can be extended to more general manifolds. Here, a typical application of the analysis on the Heisenberg groups is to questions on contact manifolds. Indeed, a contact structure defines a grading on the space of vector fields assigning them the degree of one or two. Locally, a contact manifold is diffeomorphic to the Heisenberg group, and the principal symbol of a differential operator on the contact manifold is its higher order terms, with the calculus on the contact manifold induced by that on the Heisenberg group, at least on the principal symbol level. The ellipticity condition for an operator on a contact manifold is thus replaced by the Rockland condition for the homogeneous principal part of the corresponding operator on the Heisenberg group. Such constructions can be carried out in more general settings, in particular on the so-called Heisenberg manifolds, which are smooth manifolds with a distinguished hyperplane bundle. The calculus of operators in this setting was carried out by Beals and Greiner in [BG88], in particular also generalising the calculus on CR manifolds needed for the construction of parametrices for the Kohn-Laplacian $\Box_b$. A recent advance in this direction mostly aimed at the second order operators on Heisenberg manifolds, with a more intrinsic notion of the principal symbol of such operators, was made by Ponge [Pon08]. Examples of such analysis include CR manifolds and contact manifolds, with applications to the Kohn-Laplacian, the Gover-Graham operators, the contact Laplacian associated to the Rumin complex, as well as to more general Rockland operators.

Moreover, such operators are subelliptic and their index may be calculated by the Atiyah-Singer index formula, see van Erp [vE10a]. The explicit knowledge of the Bargmann-Fock representations of the Heisenberg group allows one to construct the necessary Heisenberg calculus adapted to subelliplic operators in this setting, leading to the index formula also for subelliptic pseudo-differential operators on contact manifolds, see van Erp [vE10b].

The calculus of pseudo-differential operators on homogeneous Lie groups in terms of their kernels developed by Christ, Geller, Głowacki and Polin in [CGGP92] extended the parametrix construction of Helffer and Nourrigat in [HN79] and some properties of Taylor's calculus from [Tay84] in the case of the Heisenberg group. However, this calculus is not symbolic since it is based on the properties of the kernel. The same is true for the analysis of operators on unimodular Lie groups considered by Meladze and Shubin [MS87] where the operator classes were defined in terms of local properties of the kernels.

Recently in [MR15], Mantoiu and the second author developed a more general τ-quantization scheme on general locally compact unimodular type I groups, thus encompassing in particular the cases of compact Lie groups by the second author and Turunen [RT10a] and nilpotent Lie groups including the one developed in this monograph. Moreover, the τ-quantizations there allow one to deal with analogues of both Kohn-Nirenberg and Weyl quantizations. However, due to the generality the scope of available results at the moment is much more limited than the one presented in [RT10a] in the compact case, or in this book. The type I assumption is useful for having a rich machinery concerning the abstract Plancherel theorem (see Section 1.8.2), however it can be dropped for some questions: e.g. for an L^p-L^q Fourier multiplier theorem on general locally compact separable unimodular groups (without type I assumption) see Akylzhanov and Ruzhansky [AR15]. For nilpotent Lie groups, the relation between such quantizations and Melin's quantization described above has been also established in [MR15].

Quantization on homogeneous Lie groups and the book structure

Most of the above works that concern the non-invariant symbolic calculi of operators on nilpotent Lie groups, are restricted to the Heisenberg groups or to manifolds having the Heisenberg group as a local model (except for the calculi which are not symbolic). One of the reasons is that they rely in an essential way on the explicit formulae for representations of the Heisenberg group. However, in all the motivating aspects described above graded Lie groups appear as well as the Heisenberg group. Also, graded groups appear as local models once one is dealing with operators which are not in the form of 'sum of squares' even on manifolds such as the Heisenberg manifolds.

Recently, in [RT10a, RT13], the second author and Turunen developed a global symbolic calculus on any compact Lie group. They defined symbol classes so that the quantization procedure, analogous to the Kohn-Nirenberg quantization on $\mathbb{R}^n$, makes sense on compact Lie groups, and the resulting operators form an algebra with properties 'close enough' to the one enjoyed by the Euclidean Hörmander calculus. In particular, one can also recover the Hörmander classes of pseudo-differential operators on compact Lie groups viewed as compact manifolds through conditions imposed on global full matrix-valued symbols. This approach works for any compact Lie group and is intrinsic to the group in the sense that it does not depend on pseudo-differential calculus of (Euclidean) Hörmander classes on its Lie algebra. While relying on the representation theory of the group, the quantization and the calculus do not depend on explicit formulae for its representations. It does not depend either on the available Riemannian structure. This gives an advantage over the calculi expressed in terms of a fixed connection of a manifold, such as the one developed by Widom [Wid80], Safarov [Saf97], and Sharafutdinov [Sha05], see also the survey of McKeag and Safarov [MS11].

The crucial and new ingredient in the definition of symbol classes in [RT10a] was the introduction and systematic use of *difference operators* in order to replace the Euclidean derivatives in the Fourier variables by 'analogous' operators acting on the unitary dual of the group. These difference operators allow one to express the pseudo-differential behaviour directly on the group and are very natural from the point of view of the Calderón-Zygmund theory. These operators and their properties on compact Lie groups will be reviewed in Section 2.2.2.

It is not possible however to extend readily the results of the compact case developed in [RT10a] to the nilpotent context. Indeed, the global analysis on a non-compact setting is usually more challenging than in the case of a compact manifold. In the specific case of Lie groups, the dual of a non-compact group is no longer discrete and the unitary irreducible representations may be infinite dimensional, and are often so. More problematically there is no Laplacian and one expects to replace it by a sub-Laplacian on stratified Lie groups or, more generally, by a positive Rockland operator on graded Lie groups; such operators are not central.

Thus, in this book we study the global quantization of operators on graded Lie groups, in particular aiming at developing an intrinsic symbolic calculus of such operators. This is done in Chapter 5. As noted earlier, the graded Lie groups is a natural generality for such analysis since we can still make a full use of the Rockland operators as well as from the representation theory which is well understood e.g. by Kirillov's orbit method [Kir04]. The consequent Fourier analysis is then also well understood from earlier works on the Plancherel formula on nilpotent and even on more general locally compact unimodular type I groups; an overview of this topic is given in Section 1.7.

Summarising very briefly the results presented in Chapter 5, we introduce a global quantization on graded Lie groups and classes $S^m_{\rho,\delta}$ of symbols and of the corresponding operators in $\Psi^m_{\rho,\delta} = \mathrm{Op}\, S^m_{\rho,\delta}$ such that for each (ρ, δ) with $1 \geq \rho \geq \delta \geq 0$ and $\delta \neq 1$, we have an operator calculus, in the sense that the set $\bigcup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ forms an algebra of operators, stable under taking the adjoint, and acting on the Sobolev spaces in such a way that the loss of derivatives is controlled by the order of the operator. Moreover, the operators that are elliptic or hypoelliptic within these classes allow for a parametrix construction whose symbol can be obtained from the symbol of the original operator. Some applications of the constructed calculus are contained in Chapter 5, see also the authors' paper [FR13] for further applications to lower bounds of operators on graded Lie groups. A preliminary very brief outline of the constructions here was given in [FR14a].

To lay down the necessary foundation for the quantization of operators and symbols, we also make an exposition of the construction of the Sobolev spaces on graded Lie groups based on positive Rockland operators. Such construction has been previously done on stratified Lie groups by Folland [Fol75], for Sobolev spaces based on the (left-invariant) sub-Laplacian. Sub-Laplacians in this context are (up to a sign) a particular case of positive Rockland operators on stratified groups and our results coincide with Folland's in this case. However if we follow Folland's treatment but now in the more general context of graded Lie groups, beside the appearance of several technical problems, we would be led to make further assumptions. One of them would be that the degree of the positive Rockland operator ν must be less than the homogeneous dimension Q of the group, $\nu < Q$

(assuming $\nu < Q$ ensures the uniqueness of its homogeneous fundamental solution). In fact, Goodman makes such an assumption in his treatment of Sobolev spaces on graded Lie group in [Goo76]. In order to avoid this assumption and also to deal with other issues, we need to develop other arguments in the study of the powers of a general positive Rockland operator $\mathcal{R}$ and of $\mathrm{I}+\mathcal{R}$ and in the study of the associated Sobolev spaces. This is done under no assumptions on the relation between ν and Q in Sections 4.3 and 4.4.

The analysis of Sobolev spaces is based on the heat kernel associated with a positive Rockland operator. We make an exposition of this topic in Section 4.2.2. Our presentation there follows essentially the arguments of Folland and Stein [FS82]. The heat kernels are not necessarily positive functions and the heat semi-group does not necessarily correspond to a martingale as in the stratified case or more generally for sums of squares of vector fields with Hörmander's condition. Such sums of squares have been analysed in much more general settings. For example, we can refer to the book of Varopoulos, Saloff-Coste and Coulhon [VSCC92] for a treatment of the heat kernel on unimodular Lie groups of polynomial growth, and the usually associated to it estimates, such as Harnack and Sobolev inequalities. For another point of view, allowing dealing with heat kernels associated to more general subelliptic second order order differential operators we refer to Dungey, ter Elst and Robinson's book [DtER03].

Overall, the majority of the background material can be (sometimes even more easily) introduced in the setting of homogeneous Lie groups and we discuss these in Chapter 3. Our treatment in this chapter is inspired by those of Folland and Stein [FS82] and Ricci [Ric] but is slightly more general than that in the existing literature since we allow kernels and operators to have complex-valued homogeneity degrees. This allows us to treat complex powers of operators later on, e.g. in Section 4.3.3.

We assume that the reader is familiar with analysis at a graduate level, e.g. as presented in the books of Rudin [Rud87, Rud91], Reed and Simon [RS80, RS75], or Folland [Fol99]. Nevertheless, we make a brief exposition of topics, mostly from the representation theory of groups, to remind the reader of the necessary concepts used in later parts of the book and to fix the terminology and notation. This is done in Chapter 1, and references to more material are given throughout.

The exposition of the (matrix) quantization on compact Lie groups from [RT10a] and its related works is given in Chapter 2. This serves both as an introduction to the topic as well as provides motivation and examples for some of the concepts presented later in the book.

Chapter 6 is devoted to presenting an application of the general theory developed in Chapter 5 to the concrete setting of the Heisenberg groups. Some results from this chapter have been announced in the authors' paper [FR14b] and this chapter provides their proofs. We give the necessary preliminaries of the analysis on the Heisenberg group, including a description of its dual using Schrödinger representations, with further concrete expressions for the Plancherel measure and Plancherel formula. For the Heisenberg group $\mathbb{H}_n$, its Schrödinger representations

π_λ are acting on the space $L^2(\mathbb{R}^n)$, thus yielding symbols acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of smooth vectors of π_λ. In turn, these symbols can be conveniently described using their Weyl quantization, giving a notion of scalar-valued λ-symbols. In this particular case of the Heisenberg group, the symbol classes of Chapter 5 can be characterised by the property that these λ-symbols belong to some Shubin spaces, more precisely, a semiclassical-λ-type version of the usual Shubin classes of symbols. Consequently, this is applied to giving criteria for ellipticity and hypoellipticity of operators on the Heisenberg group in terms of the invertibility properties of their λ-symbols. We provide a list of examples to show the applicability of these results in several settings.

Notation and conventions

$\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}_* = \mathbb{R}\backslash\{0\}$
$\mathbb{R}_+ = (0, \infty)$
$\mathbb{C}_+ = \{z \in \mathbb{C}, \operatorname{Re} z > 0\}$
$0! = 1$
For $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$
$[\alpha]$ is the homogeneous degree, defined in (3.12)
$\lceil x \rceil$ is the smallest $n \in \mathbb{Z}$ such that $n > x$
$\lfloor x \rfloor$ is the largest $n \in \mathbb{Z}$ such that $n \leq x$
$\lceil M \rfloor = \max\{|\alpha| : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \leq M\}$, as defined in (3.35)
$A \asymp B$ means there is some $c > 0$ such that $c^{-1}A \leq B \leq cA$
$\overline{\sum_j} a_j := \sum_j c_j a_j$ denotes a (finite) linear combination with some (irrelevant) constants c_j
Unit elements: e on general groups, and 0 on nilpotent groups
δ_x is the delta-distribution at x: $\delta_x(\phi) = \phi(x)$
$\delta_{j,k}$ is the Kronecker delta: $\delta_{j,k} = 0$ for $j \neq k$, and $\delta_{j,j} = 1$
$\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the space of all linear continuous mappings from $\mathcal{H}_1$ to $\mathcal{H}_2$
$\mathscr{L}(\mathcal{H}) := \mathscr{L}(\mathcal{H}, \mathcal{H})$
$\mathcal{U}(\mathcal{H})$ is the space of unitary mappings in $\mathscr{L}(\mathcal{H})$
G is a Lie group, $\mathfrak{g}$ is its Lie algebra, and
$\mathfrak{U}(\mathfrak{g})$ is its universal enveloping algebra defined in Section 1.3
$\bar{T}$, T^* and T^t are defined by (1.8), (1.9), and (1.10) for an element $T \in \mathfrak{U}(\mathfrak{g})$ and in Definition 1.3.1 for an operator T on $L^2(G)$
$\operatorname{Rep} G$ is the set of all strongly continuous unitary irreducible representations of the group G
$\widehat{G}$ is the unitary dual of G, i.e. $\operatorname{Rep} G$ modulo the equivalence of representations
$L^2(\widehat{G})$ is the space of square integrable fields on $\widehat{G}$ with respect to the Plancherel measure, see (1.29)
$L^\infty(\widehat{G})$, $\mathscr{L}_L(L^2(G))$, and $\mathcal{K}(G)$ are the realisations of the von Neumann algebra of the group G as the spaces of the bounded fields of operators on $\widehat{G}$, of the left-

invariant operators in $\mathscr{L}(L^2(G))$, and of the convolution kernels corresponding to the latter, respectively, see Section 1.8.2

$L^\infty_{a,b}(\widehat{G})$, $\mathscr{L}_L(L^2_a(G), L^2_b(G))$, and $\mathcal{K}_{a,b}(G)$ are the Sobolev versions of the above, see Section 5.1.2

$\mathrm{Diff}^k(G)$ is the space of left-invariant differential operators of order k

$\mathrm{diff}^k(\widehat{G})$ is the space of difference operators on $\widehat{G}$ of order k

$C_c(G)$ is the space of continuous functions on G with compact support

$C_o(G)$ is the space of continuous functions on G vanishing at infinity (see Definition 3.1.57)

$\mathcal{D}(G) = C_c^\infty(G)$ or $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ are spaces of smooth compactly supported functions

$C^\infty(G, F)$ denotes the set of smooth functions from G to a Fréchet space F

$L^p(G)$ or simply L^p for $1 \leq p \leq \infty$ is the usual Lebesgue space on G with norm $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(G)} = \|\cdot\|_p$

$(\cdot,\cdot)_{L^2} = (\cdot,\cdot)_{L^2(G)}$ is the Hilbert sesquilinear form on $L^2(G)$ corresponding to the norm $\|\cdot\|_{L^2}$

$M(G)$ the Banach space of regular complex measures on G endowed with the total mass $\|\cdot\|_{M(G)}$

$\langle\cdot,\cdot\rangle$ denotes the distributional duality

Q denotes the homogeneous dimension of a homogeneous Lie group. After Section 6.4.2, it denotes the harmonic oscillator

sup always means the essential supremum with respect to the corresponding measure