Chapter 1

Preliminaries on Lie groups

In this chapter we provide the reader with basic preliminary facts about Lie groups that we will be using in the sequel. At the same time, it gives us a chance to fix the notation for the rest of the monograph. The topics presented here are all well-known and we decided to give a brief account without proofs referring the reader for more details to excellent sources where this material is treated from different points of view; for example, the monographs by Chevalley [Che99], Fegan [Feg91], Nomizu [Nom56], Pontryagin [Pon66], to mention only a few. Thus, this chapter can also serve as a quick and informal introduction to the subject, and we refer to monographs [RT10a] for an undergraduate level introduction to general Lie groups and their representation theory, and to Corwin and Greenleaf [CG90] or Goodman [Goo76] for a rather comprehensive treatment of nilpotent Lie groups. The groups that we are dealing with in the monograph are either compact or nilpotent Lie groups, so we can restrict our attention to unimodular Lie groups only.

The choice of material is adapted to our subsequent needs and, after giving basic definitions, we go straight to discussing convolutions, invariant differential operators, and elements of the representation theory. More information on compact or homogeneous nilpotent Lie groups will be given in relevant chapters at appropriate places. In particular Section 3.1.1 will provide examples and basic properties of graded nilpotent Lie groups. Relevant monographs to consult on invariant differential operators and related harmonic analysis may be Helgason's books [Hel84b, Hel01] or Wallach's [Wal73].

1.1 Lie groups, representations, and Fourier transform

A *Lie group* G is a smooth manifold endowed with the smooth mappings

$$G \times G \ni (x, y) \mapsto xy \in G \quad \text{and} \quad G \ni x \mapsto x^{-1} \in G$$

satisfying, for all $x, y, z \in G$, the properties

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1. $x(yz) = (xy)z$;
2. $ex = xe = x$;
3. $xx^{-1} = x^{-1}x = e$,

where $e \in G$ is an element of the group called the *unit* element. To avoid unnecessary technicalities at a few places, we will always assume that G is connected, although we sometimes will emphasise it explicitly. A *compact Lie group* is a Lie group which is compact as a manifold.

Lie groups are naturally topological groups. Recall that a *topological group* G is a topological set G endowed with the continuous mappings

$$G \times G \ni (x, y) \mapsto xy \in G \quad \text{and} \quad G \ni x \mapsto x^{-1} \in G$$

satisfying, for all $x, y, z \in G$, the same properties 1., 2. and 3. as above. When the topology of a topological group is locally compact (i.e. every point has a compact neighbourhood), we say that the group is locally compact. Lie groups are (Hausdorff) locally compact.

Representations

A *representation* π of a group G on a Hilbert space $\mathcal{H}_\pi \neq \{0\}$ is a homomorphism π of G into the group of bounded linear operators on $\mathcal{H}_\pi$ with bounded inverse. This means that

- for every $x \in G$, the linear mapping $\pi(x) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded and has bounded inverse,
- for any $x, y \in G$, we have $\pi(xy) = \pi(x)\pi(y)$.

A representation π of a group G is *unitary* when $\pi(x)$ is unitary for every $x \in G$. Hence a unitary representation π of a group G is a homomorphism $\pi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}_\pi))$, which means that

- for every $x \in G$, the linear mapping $\pi(x) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is unitary:
$$\pi(x)^{-1} = \pi(x)^*;$$
- for any $x, y \in G$, we have $\pi(xy) = \pi(x)\pi(y)$.

Here and everywhere, if $\mathcal{H}$ is a topological vector space, $\mathscr{L}(\mathcal{H})$ denotes the space of all continuous linear operators $\mathcal{H} \to \mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ the space of unitary ones, with respect to the inner product on $\mathcal{H}$. For two different topological vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all linear continuous mappings from $\mathcal{H}_1$ to $\mathcal{H}_2$.

An *invariant subspace* for a representation π is a vector subspace $W \subset \mathcal{H}_\pi$ such that $\pi(x)W \subset W$ holds for every $x \in G$. A representation π is called *irreducible* when it has no closed invariant subspaces.

Let us give the prototype example of a representation which is not irreducible. If $\pi_j \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_{\pi_j}))$ is a family of representations, then using the direct sum

$$\mathcal{H}_\pi := \bigoplus_j \mathcal{H}_{\pi_j}$$

with the induced inner product, we get a representation π which is the direct sum of π_j:

$$\pi = \bigoplus_j \pi_j \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_\pi)), \quad \pi(x)|_{\mathcal{H}_{\pi_j}} = \pi_j(x).$$

Naturally, a sum of several π_j's can not be irreducible as each $\mathcal{H}_{\pi_j}$ is a closed invariant subspace of $\mathcal{H}_\pi$.

If the space $\mathcal{H}_\pi$ is finite dimensional, the representation π is said to be *finite dimensional* and its *dimension/degree* is defined by

$$d_\pi := \dim \mathcal{H}_\pi.$$

The *trivial representation*, sometimes denoted by 1, is given by the group homomorphism $G \ni x \mapsto 1 \in \mathbb{C}$, and its dimension is one. If $\mathcal{H}_\pi$ is infinite dimensional, then the representation π is said to be *infinite dimensional.*

Two representations π_1 and π_2 are said to be *equivalent* if there exists a bounded linear mapping $A : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ between their representation spaces with a bounded inverse such that the relation

$$A\pi_1(x) = \pi_2(x)A \tag{1.1}$$

holds for all $x \in G$. In this case we write

$$\pi_1 \sim \pi_2 \quad \text{or, more precisely sometimes,} \quad \pi_1 \sim_A \pi_2$$

and denote their equivalence class by $[\pi_1] = [\pi_2]$. For unitary representations, A is assumed to be unitary as well. A bounded linear mapping with bounded inverse satisfying the relation (1.1) is sometimes called an intertwining operator or intertwiner. The set of bounded linear mappings A with bounded inverse satisfying the relation (1.1) is denoted by $\mathrm{Hom}(\pi_1, \pi_2)$.

Note that for any representation π, $\mathrm{Hom}(\pi, \pi)$ contains at least $\lambda \mathrm{I}_{\mathcal{H}_\pi}$, $\lambda \in \mathbb{C}$, where $\mathrm{I}_{\mathcal{H}_\pi}$ is the identity mapping on $\mathcal{H}_\pi$.

We now assume that the group G is topological. A representation π of G is *continuous* if the mapping

$$\begin{cases} G \times \mathcal{H}_\pi & \longrightarrow \quad \mathcal{H}_\pi \\ (x, v) & \longmapsto \quad \pi(x)v \end{cases}$$

is continuous. A representation π of G is called *strongly continuous* if the mapping $\pi : G \to \mathscr{L}(\mathcal{H}_\pi)$ is continuous for the strong operator topology in $\mathscr{L}(\mathcal{H}_\pi)$, that is, if the mapping

$$\begin{cases} G & \longrightarrow & \mathcal{H}_\pi \\ x & \longmapsto & \pi(x)v \end{cases}$$

is continuous for all $v \in \mathcal{H}_\pi$.

A continuous representation is strongly continuous. The converse is true for unitary representations. Indeed, if π is a unitary representation of G, then we have for any $x, x_0 \in G$ and $v, v_0 \in \mathcal{H}_\pi$,

$$\begin{aligned} \|\pi(x)v - \pi(x_0)v_0\|_{\mathcal{H}_\pi} &= \|\pi(x_0)(\pi(x_0^{-1}x)v - v_0)\|_{\mathcal{H}_\pi} = \|\pi(x_0^{-1}x)v - v_0\|_{\mathcal{H}_\pi} \\ &= \|\pi(x_0^{-1}x)(v - v_0) + (\pi(x_0^{-1}x)v_0 - v_0)\|_{\mathcal{H}_\pi} \\ &\leq \|\pi(x_0^{-1}x)(v - v_0)\|_{\mathcal{H}_\pi} + \|\pi(x_0^{-1}x)v_0 - v_0\|_{\mathcal{H}_\pi} \\ &\quad = \|v - v_0\|_{\mathcal{H}_\pi} + \|\pi(x_0^{-1}x)v_0 - v_0\|_{\mathcal{H}_\pi}, \end{aligned}$$

having used only the unitarity of π and the triangle inequality. This shows that if a representation of G is unitary and strongly continuous then it is continuous.

Schur's lemma: Let π be a strongly continuous unitary representation of a topological group G on a Hilbert space $\mathcal{H}_\pi$. The representation π is irreducible if and only if the only bounded linear operators on $\mathcal{H}_\pi$ commuting with all $\pi(x)$, $x \in G$, are the scalar operators. Equivalently,

$$\pi \text{ irreducible} \iff \mathrm{Hom}(\pi, \pi) = \{\lambda \mathrm{I}_{\mathcal{H}_\pi} : \lambda \in \mathbb{C}\}.$$

The set of all equivalence classes of strongly continuous irreducible unitary representations of G is called the *unitary dual of* G or just dual of G and is denoted by $\widehat{G}$.

Later, we will give more details on representations of compact or nilpotent Lie groups and their dual.

The unitary dual of G is never a group unless G is commutative. However, if G is a commutative locally compact group, then $\widehat{G}$ has a natural structure of a commutative locally compact group and we have

Pontryagin duality: if G is a commutative locally compact group, then $\widehat{\widehat{G}} \simeq G$.

For most of the statements in the sequel, if they hold for one representation, they will also hold for all equivalent representations. That is why we may simplify the notation a little writing $\pi \in \widehat{G}$ instead of $[\pi] \in \widehat{G}$ for its equivalence class. In this case we can think of π as either any representative from its class or the equivalence class itself. If we need to work with a particular representation from an equivalence class (for example the one diagonalising certain operators in a particular choice of the basis in $\mathcal{H}_\pi$) we will specify this explicitly.

Haar measure

A fundamental fact, valid on general locally compact groups, is the existence of an invariant measure, called *Haar measure*:

Theorem 1.1.1. *Let G be a locally compact group. Then there exists a non-zero left-invariant measure on G, and it is unique up to a positive constant. More precisely, there exists a positive Radon measure on G satisfying*

$$|xA| = |A| \text{ for every Borel set } A \subset G \text{ and every } x \in G,$$

where $|A|$ denotes the measure of the set A with respect to this Radon measure. In the sequel, we denote this measure by dx, dy, etc., depending on the variable of integration. Then, for every $x \in G$ and every continuous compactly supported function f on G, we have

$$\int_G f(xy)dy = \int_G f(y)dy.$$

We fix one of such measures. In this monograph, we will be only dealing with either compact or nilpotent Lie groups, in which case it can be shown that the Haar measure is also right-invariant:

$$|Ax| = |A| \text{ for every Borel set } A \subset G \text{ and every } x \in G,$$

and also

$$\int_G f(yx)dy = \int_G f(y)dy;$$

such groups are called *unimodular*. Since the mapping $f \mapsto \int_G f(y^{-1})dy$ is positive, left-invariant, and normalised, by uniqueness we must also have

$$\int_G f(y^{-1})dy = \int_G f(y)dy.$$

For a more general definition of a modular function we can refer to Definition B.2.10. Here we can summarise a few properties of (unimodular) groups:

- Any Lie group is a locally compact (Hausdorff) group.
- Any compact (Hausdorff) group is a locally compact (Hausdorff) group and it is also unimodular.
- Any abelian locally compact (Hausdorff) group is unimodular.
- Any nilpotent or semi-simple Lie group is unimodular.

If $1 \leq p \leq \infty$, $L^p(G)$ or simply L^p denote the usual *Lebesgue space* on G with respect to the Haar measure, with the norm

$$\|\cdot\|_{L^p} = \|\cdot\|_{L^p(G)} = \|\cdot\|_p,$$

given for $p \in [1, \infty)$ by

$$\|f\|_p = \left(\int_G |f(x)|^p dx \right)^{1/p},$$

and for $p = \infty$ by

$$\|f\|_\infty = \sup_{x \in G} |f(x)|.$$

Here the supremum refers to the essential supremum with respect to the Haar measure.

The Hilbert sesquilinear form on $L^2(G)$ is denoted by

$$(f_1, f_2)_{L^2} = (f_1, f_2)_{L^2(G)} = \int_G f_1(x) \overline{f_2(x)} dx.$$

Example 1.1.2. Let us give an important example of so-called left and right regular representations leading to the notions of left- and right-invariant operators. We define the *left* and *right regular representations* of G on $L^2(G)$, $\pi_L, \pi_R : G \to \mathcal{U}(L^2(G))$, respectively, by

$$\pi_L(x) f(y) := f(x^{-1} y) \quad \text{and} \quad \pi_R(x) f(y) := f(yx).$$

Definition 1.1.3. An operator A is called *left* (*right*, resp.) *invariant* if it commutes with the left (right, resp.) regular representation of G.

Fourier analysis

For $f \in L^1(G)$ we define its *Fourier coefficient* or *group Fourier transform* at the strongly continuous unitary representation π as

$$\mathcal{F}_G f(\pi) \equiv \widehat{f}(\pi) \equiv \pi(f) := \int_G f(x) \pi(x)^* dx. \tag{1.2}$$

More precisely, we can write

$$(\widehat{f}(\pi) v_1, v_2)_{\mathcal{H}_\pi} = \int_G f(x) (\pi(x)^* v_1, v_2)_{\mathcal{H}_\pi} dx.$$

This gives a linear mapping $\widehat{f}(\pi) : \mathcal{H}_\pi \to \mathcal{H}_\pi$. If the representation π is finite dimensional, then after a choice of a basis in the representation space $\mathcal{H}_\pi$, the Fourier coefficient $\widehat{f}(\pi)$ can be also viewed as a matrix $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$.

Remark 1.1.4. The choice of taking the adjoint $\pi(x)^*$ in (1.2) is natural if we think of the unitary dual of the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ being $\widehat{\mathbb{T}^n} = \{\pi_\xi(x) = e^{2\pi i x \cdot \xi}\}_{\xi \in \mathbb{Z}^n} \simeq \mathbb{Z}^n$, and the Fourier transform on the torus defined by

$$\widehat{f}(\pi_\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i x \cdot \xi} f(x) dx = \int_{\mathbb{T}^n} f(x) \pi_\xi(x)^* dx.$$

In other contexts, the other choice, that is, integrating against $\pi(x)$ instead of $\pi(x)^*$, may be made. This is the case for instance in the study of C^*-algebras associated with groups.

Remark 1.1.5. We note that the Fourier coefficient $\widehat{f}(\pi)$ depends on the choice of the representation π from its equivalence class $[\pi]$. Namely, if $\pi_1 \sim \pi_2$, so that

$$\pi_2(x) = U^{-1}\pi_1(x)U$$

for some unitary U and all $x \in G$, then

$$\widehat{f}(\pi_2) = U^{-1}\widehat{f}(\pi_1)U.$$

This means that strictly speaking, we need to look at Fourier coefficients modulo conjugations induced by the equivalence of representations. This should, however, cause no problems, and we refer to Remark 2.2.1 for more discussion on this.

Recalling that the Fourier transform on $\mathbb{R}^n$ maps translations to modulations, here we have an analogous property, namely, if $\pi \in \widehat{G}$, $f \in L^1(G)$ and $x \in G$, then

$$\widehat{f(\cdot\, x)}(\pi) = \pi(x)\widehat{f}(\pi) \text{ and } \widehat{f(x\,\cdot)}(\pi) = \widehat{f}(\pi)\pi(x), \tag{1.3}$$

whenever the right hand side makes sense. Let us show these properties by a formal argument, which can be made rigorous on Lie groups, see the proof of Proposition 1.7.6, (iv). We have

$$\begin{aligned} \pi(x)\widehat{f}(\pi) &= \int_G f(y)\pi(x)\pi(y)^* dy \\ &= \int_G f(y)\pi(yx^{-1})^* dy \\ &= \int_G f(yx)\pi(y)^* dy \\ &= \widehat{f(\cdot\, x)}(\pi), \end{aligned} \tag{1.4}$$

as well as

$$\begin{aligned} \widehat{f(x\,\cdot)}(\pi) &= \int_G f(xy)\pi(y)^* dy \\ &= \int_G f(y)\pi(x^{-1}y)^* dy \\ &= \int_G f(y)\pi(y)^*\pi(x) dy \\ &= \widehat{f}(\pi)\pi(x). \end{aligned} \tag{1.5}$$

We will continue with a more detailed discussion of the Fourier transform on compact Lie groups in Section 2.1, on nilpotent Lie groups in Section 1.8.1 and, more generally, on a separable locally compact connected, unimodular, amenable group G of type I in Section 1.8.2.

1.2 Lie algebras and vector fields

A (real) *Lie algebra* is a real vector space V endowed with a bilinear mapping

$$V \times V \ni (a, b) \mapsto [a, b] \in V,$$

called *the commutator of* a *and* b, such that

- $[a, a] = 0$ for every $a \in V$;
- *Jacobi identity:* $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in V$.

By writing $[a+b, a+b] = [a, a] + [a, b] + [b, a] + [b, b]$ we see that the first property is equivalent to the condition that

$$\forall a, b \in V \quad [a, b] = -[b, a].$$

We now proceed to equip the tangent space of G (at every point) with a Lie algebra structure. A map $X_{(x)} : C^\infty(G) \to \mathbb{R}$ is called a *tangent vector* to G at $x \in G$ if

- $X_{(x)}(f + g) = X_{(x)}f + X_{(x)}g$;
- $X_{(x)}(fg) = X_{(x)}(f)g(x) + f(x)X_{(x)}(g)$.

The notation $X_{(x)}$ is used only in this section and the reason for its choice is that we want to reserve the notation X_x for derivatives, to be used later.

The space of all tangent vectors at x is a finite dimensional vector space of dimension equal to the dimension of G as a manifold; the finite dimensionality can be seen by passing to local coordinates. This vector space is denoted by T_xG. The disjoint union,

$$TG := \bigcup_{x \in G} T_xG$$

is a vector bundle over X, called *the tangent bundle*. The canonical projection $\mathrm{proj} : TG \to G$ is given by $\mathrm{proj}\, X_{(x)} := x$. If U_x is a (sufficiently small) open neighbourhood of x in G, we can trivialise the vector bundle TG by $\mathrm{proj}^{-1}(U_x) \simeq U_x \times E$ with a vector space E of dimension equal to that of G. This induces the manifold structure on TG.

A *(smooth) vector field* on G is a (smooth) section of TG, i.e. a (smooth) mapping $X : G \to TG$ such that $X(x) \equiv X_{(x)} \in T_xG$. It acts on $C^\infty(G)$ by

$$(Xf)(x) := (X_{(x)}f)(x), \quad f \in C^\infty(G).$$

There is a bracket structure on the space of vector fields acting on $C^\infty(G)$ given by

$$[X, Y](x)(f) := X(x)Yf - Y(x)Xf, \quad x \in G,$$

leading to the corresponding (smooth) vector field $[X, Y] : G \to TG$ given by $x \mapsto [X, Y](x)$. One can readily check that $[X, X] = 0$ for every vector field X and that the introduced bracket satisfies the Jacobi identity. This bracket $[\cdot,\cdot]$ is called the *commutator bracket* for vector fields.

We now recall that G is also a group, and relate vector fields to the group structure. First, we define the left and right translations by an element $y \in G$:

$$L_y, R_y : G \to G, \quad L_y(x) := yx,\ R_y(x) := xy.$$

Consequently, their derivatives are the mappings

$$dL_y, dR_y : TG \to TG \text{ such that } dL_y \in \mathscr{L}(T_xG, T_{yx}G),\ dR_y \in \mathscr{L}(T_xG, T_{xy}G).$$

Now, a vector field $X : G \to TG$ is called *left-invariant* if it commutes with the left translations, in the sense that

$$X \circ L_y = dL_y \circ X \quad \forall y \in G. \tag{1.6}$$

A similar construction leads to the notion of *right-invariant* vector fields, satisfying

$$X \circ R_y = dR_y \circ X$$

for all $y \in G$.

It follows that once a left-invariant vector field is defined at any one point, by the left-invariance it is uniquely determined at all points. Thus, the mapping $X \mapsto X_{(e)}$ is a one-to-one correspondence between left-invariant vector fields on G and the tangent space T_eG at the unit element $e \in G$. Conversely, given $X_{(e)} \in T_eG$, the vector field X defined by (1.6) is automatically smooth and, by definition, left-invariant. With this identification, we can now simplify the notation for left-invariant vector fields X, writing X also for its value $X_{(e)}$ at the unit element. It can be readily checked that if X and Y are left-invariant vector fields, so is also their commutator $[X, Y]$.

Definition 1.2.1. The *Lie algebra* $\mathfrak{g}$ *of the Lie group* G is the space T_eG equipped with the commutator $[\cdot,\cdot]$ induced by the commutator bracket of vector fields.

We now define the *exponential mapping* $\exp_G$. For $X \in \mathfrak{g}$, consider the initial value problem for a function $\gamma : [0, \epsilon) \to G$, $\epsilon > 0$, given by the ordinary differential equation determined by the left-invariant vector field associated with X:

$$\gamma'(t) = X_{(\gamma(t))}, \quad \gamma(0) = e.$$

From the theory of ordinary differential equations we know that this equation is uniquely solvable on some interval $[0, \epsilon)$ and the solution depends smoothly on $X_{(e)}$. Moreover, we notice that we can increase the interval of existence by taking smaller vectors $X_{(e)}$, in particular, in such a way that the solution exists on the interval $[0, 1]$. In this case we set $\exp_G X := \gamma(1)$. Altogether, it follows that the mapping $\exp_G$ is a smooth diffeomorphism from some open neighbourhood of $0 \in \mathfrak{g}$ to some open neighbourhood of $e \in G$.

Now, each vector $X \in \mathfrak{g}$ can be viewed as a left-invariant differential operator on $C^\infty(G)$ defined by

$$Xf(x) := \frac{d}{dt} f(x \ \exp_G(tX))|_{t=0}. \tag{1.7}$$

Indeed, it can be readily checked that $X\pi_L(y) = \pi_L(y)X$ for all $y \in G$. Analogously, the same vector $X \in \mathfrak{g}$ defines a right-invariant differential operator, which we denote by

$$\tilde{X}f(x) := \frac{d}{dt} f(\exp_G(tX)\, x)|_{t=0}.$$

Thus, throughout this book, we will be interpreting the Lie algebra $\mathfrak{g} = T_eG$ of G as the vector space of first order left-invariant partial differential operators on G. The space of all left-invariant vector fields will be sometimes denoted by $\mathbb{D}(G)$ or by $\mathrm{Diff}^1(G)$, and the space of all right-invariant vector fields by $\widetilde{\mathbb{D}}(G)$.

1.3 Universal enveloping algebra and differential operators

Roughly speaking, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is the natural non-commutative polynomial algebra on $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then, similarly to the interpretation of $\mathfrak{g}$ as the space of left-invariant derivatives on G, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra of G will be also interpreted as the vector space of left-invariant partial differential operators on G of finite order. The associative algebra will be generated as a complex algebra over $\mathfrak{g}$, so that we could write $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ for it, where $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$. However, we will simplify the notation writing $\mathfrak{U}(\mathfrak{g})$, and will later use the Poincaré-Birkhoff-Witt theorem to identify it with the left-invariant differential operators on G with complex coefficients. Let us now formalise these statements.

The following construction is algebraic and works for any real Lie algebra $\mathfrak{g}$. Let us denote the m-fold tensor product of $\mathfrak{g}^{\mathbb{C}}$ by $\otimes^m \mathfrak{g}^{\mathbb{C}} := \mathfrak{g}^{\mathbb{C}} \otimes \cdots \otimes \mathfrak{g}^{\mathbb{C}}$, and let

$$\mathcal{T} := \bigoplus_{m=0}^{\infty} \otimes^m \mathfrak{g}^{\mathbb{C}}$$

be the tensor product algebra of $\mathfrak{g}$, which means that $\mathcal{T}$ is the linear span of the elements of the form

$$\lambda_{00}\mathbf{1} + \sum_{m=1}^{M} \sum_{k=1}^{K_m} \lambda_{mk} X_{mk1} \otimes \cdots \otimes X_{mkm},$$

where $\mathbf{1}$ is the formal unit element of $\mathcal{T}$, $\lambda_{mk} \in \mathbb{C}$, $X_{mkj} \in \mathfrak{g}$, and $M, K_M \in \mathbb{N}$. This $\mathcal{T}$ becomes an associative algebra with the product

$$(X_1 \otimes \cdots \otimes X_p)(Y_1 \otimes \cdots \otimes Y_q) := X_1 \otimes \cdots \otimes X_p \otimes Y_1 \otimes \cdots \otimes Y_q$$

extended to a uniquely determined bilinear mapping $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$. We now want to induce the commutator structure on $\mathcal{T}$: let $\mathcal{I}$ be the two-sided ideal in $\mathcal{T}$ spanned by the set

$$\mathcal{O} := \{X \otimes Y - Y \otimes X - [X, Y] : X, Y \in \mathfrak{g}\},$$

i.e. $\mathcal{I}$ is the smallest vector subspace of $\mathcal{T}$ such that

- $\mathcal{O} \subset \mathcal{I}$;
- for every $J \in \mathcal{I}$ and $T \in \mathcal{T}$ we have $JT, TJ \in \mathcal{I}$.

The quotient algebra

$$\mathfrak{U}(\mathfrak{g}) := \mathcal{T}/\mathcal{I}$$

is called the *universal enveloping algebra* of $\mathfrak{g}$; the quotient mapping

$$\iota : \mathcal{T} \ni T \mapsto T + \mathcal{I} \in \mathfrak{U}(\mathfrak{g}) = \mathcal{T}/\mathcal{I},$$

restricted to $\mathfrak{g}$, $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$, is called the canonical mapping of $\mathfrak{g}$. This gives the embedding of $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$:

Ado-Iwasawa theorem: the canonical mapping $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is injective.

Let $n = \dim G$ and let $\{X_j\}_{j=1}^n$ be a basis of the Lie algebra $\mathfrak{g}$ of G. Regarded as first order left-invariant derivatives, they give rise to higher order left-invariant differential operators

$$X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n.$$

The converse is also true (for a stronger version of this see e.g. [Bou98, Ch 1, Sec. 2.7]):

Poincaré-Birkhoff-Witt theorem: any left-invariant differential operator T on G can be written in a unique way as a finite sum

$$T = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha X^\alpha,$$

where all but a finite number of the coefficients $c_\alpha \in \mathbb{C}$ are zero. This gives an identification between the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ and the space of left-invariant differential operators on G.

We denote the space of all left-invariant differential operators of order k by $\mathrm{Diff}^k(G)$.

If T is as above, we define three new elements $\bar{T}$, T^*, and T^t of $\mathfrak{U}(\mathfrak{g})$ via

$$\bar{T} := \sum_{\alpha \in \mathbb{N}_0^n} \bar{c}_\alpha (X_n)^{\alpha_n} \dots (X_1)^{\alpha_1}, \tag{1.8}$$

$$T^* := \sum_{\alpha \in \mathbb{N}_0^n} \bar{c}_\alpha (-X_n)^{\alpha_n} \dots (-X_1)^{\alpha_1}, \tag{1.9}$$

and

$$T^t := \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha (-X_n)^{\alpha_n} \dots (-X_1)^{\alpha_1}. \tag{1.10}$$

These T^* and T^t are called the (formal) *adjoint* and *transpose* operators of T, respectively. Naturally, they coincide with the natural transpose and formal adjoint operators of their corresponding left-invariant vector fields. Recall that the latter operators are defined via:

Definition 1.3.1. Let T be an operator T on $L^2(G)$ with domain $\mathcal{D}(G)$ (T may be unbounded, $\mathcal{D}(G) \subset \operatorname{Dom} T$). The natural transpose and formal adjoint operators of T are the operators T^t and T^* on $L^2(G)$ defined via

$$\langle T\phi, \psi\rangle = \langle \phi, T^t\psi\rangle \quad \text{and} \quad (T\phi,\psi)_{L^2(G)} = (\phi, T^*\psi)_{L^2(G)}, \quad \phi,\psi\in\mathcal{D}(G).$$

We also define the operator $\bar{T}$ on $L^2(G)$ via

$$\bar{T}\phi := \overline{T\bar{\phi}},$$

for $\phi, \bar{\phi} \in \operatorname{Dom} T$.

Note that we also have, e.g.,

$$T^* = \overline{\{T^t\}} = \{\bar{T}\}^t$$

and so on. Denoting

$$\tilde{f}(x) := f(x^{-1}),$$

the left- and right- invariant differential operators are related by

$$\tilde{X} f(x) = -(X\tilde{f})(x^{-1}) \quad \text{and hence} \quad \tilde{X}^\alpha f(x) = (-1)^{|\alpha|}(X^\alpha \tilde{f})(x^{-1}). \tag{1.11}$$

Indeed, we can write

$$X\tilde{f}(x) = \frac{d}{dt} f((x\exp_G(tX))^{-1})|_{t=0} = \frac{d}{dt} f(\exp_G(-tX)x^{-1})|_{t=0} = -(\tilde{X}f)(x^{-1}),$$

implying (1.11).

For any $X\in\mathfrak{g}$ identified with a left-invariant vector field, we have

$$\tilde{X}_y\{f(xy)\} = \frac{d}{dt} f(xe^{tX}y)_{t=0} = X_x\{f(xy)\}.$$

Recursively, we obtain

$$\tilde{X}_y^\alpha\{f(xy)\} = X_x^\alpha\{f(xy)\}. \tag{1.12}$$

The first order differential operators are formally skew-symmetric:

$$\int_G (Xf_1)f_2 = -\int_G f_1(Xf_2) \quad \text{and} \quad \int_G (\tilde{X}f_1)f_2 = -\int_G f_1(\tilde{X}f_2),$$

so that from (1.11) we also have

$$\tilde{X}f(x) = -(X\tilde{f})(x^{-1}) = (X^t\tilde{f})(x^{-1}).$$

We now summarise several further notions and their properties that will be of use to us in the sequel:

- there is a natural representation of the Lie group G acting on its Lie algebra $\mathfrak{g}$, called the *adjoint representation*. To introduce it, first define the inner automorphism $I_x(y) := xyx^{-1}$. We have $I_x : G \to G$ and $I_{xy} = I_x I_y$. Its differential at e gives a linear mapping from T_eG to T_eG, and we denote it by
$$\mathrm{Ad}(x) := (dI_x)_e : \mathfrak{g} \to \mathfrak{g}.$$
We have $\mathrm{Ad}(e) = \mathrm{I}$ and $\mathrm{Ad}(xy) = \mathrm{Ad}(x)\mathrm{Ad}(y)$, so that $\mathrm{Ad} : G \to \mathscr{L}(\mathfrak{g})$ becomes a representation of G on $\mathfrak{g}$;
- the left and right multiplications on G are related by
$$x \exp_G X = \exp(\mathrm{Ad}(x)X)x, \quad x \in G,\ X \in \mathfrak{g};$$
- a Lie group G is called a *linear Lie group* if it is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$; the adjoint representation of such G is given by
$$\mathrm{Ad}(X)Y = XYX^{-1}$$
as multiplication of matrices;
- *universality of unitary groups*: any compact Lie group is isomorphic to a subgroup of $U(N)$, the group of $(N \times N)$-unitary matrices, for some $N \in \mathbb{N}$;
- let $\mathrm{ad} : \mathfrak{g} \to \mathcal{L}(\mathfrak{g})$ be the linear mapping defined by
$$\mathrm{ad}(X)Y := [X, Y];$$
then $d(\mathrm{Ad})_e = \mathrm{ad}$; see also Definition 1.7.4;
- the *Killing form* of the Lie algebra $\mathfrak{g}$ is the bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by
$$B(X, Y) := \mathrm{Tr}(\mathrm{ad}(X)\ \mathrm{ad}(Y));$$
it satisfies
$$B(X, Y) = B(Y, X) \text{ and } B(X, [Y, Z]) = B([X, Y], Z)$$
and is invariant under the adjoint representation of G, namely,
$$B(X, Y) = B(\mathrm{Ad}(x)(X), \mathrm{Ad}(x)(Y)) \text{ for all } x \in G,\ X, Y \in \mathfrak{g};$$

- A connected Lie group G is called *semi-simple* if B is non-degenerate; a connected semi-simple group G is compact if and only if B is negative definite.

The Ad-invariance of the Killing form has its consequences. On one hand, any bilinear form on $\mathfrak{g}$ can be extended to a bilinear (non-necessarily positive definite) metric on G by left translations. It is automatically left-invariant. On the other hand, if the form on $\mathfrak{g}$ is Ad-invariant, then the extended metric is also right-invariant. Thus, we can conclude that the Killing form induces a bi-invariant metric on G. By the last property above, if G is semi-simple, the Killing form is non-degenerate, and hence the corresponding metric is pseudo-Riemannian. Moreover, if G is a connected semi-simple compact Lie group, the positive-definite form $-B$ induces the bi-invariant Riemannian metric on G.

For the basis $\{X_j\}_{j=1}^n$ as above, let us define $R_{ij} := B(X_i, X_j)$. If the group G is semi-simple, the matrix (R_{ij}) is invertible, and we denote its inverse by R^{-1}. This leads to another vector space basis on $\mathfrak{g}$ given by

$$X^i := \sum_{j=1}^{n} (R^{-1})_{ij} X_j,$$

and to the so-called *Casimir element* of $\mathfrak{U}(\mathfrak{g})$ defined by

$$\Omega := \sum_{i=1}^{n} X_i X^i.$$

It has the crucial property: Ω is independent of the choice of the basis $\{X_j\}$, and

$$\Omega T = T\Omega \quad \text{for all} \quad T \in \mathfrak{U}(\mathfrak{g}).$$

We finish this section with the formula for the group product which will be useful for us, especially in the nilpotent case:

Theorem 1.3.2 (Baker-Campbell-Hausdorff formula)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There exists a neighbourhood V of 0 in $\mathfrak{g}$ such that for any $X, Y \in V$, we have*

$$\begin{aligned} \exp_G X \exp_G Y \quad &= \quad \exp_G \Big(\sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{p,q \in \mathbb{N}_0^n \\ p_i + q_i > 0}} \frac{\left(\sum_{j=1}^n (p_j + q_j)\right)^{-1}}{p_1! q_1! \dots p_n! q_n!} \\ &\qquad \times (\mathrm{ad} X)^{p_1} (\mathrm{ad} Y)^{q_1} \dots (\mathrm{ad} X)^{p_n} (\mathrm{ad} Y)^{q_n - 1} Y \Big). \end{aligned}$$

The equality holds whenever the sum on the right-hand side is convergent.

Writing first few terms explicitly, we have

$$\begin{aligned} &\exp_G X \exp_G Y \\ &\quad = \exp_G \left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[[X, Y], Y] - \frac{1}{12}[[X, Y], X] + \dots \right). \end{aligned}$$

1.4 Distributions and Schwartz kernel theorem

Here we fix the notation concerning distributions. For an extensive analysis of spaces of distributions and their properties on manifolds we refer to [Hör03].

The space of smooth functions compactly supported in a smooth manifold M will be denoted by $\mathcal{D}(M)$. Throughout the book, any smooth manifold is assumed to be paracompact (i.e. every open cover has an open refinement that is locally finite) and this allows us to consider the space of distributions $\mathcal{D}'(M)$ as the dual of $\mathcal{D}(M)$. Note that any Lie group is paracompact.

If $u \in \mathcal{D}'(M)$ and $\phi \in \mathcal{D}(M)$, we shall denote the evaluation of u on ϕ by $\langle u, \phi \rangle$, or even by $\langle u, \phi \rangle_M$ when we wish to be precise; however, we shall usually pretend that the distributions are functions and write

$$\langle u, \phi \rangle = \int_M u(x)\phi(x)dx, \quad u \in \mathcal{D}'(M),\ \phi \in \mathcal{D}(M).$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions will be equipped with a family of seminorms defined by

$$\|f\|_{\mathcal{S}(\mathbb{R}^n),N} := \sup_{|\alpha| \leq N,\, x \in \mathbb{R}^n} (1+|x|)^N \left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right|. \tag{1.13}$$

Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.4.1 (Schwartz kernel theorem)**.** *We have the following statements:*

- *Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator. Then there exists a unique distribution $\kappa \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

  $$T\phi(x) = \int_{\mathbb{R}^n} \kappa(x,y)\phi(y)dy.$$

  In other words, T is an integral operator with kernel κ. The converse is also true.

- *Let M be a smooth connected manifold and let $T : \mathcal{D}(M) \to \mathcal{D}'(M)$ be a continuous linear operator. There exists a unique distribution $\kappa \in \mathcal{D}'(M \times M)$ such that*

  $$T\phi(x) = \int_M \kappa(x,y)\phi(y)dy.$$

  In other words, T is an integral operator with kernel κ. The converse also is true.

  In both cases, the map $\kappa \mapsto T$ is an isomorphism of topological vector space.

We refer to e.g. [Tre67] for further details. We will also give a version of this theorem on Lie groups for left-invariant operators in Corollary 3.2.1.

Let Ω be an open set in $\mathbb{R}^n$ or in M. We say that $u \in \mathcal{D}'(\Omega)$ is supported in the set $K \subset \Omega$ if $\langle u, \phi \rangle = 0$ for all $\phi \in \mathcal{D}(\Omega)$ such that $\phi = 0$ on K. The smallest closed set in which u is supported is called the *support* of u and is denoted by $\operatorname{supp} u$. The space of compactly supported distributions on M is denoted by $\mathcal{E}'(M)$, and the duality between $\mathcal{E}'(M)$ and $C^\infty(M)$ will still be denoted by $\langle \cdot, \cdot \rangle$.

We write $u \in \mathcal{D}'_j(\Omega)$ for the space of distributions of order j on Ω, which means that for any compact subset K of Ω,

$$\exists C > 0 \qquad \forall \phi \in \mathcal{D}(K) \qquad |\langle u, \phi \rangle| \leq C \|\phi\|_{C^j(K)},$$

but j does not depend on K. An important property of such distributions, useful for us, is the following

Proposition 1.4.2. *If a distribution $u \in \mathcal{D}'_j(\mathbb{R}^n)$ has support $\operatorname{supp} u = \{0\}$, then there exist constants $a_\alpha \in \mathbb{C}$ such that*

$$u = \sum_{|\alpha| \leq j} a_\alpha \partial^\alpha \delta_0,$$

where $\delta_0(\phi) = \phi(0)$ is the delta-distribution at zero.

1.5 Convolutions

Let $f, g \in L^1(G)$ be integrable function on a locally compact group. The *convolution* $f * g$ is defined by

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) dy.$$

In this monograph we consider only unimodular groups. This means that the Haar measure is both left- and right-invariant. Consequently we also have

$$(f * g)(x) = \int_G f(xy^{-1}) g(y) dy.$$

On a nilpotent or compact Lie group which is not abelian, the convolution is not commutative: in general, $f * g \neq g * f$. However, apart from the lack of commutativity, group convolution and the usual convolution on $\mathbb{R}^n$ share many properties. For example, we have

$$\begin{aligned} \langle f * g, h \rangle &= \int_G (f * g)(x)\ h(x)\ dx \\ &= \int_G \int_G f(y)\ g(y^{-1}x)\ h(x)\ dy\, dx \\ &= \langle f, h * \tilde{g} \rangle, \quad \text{with} \quad \tilde{g}(x) = g(x^{-1}). \end{aligned} \tag{1.14}$$

We also have

$$\begin{aligned}
\langle f * g, h\rangle &= \int_G \int_G f(y)\ g(y^{-1}x)\ h(x)\ dy\, dx \\
&= \int_G \int_G f(y)\ g(z)\ h(yz)\ dy\, dz \\
&= \int_G \int_G f(wz^{-1})\ g(z)\ h(w)\ dz\, dw \\
&= \langle g, \tilde{f} * h\rangle. \qquad (1.15)
\end{aligned}$$

With the notation $\tilde{\cdot}$ for the operation given by $\tilde{g}(x) = g(x^{-1})$, we also have

$$(f * g)\tilde{} = \tilde{g} * \tilde{f}. \qquad (1.16)$$

One can readily check the following simple properties:

- if $f, g \in L^1(G)$ then $f * g \in L^1(G)$, and we have $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$;
- under the assumptions above, we have

  $$(f * g)(x) = \int_G f(y^{-1})g(yx)dy = \int_G f(xy)g(y^{-1})dy$$

  for almost every $x \in G$;
- if either f or g are continuous on G then $f * g$ is continuous on G;
- $\|f * g\|_{L^\infty} \leq \|f\|_{L^2}\|g\|_{L^2}$;
- the convolution is associative: $f * (g * h) = (f * g) * h$, for $f, g, h \in L^1(G)$;
- the convolution is commutative if and only if G is commutative;
- (if G is a Lie group and) if X is a left-invariant vector field, whenever it makes sense, we have

  $$X(f * g) = f * (Xg) \quad \text{and} \quad \tilde{X}(f * g) = (\tilde{X}f) * g;$$

  moreover, we also have

  $$(Xf) * g = f * (\tilde{X}g);$$
- the right convolution operator $f \mapsto f * \kappa$ is left-invariant; the left convolution operator $f \mapsto \kappa * f$ is right-invariant.

To check the last statement, let us show that the right convolution operator given via $Af = f * \kappa$ is left-invariant:

$$\begin{aligned}
\pi_L(z)Af(x) = (f * \kappa)(z^{-1}x) &= \int_G f(y)\ \kappa(y^{-1}z^{-1}x)dy \\
&= \int_G f(z^{-1}y)\ \kappa(y^{-1}x)dy = (\pi_L(z)f) * \kappa(x) = A\pi_L(z)f(x).
\end{aligned}$$

Conversely, it follows from the Schwartz integral kernel theorem that if A is left-invariant, it can be written as a right convolution $Af = f * \kappa$, and if A is right-invariant, it can be written as a left convolution $Af = f * \kappa$, see Section 1.4 and later Corollary 3.2.1.

With our choice of the definition of the convolution and the Fourier transform in (1.2), one can readily check that for $f, g \in L^1(G)$, we have

$$\widehat{f * g}(\pi) = \widehat{g}(\pi)\widehat{f}(\pi) \tag{1.17}$$

or, in the other notation,

$$\pi(f * g) = \pi(g)\pi(f).$$

We say that an operator A is of *weak type* (p,p) if there is a constant $C > 0$ such that for every $\lambda > 0$ we have

$$|\{x \in G : |Af(x)| > \lambda\}| \leq C \frac{\|f\|^p_{L^p(G)}}{\lambda^p},$$

where $|\{\cdot\}|$ denotes the Haar measure of a set in G.

Proposition 1.5.1 (Marcinkiewicz interpolation theorem). *Let $r < q$ and assume that operator A is of weak types (r,r) and (q,q). Then A is bounded on $L^p(G)$ for all $r < p < q$.*

An important fact, the Young inequality, relates convolution to L^p-spaces:

Proposition 1.5.2 (Young's inequality). *Suppose*

$$1 \leq p, q, r \leq \infty \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f_1 \in L^p(G)$ and $f_2 \in L^q(G)$ then $f_1 * f_2 \in L^r(G)$ and

$$\|f_1 * f_2\|_r \leq \|f_1\|_p \|f_2\|_q.$$

If $p, q \in (1, \infty)$ are such that $\frac{1}{p} + \frac{1}{q} > 1$, $f_1 \in L^p(G)$, and f_2 satisfies the weak-$L^q(G)$ condition:

$$\sup_{s>0} s^q \, |\{x \ : \ |f_2(x)| > s\}| \ =: \|f_2\|^q_{w-L^q(G)} < \infty,$$

then $f_1 * f_2 \in L^r$ with r as above and

$$\|f_1 * f_2\|_r \leq \|f_1\|_p \|f_2\|_{w-L^q(G)}.$$

The proof is an easy adaptation of the Euclidean case which can be found e.g., in [SW71] or, in the nilpotent case, in [FS82, Proposition 1.18] and [Fol75, Proposition 1.10].

Convolution of distributions

We now define the convolution of distributions on a Lie group G. For $\phi \in C^\infty(G)$, we recall that

$$\tilde{\phi}(x) = \phi(x^{-1})$$

and

$$\pi_L(x)\phi(y) = \phi(x^{-1}y).$$

Consequently, we note that

$$(\pi_L(x)\tilde{\phi})(y) = \tilde{\phi}(x^{-1}y) = \phi(y^{-1}x).$$

It follows that we can write the convolution as

$$(f * g)(x) = \langle f, \pi_L(x)\tilde{g}\rangle,$$

and hence it make sense to define

Definition 1.5.3. Let $v \in \mathcal{D}'(G)$ and $\phi \in \mathcal{D}(G)$. Then we define their convolution as

$$(v * \phi)(x) := \langle v, \pi_L(x)\tilde{\phi}\rangle \equiv \langle v, \tilde{\phi}(x^{-1}\,\cdot)\rangle.$$

We also define

$$(\phi * v)(x) := \langle v, \pi_R(x^{-1})\tilde{\phi}\rangle \equiv \langle v, \tilde{\phi}(\cdot\, x^{-1})\rangle,$$

where

$$\pi_R(x^{-1})\tilde{\phi}(y) = \tilde{\phi}(yx^{-1}),$$

and which is also consistent with the convolution of functions.

We note that this expression makes since since $\pi_L(x), \pi_R(x^{-1})$ and $\phi \mapsto \tilde{\phi}$ are continuous mappings from $\mathcal{D}(G)$ to $\mathcal{D}(G)$.

For example, for the delta-distribution δ_e at the unit element $e \in G$, it follows that

$$\delta_e * \phi = \phi \quad \text{for every } \ \phi \in \mathcal{D}(G),$$

since we can calculate

$$(\delta_e * \phi)(x) = \langle \delta_e, \pi_L(x)\tilde{\phi}\rangle = \phi(y^{-1}x)|_{y=e} = \phi(x).$$

The following properties are easy to check using Definition 1.5.3:

- if $v \in \mathcal{D}'(G)$ and $\phi \in \mathcal{D}(G)$, then $v * \phi \in C^\infty(G)$;
- if $u, v, \phi \in \mathcal{D}(G)$, then $\langle u * v, \phi\rangle = \langle u, \phi * \tilde{v}\rangle$, in consistency with (1.14).

For $v \in \mathcal{D}'(G)$, we now define $\tilde{v} \in \mathcal{D}'(G)$ by

$$\langle \tilde{v}, \phi\rangle := \langle v, \tilde{\phi}\rangle.$$

In particular, if $v \in \mathcal{D}'(G)$ and $\phi \in \mathcal{D}(G)$, then $\phi * \tilde{v} \in C^\infty(G)$. This shows that the following convolution of distributions is correctly defined:

Definition 1.5.4. Let $u \in \mathcal{E}'(G)$ and $v \in \mathcal{D}'(G)$. Then we define their convolution as

$$\langle u * v, \phi \rangle := \langle u, \phi * \tilde{v} \rangle, \quad \forall \phi \in \mathcal{D}(G).$$

This gives $u * v \in \mathcal{D}'(G)$ which is consistent with the convolution of functions in view of (1.15). If we start with a compactly supported distribution $v \in \mathcal{E}'(G)$ in Definition 1.5.3, we arrive at the definition of the composition $u * v$ for $u \in \mathcal{D}'(G)$ and $v \in \mathcal{E}'(G)$, given by the same formula as in Definition 1.5.4.

A word of caution has to be said about convolution of distributions, namely, it is not in general associative for distributions, although it is associative for functions.

1.6 Nilpotent Lie groups and algebras

From now on, any Lie algebra $\mathfrak{g}$ is assumed to be real and finite dimensional.

Proposition 1.6.1. *The following are equivalent:*

- ad *is a nilpotent endomorphism over* $\mathfrak{g}$, *i.e.*

$$\exists k \in \mathbb{N} \quad \forall X \in \mathfrak{g} \qquad (\text{ad}X)^k = 0;$$

- *the lower central series of* $\mathfrak{g}$, *defined inductively by*

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \quad \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}], \tag{1.18}$$

terminates at 0 in a finite number of steps.

Definition 1.6.2. (i) If a Lie algebra $\mathfrak{g}$ satisfies any of the equivalent conditions of Proposition 1.6.1, then it is called *nilpotent*.

(ii) Moreover, if $\mathfrak{g}_{(s+1)} = \{0\}$ and $\mathfrak{g}_{(s)} \neq \{0\}$, then $\mathfrak{g}$ is said to be nilpotent of *step* s.

(iii) A Lie group G is *nilpotent* (of step s) whenever its Lie algebra is nilpotent (of step s).

Here are some examples of nilpotent Lie groups and their Lie algebras.

Example 1.6.3. The abelian group $\mathbb{R}^n$ equipped with the usual addition is nilpotent. Its Lie algebra is $\mathbb{R}^n$ equipped with the trivial Lie bracket.

Example 1.6.4. If $n_o \in \mathbb{N}$, the Heisenberg group $\mathbb{H}_{n_o}$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n_o+1}$ and whose law is

$$h_1 h_2 = \big(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(x_1 y_2 - y_1 x_2)\big), \tag{1.19}$$

for $h_1 = (x_1, y_1, t_1)$ and $h_2 = (x_2, y_2, t_2)$ in $\mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathbb{R}$. Here, for vectors $x_1, y_1, x_2, y_2 \in \mathbb{R}^{n_o}$, we denote by $x_1 y_2$ and $y_1 x_2$ their usual inner products on $\mathbb{R}^{n_o}$.

Its Lie algebra $\mathfrak{h}_{n_o}$ is $\mathbb{R}^{2n_o+1}$ equipped with the Lie bracket given by the commutator relations of its canonical basis $\{X_1, \ldots, X_{n_o}, Y_1, \ldots, Y_{n_o}, T\}$:

$$[X_j, Y_j] = T \qquad \text{for } j = 1, \ldots, n_o,$$

and all the other Lie brackets (apart from those obtained by anti-symmetry) are trivial.

In the case $n_o = 1$, we will often simplify the notation and write X, Y, T for the basis of $\mathfrak{h}_1$, etc...

Example 1.6.5. Let T_{n_o} be the group of $n_o \times n_o$ matrices which are upper triangular with 1 on the diagonal. The matrix group T_{n_o} is a nilpotent Lie group.

It can be proved that any (connected simply connected) nilpotent Lie group can be realised as a subgroup of T_{n_o}.

Its Lie algebra $\mathfrak{t}_{n_o}$ is the space of $n_o \times n_o$ matrices which are upper triangle with 0 on the diagonal. A basis is $\{E_{i,j}, 1 \le i < j \le n_o\}$ where $E_{i,j}$ is the matrix with all zero entries except the i-th row and j-th column which is 1.

Proposition 1.6.6. *Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then*

(a) The exponential map $\exp_G$ is a diffeomorphism from $\mathfrak{g}$ onto G.

(b) If G is identified with $\mathfrak{g}$ via $\exp_G$, the group law $(x, y) \mapsto xy$ is a polynomial map.

(c) If $d\lambda_{\mathfrak{g}}$ denotes a Lebesgue measure on the vector space $\mathfrak{g}$, then $d\lambda_{\mathfrak{g}} \circ \exp_G^{-1}$ is a bi-invariant Haar measure on G.

This proposition can be found in, e.g. [FS82, Proposition 1.2] or [CG90, Sec. 1.2].

After the choice of a basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$, Proposition 1.6.6, Part (a), implies that the group G is identified with $\mathbb{R}^n$ via the exponential mapping; this means that a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is identified with the point

$$\exp_G(x_1 X_1 + \ldots + x_n X_n)$$

of the group. Part (b) implies that the law can be written as

$$x \cdot y = (P_1(x, y), P_2(x, y), \ldots, P_n(x, y)), \tag{1.20}$$

where $P_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, n$, are polynomial mappings given via the Baker-Campbell-Hausdorff formula (see Theorem 1.3.2). Indeed in the nilpotent case, since ad is nilpotent, the Baker-Campbell-Hausdorff formula is finite and holds for any two elements of the Lie algebra.

Remark 1.6.7. More is known.

1. Certain choices of bases, namely the so-called Jordan-Hölder or strong-Malcev bases ([Puk67, CG90]), lead to a 'triangular' shaped law, that is,

$$\begin{array}{rcl} P_1(x,y) & = & x_1 + y_1, \\ P_2(x,y) & = & x_2 + y_2 + Q_2(x_1,y_1), \\ \vdots & & \\ P_n(x,y) & = & x_n + y_n + Q_n(x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}), \end{array}$$

   with $Q_1, \ldots, Q_n$ polynomials.

   In Chapter 3 we will see that in the particular case of homogeneous Lie groups, with the choice of the basis made in Section 3.1.3, this fact together with some additional homogeneous properties is proved in Proposition 3.1.24.

2. The second type of exponential coordinates

$$\mathbb{R}^n \ni (x_1, \ldots, x_n) \longmapsto \exp_G(x_1 X_1) \ldots \exp_G(x_n X_n) \in G,$$

   may be used to identify a nilpotent Lie group with $\mathbb{R}^n$ after the choice of a suitable basis as in Part 1.

   In the particular case of homogeneous Lie groups, with the choice of the basis made in Section 3.1.3, this fact together with some additional homogeneous properties is proved in Lemma 3.1.47.

3. The converse of (a) and (b) in Proposition 1.6.6 holds in the following sense: if a Lie group G can be identified with $\mathbb{R}^n$ such that

   (a) its law is a polynomial mapping (as in (1.20)),

   (b) and for any $s,t \in \mathbb{R}$, $x \in \mathbb{R}^n$, the product of the two points sx and tx is the point $(s+t)x$,

   then the Lie group G is nilpotent [Puk67, Part. II chap. I].

However, we will not use these general facts.

Setting aside the abelian case $(\mathbb{R}^n, +)$, we use the multiplicative notation for the group law of any other connected simply connected nilpotent Lie group G. The identification of G with $\mathfrak{g}$ leads to *consider the origin* 0 *as the unit element* (even if the equality $xx^{-1} = 0$ may look surprising at first sight). Because of the Baker-Campbell-Hausdorff formula (see Theorem 1.3.2), the inverse of an element is in fact its opposite, that is, with the notation above,

$$x^{-1} = (-x_1, \ldots, -x_n).$$

The identification of G with $\mathfrak{g}$ allows us to define objects which usually live on a vector space, for example the Schwartz class:

Definition 1.6.8. A Schwartz function f on G is a function f such that $f \circ \exp_G$ is a Schwartz function on $\mathfrak{g}$. We denote by $\mathcal{S}(G)$ the class of Schwartz functions. It is naturally a Fréchet space and its dual space is the space of tempered distribution $\mathcal{S}'(G)$.

Formally a distribution $T \in \mathcal{D}'(G)$ is tempered when $T \circ \exp_G$ is a tempered distribution on $\mathfrak{g}$. The distribution duality is formally given by

$$\langle f, \phi \rangle = \int_G f(x)\phi(x)dx, \quad f \in \mathcal{S}'(G), \ \phi \in \mathcal{S}(G).$$

The Schwartz space and the tempered distributions on a nilpotent homogeneous Lie group will be studied more thoroughly in Section 3.1.9.

1.7 Smooth vectors and infinitesimal representations

In this section we describe the basics of the part of the representation theory of non-compact Lie groups that is relevant to our context. For most statements of this section we give proofs since understanding of these ideas will be important for the developments of pseudo-differential operators in Chapter 5. Thus, the setting that we have in mind is that of nilpotent Lie groups, although we do not need to make this assumption for the following discussion. For the general representation theory of locally compact groups we can refer to, for example, the books of Knapp [Kna01], Wallach [Wal92, Chapter 14] or Folland [Fol95].

Let us first recall some basic definitions about differentiability of a Banach space-valued function.

Definition 1.7.1. Let f be a function from on open subset Ω of $\mathbb{R}^n$ to a Banach space B with norm $|\cdot|_B$.

The function f is said to be *differentiable* at $x_o \in \Omega$ if there exists a (necessarily unique) linear map $f'(x_o) : \mathbb{R}^n \to B$ such that

$$\frac{1}{|x - x_o|_{\mathbb{R}^n}} |f(x) - f(x_o) - f'(x_o)(x - x_o)|_B \underset{x \to x_o}{\longrightarrow} 0.$$

We call $f'(x_o)$ the differential of f at x_o.

If f is differentiable at each point of Ω, then $x \mapsto f'(x)$ is a function from Ω to the Banach space $\mathscr{L}(\mathbb{R}^n, B)$ of linear mappings from $\mathbb{R}^n$ to B (recall that linear mappings from $\mathbb{R}^n$ to B are automatically bounded.) We say that f is *of class* C^1 if $x \mapsto f'(x)$ is continuous, and that f is *of class* C^2 if $x \mapsto f'(x)$ is of class C^1 and so on. We say that f is of *class* C^∞ if f is of class C^k for all $k \in \mathbb{N}$.

These definitions extend to any open set of any smooth manifold.

As in the case of functions valued in a finite dimensional Euclidean space, we have the basic properties for a function f as in Definition 1.7.1:

- The function f is of class C^k if and only if all of its partial derivatives of order $1, 2, \ldots, k$ exist and are continuous.
- The chain rule holds for a composition $f \circ h$ where h is a mapping from an open subset of a finite dimensional Euclidean space into Ω.

We can now define the smooth vectors of a representation.

Definition 1.7.2. Let G be a Lie group and let π be a representation of G on a Hilbert space $\mathcal{H}_\pi$. A vector $v \in \mathcal{H}_\pi$ is said to be *smooth* or *of type* C^∞ if the function

$$G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$$

is of class C^∞.

We denote by $\mathcal{H}_\pi^\infty$ the space of all smooth vectors of π.

The following is a necessary preparation to introduce the notion of the *infinitesimal representation* and of the operator $d\pi(X)$. This will be of fundamental importance in the sequel.

Proposition 1.7.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. Then for any $X \in \mathfrak{g}$ and $v \in \mathcal{H}_\pi^\infty$, the limit*

$$\lim_{t\to 0} \frac{1}{t} \left(\pi(\exp_G(tX))v - v\right)$$

exists in the norm topology of $\mathcal{H}_\pi$ and is denoted by $d\pi(X)v$. Each $d\pi(X)$ leaves $\mathcal{H}_\pi^\infty$ invariant, and $d\pi$ is a representation of $\mathfrak{g}$ on $\mathcal{H}_\pi^\infty$ satisfying

$$\forall X, Y \in \mathfrak{g} \qquad d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X) - d\pi\left([X,Y]\right) = 0. \tag{1.21}$$

Consequently, $d\pi$ extends to a representation of the Lie algebra $\mathfrak{U}(\mathfrak{g})$ on $\mathcal{H}_\pi^\infty$ with $d\pi(0) = 0$ and $d\pi(1) = 0$.

Recalling the derivative with respect to X in (1.7), we may formally abbreviate writing

$$d\pi(X)v = X(\pi(x)v)|_{x=e} \quad \text{or even} \quad d\pi(X) = X\pi(e). \tag{1.22}$$

Sketch of the proof of Proposition 1.7.3. Let $v \in \mathcal{H}_\pi^\infty$. The function $f : \mathfrak{g} \to \mathcal{H}_\pi$ defined by $f(X) := \pi(\exp X)v$ is of class C^∞, and for any $X \in \mathfrak{g}$ we have

$$f'(0)(X) = \lim_{t\to 0} \frac{1}{t} \left(\pi(\exp_G(tX))v - v\right).$$

By definition $f'(0)(X) = d\pi(X)$.

Since π is continuous we have, using the identification of $\mathfrak{g}$ with the space of left-invariant vector fields,

$$\begin{aligned} \pi(x)d\pi(X)v &= \lim_{t\to 0} \frac{1}{t}\pi(x)\left(\pi(\exp_G(tX))v - v\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left(\pi(x\exp_G(tX))v - \pi(x)v\right) = XF(x), \end{aligned}$$

where $F : G \to \mathcal{H}$ is the function defined by $F(x) := \pi(x)v$. By assumption F is of type C^∞ thus $x \mapsto XF(x)$ is also of type C^∞ and the equality above says that $d\pi(X)v$ is smooth. Hence $d\pi(X)$ leaves $\mathcal{H}_\pi^\infty$ stable. Consequently $X \mapsto d\pi(X)$ can be extended to an algebra homomorphism $\mathfrak{U}(\mathfrak{g}) \to \mathcal{H}_\pi^\infty$ as in the statement.

It remains to prove (1.21), i.e. that

$$\forall X, Y \in \mathfrak{g} \qquad d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X) - d\pi([X,Y]) = 0.$$

We fix $X, Y \in \mathfrak{g}$ and define a path c by

$$c(t) := \exp_G\left((-\mathrm{sgn}t)|t|^{\frac{1}{2}}X\right)\exp_G\left(-|t|^{\frac{1}{2}}Y\right)\exp_G\left((\mathrm{sgn}t)|t|^{\frac{1}{2}}X\right)\exp_G\left(|t|^{\frac{1}{2}}Y\right).$$

Clearly c is defined on a neighbourhood of 0 in $\mathbb{R}$ and valued in G, and is of class C^1 with $c'(0) = [X, Y]$. Let $v \in \mathcal{H}_\pi^\infty$. By the chain rule the map $t \mapsto \pi(c(t))v$ has differential $F'(e)([X,Y])$ at $t = 0$, where F is $F(x) = \pi(x)v$ as above and e is the neutral element. Thus

$$d\pi([X,Y]) = \lim_{t\to 0}\frac{1}{t}\left(\pi(c(t))v - v\right) = \lim_{t\to 0}\frac{1}{t^2}\left(\pi(c(t^2))v - v\right).$$

The strong continuity of π implies then

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t^2}\left(\pi(\exp_G(tX)\exp_G(tY))v - \pi(\exp_G(tY)\exp_G(tX))v\right)\\
&\qquad = \lim_{t\to 0}\pi(\exp_G(tY)\exp_G(tX))\frac{1}{t^2}\left(\pi(c(t^2))v - v\right)\\
&\qquad = d\pi([X,Y])v. \qquad (1.23)
\end{aligned}$$

But we can also compute

$$\begin{aligned}
&(d\pi(X)d\pi(Y)v, u) = \partial_{s=0}\partial_{t=0}(\pi(\exp_G(sX)\exp_G(tY))v, u)\\
&\quad = \lim_{t\to 0}\left(\frac{1}{t^2}\left\{\pi(\exp_G(tX)\exp_G(tY)) - \pi(\exp_G(tX)) - \pi(\exp_G(tY)) + \mathrm{I}\right\}v, u\right).
\end{aligned}$$

Interchanging X and Y and subtracting we find

$$\begin{aligned}
&((d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X))v, u) \qquad (1.24)\\
&\quad = \lim_{t\to 0}\left(\frac{1}{t^2}\left\{\pi(\exp_G(tX)\exp_G(tY)) - \pi(\exp_G(tY)\exp_G(tX))\right\}v, u\right).
\end{aligned}$$

Comparing this with (1.23), we obtain (1.21). This concludes the proof of Proposition 1.7.3. □

Definition 1.7.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. The representation $d\pi$ defined in Proposition 1.7.3 is called the *infinitesimal representation* associated to π. We will often denote it also by π. Consequently, for $T \in \mathfrak{U}(\mathfrak{g})$ or for its corresponding left-invariant differential operator, we write

$$\pi(T) := d\pi(T).$$

Example 1.7.5. For example, the infinitesimal representation of Ad is ad, see Section 1.3.

We now collect some properties of the infinitesimal representations.

Proposition 1.7.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let π be a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Then we have the following properties.*

(i) For the infinitesimal representation $d\pi$ of $\mathfrak{g}$ on $\mathcal{H}_\pi^\infty$ each $d\pi(X)$ for $X \in \mathfrak{g}$ is skew-hermitian: $d\pi(X)^* = -d\pi(X)$.

(ii) The space $\mathcal{H}_\pi^\infty$ of smooth vectors is invariant under $\pi(x)$ for every $x \in G$, and

$$\forall D \in \mathfrak{U}(\mathfrak{g})\ \forall v \in \mathcal{H}_\pi^\infty \qquad \pi(x)d\pi(D)\pi(x)^{-1}v = d\pi(\mathrm{Ad}(x)D)v.$$

(iii) If S is a vector subspace of $\mathcal{H}_\pi$ such that for all $v \in S$ and $X \in \mathfrak{g}$, the limits of $t^{-1}\{\pi(\exp_G(tX))v - v\}$ as $t \to 0$ exist, then $S \subset \mathcal{H}_\pi^\infty$.

(iv) Let $\phi \in \mathcal{D}(G)$. For any $X \in \mathfrak{g}$, viewed as a left-invariant vector field,

$$\forall v \in \mathcal{H}_\pi \qquad \pi(\phi)v \in \mathcal{H}_\pi^\infty \quad \text{and} \quad d\pi(X)\pi(\phi)v = \pi(X\phi)v,$$

and viewing X as a right-invariant vector field $\tilde{X}$,

$$\forall v \in \mathcal{H}_\pi^\infty \qquad \pi(\phi)d\pi(X)v = \pi(\tilde{X}\phi)v.$$

If G is a connected simply connected nilpotent Lie group, one can replace $\mathcal{D}(G)$ by the Schwartz space $\mathcal{S}(G)$.

Proof. Let us prove Part (i). Let $u, v \in \mathcal{H}_\pi^\infty$. The unitarity of π implies

$$\left(v, \frac{i}{t}\left(\pi(\exp_G(tX))u - u\right)\right) = \left(\frac{i}{-t}\left(\pi(\exp_G(-tX))v - v\right), u\right).$$

By definition of $d\pi(X)u$ and $d\pi(X)v$, the limits as $t \to 0$ of the left and right hand sides are $(v, id\pi(X)u)$ and $(id\pi(X)v, u)$, respectively. Hence they are equal and $d\pi(X)$ is skew-hermitian. This proves Part (i).

For (ii), we first observe that the map $x \mapsto \pi(x)\pi(x_o)v$ is the composition of $x \mapsto xx_o$ and $x \mapsto \pi(x)v$. Hence $\mathcal{H}_\pi^\infty$ is an invariant subspace for $\pi(x_o)$.

Now let $X \in \mathfrak{g}$, $x \in G$ and $v \in \mathcal{H}_\pi^\infty$. Then we compute easily

$$\begin{aligned} \frac{1}{t}\left(\pi(\exp_G(tX)) - \mathrm{I}\right)\pi(x)^{-1}v &= \pi(x)^{-1}\frac{1}{t}\left(\pi(x\exp_G(tX)x^{-1}) - \mathrm{I}\right)v \\ &= \pi(x)^{-1}\frac{1}{t}\left(\pi(\exp_G(\mathrm{Ad}(x)(tX)) - \mathrm{I}\right)v. \end{aligned}$$

Passing to the limit as $t \to 0$, we obtain

$$d\pi(X)\pi(x)^{-1}v = \pi(x)^{-1}d\pi(\mathrm{Ad}(x)(tX))v.$$

Hence

$$\pi(x)d\pi(X)\pi(x)^{-1} = d\pi(\mathrm{Ad}(x)(tX))$$

on $\mathcal{H}_\pi^\infty$. Using Proposition 1.7.3, we obtain a similar property for $D \in \mathfrak{U}(\mathfrak{g})$ instead of X. This shows (ii).

For (iii), by assumption for $v \in S$ the map $F_v : G \ni x \mapsto \pi(x)v$ is differentiable at the neutral element e, the partial derivative in the $X \in \mathfrak{g}$ direction being

$$XF_v(e) = \lim_{t\to 0} \frac{1}{t} \left\{\pi(\exp_G(tX))v - v\right\}.$$

More generally, since π is strongly continuous, we have for any $x \in G$,

$$\pi(x)XF_v(e) = \lim_{t\to 0} \frac{1}{t}\pi(x) \left\{\pi(\exp_G(tX))v - v\right\} = \lim_{t\to 0} \frac{1}{t} \left\{F_v(x \exp_G(tX)) - F_v(x)\right\}.$$

Thus F_v is also differentiable at $x \in G$ and

$$XF_v(x) = \pi(x)XF_v(e)$$

for any $X \in \mathfrak{g}$. This shows that the first derivatives of F_v are continuous, thus F_v must be of class C^1. Furthermore,

$$F_v'(x)(X) = \pi(x)XF_v(e).$$

If F_v is of class C^k for $k \in \mathbb{N}$, then the map $x \mapsto XF_v(x) = \pi(x)XF_v(e)$ is of class C^k and F_v must be of class C^{k+1}. Inductively this shows that F_v is of type C^∞. This shows Part (iii).

For (iv), for any $\phi \in L^1(G)$ and $x \in G$, recalling (1.3), we have

$$\pi(x)\pi(\phi) = \pi(\phi(\cdot\, x)).$$

Hence for any $\phi \in \mathcal{D}(G)$, $v \in \mathcal{H}_\pi$ and $X \in \mathfrak{g}$,

$$\frac{1}{t}\left(\pi(\exp_G(tX))\pi(\phi)v - \pi(\phi)v\right) = \pi\left(\frac{\phi(\cdot\, \exp_G(tX)) - \phi}{t}\right)v.$$

This last expression tends to $\pi(X\phi)v$ as $t \to 0$. Applying (iii) to $S = \pi(\phi)\mathcal{H}_\pi$, we see that $S \subset \mathcal{H}_\pi^\infty$. We also have

$$d\pi(X)\pi(\phi)v = \pi(X\phi)v.$$

For the right-invariant case, again by (1.3), we have

$$\pi(\phi)\pi(x) = \pi(\phi(x\,\cdot))$$

for any $\phi \in L^1(G)$ and $x \in G$. Hence for any $\phi \in \mathcal{D}(G)$, $v \in \mathcal{H}_\pi$ and $X \in \mathfrak{g}$,

$$\frac{1}{t}\left(\pi(\phi)\pi(\exp_G(tX))v - \pi(\phi)v\right) = \pi\left(\frac{\phi(\exp_G(tX)\,\cdot) - \phi}{t}\right)v.$$

This last expression tends to $\pi(\tilde{X}\phi)v$ as $t \to 0$ while the left-hand side tends to $\pi(\phi)d\pi(X)v$ if $v \in \mathcal{H}_\pi^\infty$. This proves Part (iv) in the general case. The changes for G connected simply connected nilpotent Lie group, and to replace $\mathcal{D}(G)$ by $\mathcal{S}(G)$ are straightforward. This concludes the proof of Proposition 1.7.6. □

In the following proposition, we show that the space of smooth vectors is dense in the space of a strongly continuous representation. The argument is famously due to Gårding.

Proposition 1.7.7. *Let G be a Lie group and let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$. Then the subspace $\mathcal{H}_\pi^\infty$ of smooth vectors is dense in $\mathcal{H}_\pi$.*

Proof. Let $v \in \mathcal{H}_\pi$ and $\epsilon > 0$ be given. Since π is strongly continuous, the set

$$\Omega := \{x \in G \,:\, |\pi(x)^* v - v|_{\mathcal{H}_\pi} < \epsilon\}$$

is open. We can find a non-negative function $\phi \in \mathcal{D}(G)$ supported in Ω satisfying $\int_G \phi(x)dx = 1$. Then

$$\begin{aligned} |\pi(\phi)v - v|_{\mathcal{H}_\pi} &= \left|\int_G \phi(x)(\pi(x)^* v - v)dx\right|_{\mathcal{H}_\pi} \\ &\leq \int_\Omega \phi(x)|\pi(x)^* v - v|_{\mathcal{H}_\pi} dx \leq \int_G \phi(x)\epsilon dx = \epsilon. \end{aligned}$$

By Proposition 1.7.6, we know that $\pi(\phi)v$ is a smooth vector. This shows that $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$. □

In the proof above, we have in fact showed that the vectors $\pi(\phi)v$ for $v \in \mathcal{H}_\pi$ and $\phi \in \mathcal{D}(G)$ form a dense subspace of $\mathcal{H}_\pi$. If G is nilpotent connected simply connected, the same property holds with $\phi \in \mathcal{S}(G)$. The finite linear combinations of those vectors form a subspace called the *Gårding subspace*, which is included in $\mathcal{H}_\pi^\infty$ by Proposition 1.7.6 (iv).

It turns out that the Gårding subspace is not only included in the subspace $\mathcal{H}_\pi^\infty$ but is in fact equal to $\mathcal{H}_\pi^\infty$. This is a consequent of the following theorem, due to Dixmier and Malliavin [DM78]:

Theorem 1.7.8 (Dixmier-Malliavin)**.** *Let G be a Lie group and let π be a strongly continuous representation of G on a Hilbert space $\mathcal{H}_\pi$.*

The space $\mathcal{H}_\pi^\infty$ of smooth vectors is spanned by all the vectors of the form $\pi(\phi)v$ for $v \in \mathcal{H}_\pi^\infty$ and $\phi \in \mathcal{D}(G)$. This means that any smooth vector can be written as a finite linear combination of vectors of the form $\pi(\phi)v$.

If G is a connected simply connected nilpotent Lie group, one can replace $\mathcal{D}(G)$ by the Schwartz space $\mathcal{S}(G)$.

1.8 Plancherel theorem

Here we discuss the Plancherel theorem for locally compact groups and for the special case of nilpotent Lie groups. Our presentation will be rather informal. One reason is that we decided not to present here in full detail the orbit method yielding the representations of the nilpotent Lie groups but to limit ourselves only to its consequences useful for our subsequent analysis. The reason behind this choice is that it could take quite much space to prove the general results for the orbit method and would lead us too much away from our main exposition also risking overwhelming the reader with technical discussions somewhat irrelevant for our purposes. In general, this subject is well-known and we can refer to books by Kirillov [Kir04] or by Corwin and Greenleaf [CG90] for excellent expositions of this topic. The same reasoning applies to the abstract Plancherel theorem: it is known in a much more general form, due to e.g. Dixmier [Dix77, Dix81], and we will limit ourselves to describing its implications for nilpotent Lie groups relevant to our subsequent work.

As we will see in Chapter 2, all the results of the abstract Plancherel theorem in the case of compact groups can be recaptured there thanks to the Peter-Weyl theorem (see Theorem 2.1.1). However, for nilpotent Lie groups, even if the orbit method provides a description of the dual of the group and of the Plancherel measure, in our analysis we will need to use the properties of the von Neumann algebra of the group provided by the general abstract Plancherel theorem. This will replace the use of the Fourier coefficients in the compact case.

Before we proceed, let us adopt two useful conventions. First, the set of all strongly continuous unitary irreducible representations of a locally compact group G will be denoted by $\mathrm{Rep}\, G$, i.e.

$$\mathrm{Rep}\, G = \{\text{all strongly continuous unitary irreducible representations of } G\}.$$

The equivalence of representations in $\mathrm{Rep}\, G$ leads to the unitary dual $\widehat{G}$. We have already agreed to write $\pi \in \widehat{G}$ meaning that the expressions, when dealing with Fourier transforms, may depend on π as described in Remark 1.1.5. However, in this section we will sometimes want to show that certain expressions do not depend on the equivalence class of π, and for this purpose we will be sometimes distinguishing between the sets $\mathrm{Rep}\, G$ and $\widehat{G}$.

The second useful convention that we will widely use especially in Chapter 5 is that we may denote the Fourier transform in three ways, namely, we have

$$\widehat{\phi}(\pi) \equiv \pi(\phi) \equiv \mathcal{F}_G(\phi)(\pi).$$

Although this may seem as too much notation for the same object, the reason for this is two-fold. Firstly, the notation $\pi(\phi)$ is widely adopted in the representation theory of C^*-algebra associated with groups. Secondly, it becomes handy for longer expressions as well as for expressing properties like

$$\pi(T\phi) = \pi(T)\pi(\phi)$$

where $\pi(T)$ is the infinitesimal representation given in Definition 1.7.4. The notation $\widehat{\phi}(\pi)$ is useful as an analogy for the Euclidean case and will be extensively used in the case of compact groups. When we want to write the Fourier transform as a mapping between different spaces, the notation $\mathcal{F}_G$ becomes useful.

1.8.1 Orbit method

In this section we briefly discuss the idea of the orbit method and its implications for our analysis. In general, we will not use the orbit method by itself in our analysis, but only the existence of a Plancherel measure and some Fourier analysis similar to the compact case as described in Section 2.1.

Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. The orbit method describes a way to associate to a given linear functional on $\mathfrak{g}$ a collection of unitary irreducible representations of G which are all unitarily equivalent between themselves. Consequently, to any element of the dual $\mathfrak{g}'$ of $\mathfrak{g}$, one can associate an equivalence class of unitary irreducible representations. It turns out that any such class is realised in this way. Furthermore, two elements $f_1, f_2 \in \mathfrak{g}'$ lead to the same class if and only if the two elements are in the same orbit under the natural action of G on $\mathfrak{g}'$; this natural action is the so-called *co-adjoint representation*: since the group G acts on $\mathfrak{g}$ by the adjoint representation Ad, it also acts on its dual $\mathfrak{g}'$ by

$$\text{co-Ad} : G \times \mathfrak{g}' \ni (g, f) \longmapsto f(\text{Ad}^{-1} g\, \cdot) \in \mathfrak{g}'.$$

This gives a one-to-one correspondence between

- on the one hand, the dual $\widehat{G}$ of the group, that is, the collection of unitary irreducible representations modulo unitary equivalence, and
- on the other hand, $\mathfrak{g}'/\text{co-Ad}(G)$, that is, the set of co-adjoint orbits.

Example 1.8.1. In the case of the Heisenberg group $\mathbb{H}_{n_o}$ presented in Example 1.6.4, a family of representatives of all co-adjoint orbits is

1. either of the form $\lambda T'$ if $\lambda \in \mathbb{R}\backslash\{0\}$,
2. or of the form $\sum_{j=1}^{n_o} \left(x_j' X_j' + y_j' Y_j'\right)$ with $x_j', y_j' \in \mathbb{R}$,

where $\{X_1', \ldots, X_{n_o}', Y_1', \ldots, Y_{n_o}', T'\}$ is the dual basis to the canonical basis of $\mathfrak{h}_{n_o}$ given in Example 1.6.4. To $\lambda T'$ is associated the Schrödinger representation π_λ, and to $\sum_{j=1}^{n_o} x_j' X_j' + y_j' Y_j'$ is associated the 1-dimensional representation $(x, y, t) \mapsto \exp\left(i(xx' + yy')\right)$, where xx' and yy' denote the canonical scalar product on $\mathbb{R}^n$. See Section 6.2.

As for Schrödinger representations, the representations constructed via the orbit method can be realised as acting on some $L^2(\mathbb{R}^m)$ and the dual $\widehat{G}$ may be identified with $\mathfrak{g}'/\text{co-Ad}(G)$, or even with suitable representatives of this quotient.

Thus, by the orbit method the unitary dual $\widehat{G}$ is 'concretely' described as a subset of some Euclidean space. It is then possible to construct 'explicitly' a measure μ on $\widehat{G}$ such that we have the Fourier inversion theorem (where we recall once more the notation and conventions described in the beginning of Section 1.8):

Theorem 1.8.2. *Let G be a connected simply connected nilpotent Lie group. The dual $\widehat{G}$ is then equipped with a measure μ called* the Plancherel measure *satisfying the following property for any $\phi \in \mathcal{S}(G)$.*

The operator $\pi(\phi) \equiv \widehat{\phi}(\pi)$ is trace class for any strongly continuous unitary irreducible representation $\pi \in \operatorname{Rep} G$, and $\operatorname{Tr}(\pi(\phi))$ depends only on the class of π; the function $\widehat{G} \ni \pi \mapsto \operatorname{Tr}(\pi(\phi))$ is integrable against μ and the following formula holds:

$$\phi(0) = \int_{\widehat{G}} \operatorname{Tr}(\pi(\phi))\, d\mu(\pi). \tag{1.25}$$

For the explicit expression of the Plancherel measure μ, see, e.g., [CG90, Theorem 4.3.9].

Applying formula (1.25) to $\phi(\cdot) = f(\cdot\, x)$ and using $\pi(\phi) = \pi(x)\pi(f)$ in view of (1.3), we obtain:

Corollary 1.8.3 (Fourier inversion formula)**.** *Let G be a connected simply connected nilpotent Lie group and let μ be the Plancherel measure on $\widehat{G}$.*

If $f \in \mathcal{S}(G)$, then $\pi(x)\pi(f)$ and $\pi(f)\pi(x)$ are trace class for every $x \in G$, the function $\widehat{G} \ni \pi \mapsto \operatorname{Tr}(\pi(x)\pi(f))$ is integrable against μ, and we have

$$f(x) = \int_{\widehat{G}} \operatorname{Tr}(\pi(x)\pi(f))\, d\mu(\pi) = \int_{\widehat{G}} \operatorname{Tr}(\pi(f)\pi(x))\, d\mu(\pi). \tag{1.26}$$

The latter equality can be seen by the same argument as above, applied to the function $f(x\,\cdot)$.

Example 1.8.4. In the case of the Heisenberg group $\mathbb{H}_{n_o}$, the Plancherel measure is given by integration over $\mathbb{R}\backslash\{0\}$ against $c_{n_0}|\lambda|^{n_o} d\lambda$, with a suitable constant c_{n_o} (depending on normalisations):

$$\phi(0) = c_{n_o} \int_{\mathbb{R}\backslash\{0\}} \operatorname{Tr}\big(\pi_\lambda(\phi)\big) |\lambda|^{n_o} d\lambda.$$

An orthonormal basis for $\mathcal{H}_{\pi_\lambda} = L^2(\mathbb{R}^{n_o})$ is given by the Hermite functions. The subset of $\widehat{G}$ formed by the 1-dimensional representations is negligible with respect to the Plancherel measure. We refer to Section 6.2.3 for a more detailed discussion as well as for the constant c_{n_o}.

Applying the inversion formula to $\phi * (\phi^*)$, where $\phi^*(x) = \bar{\phi}(x^{-1})$, one obtains:

Theorem 1.8.5 (Plancherel formula). *We keep the notation of Theorem 1.8.2. Let $\phi \in \mathcal{S}(G)$. Then the operator $\pi(\phi)$ is Hilbert-Schmidt, that is,*

$$\|\pi(\phi)\|_{\mathtt{HS}}^2 = \mathrm{Tr}\,(\pi(\phi)\pi(\phi)^*) < \infty$$

for any $\pi \in \mathrm{Rep}\, G$, and its Hilbert-Schmidt norm is constant on the equivalence class of π. The function $\widehat{G} \ni \pi \mapsto \|\pi(\phi)\|_{\mathtt{HS}}^2$ is integrable against μ and

$$\int_G |\phi(x)|^2 dx = \int_{\widehat{G}} \|\pi(\phi)\|_{\mathtt{HS}}^2 \, d\mu(\pi). \tag{1.27}$$

Formula (1.27) can be extended unitarily to hold for any $\phi \in L^2(G)$, permitting the definition of the group Fourier transform of a square integrable function on G.

Applying the inversion formula to $\phi * (\psi^*)$, or bilinearising the Plancherel formula, we also obtain:

Corollary 1.8.6. *Let $\phi, \psi \in \mathcal{S}(G)$. Then the operator $\pi(\phi)\pi(\psi)^*$ is trace class for any $\pi \in \mathrm{Rep}\, G$, and its trace is constant on the equivalence class of π. The function $\widehat{G} \ni \pi \mapsto \mathrm{Tr}\,(\pi(\phi)\pi(\psi)^*)$ is integrable against μ and*

$$(\phi, \psi)_{L^2(G)} = \int_G \phi(x)\overline{\psi(x)} dx = \int_{\widehat{G}} \mathrm{Tr}\,(\pi(\phi)\pi(\psi)^*) \, d\mu(\pi).$$

1.8.2 Plancherel theorem and group von Neumann algebras

In this section we describe the concept of the group von Neumann algebra that becomes handy in associating symbols with convolution kernels of invariant operators on G. For the details of the constructions described below we refer to Dixmier's books [Dix77, Dix81] and to Section B in the appendix of this monograph. For the Plancherel theorem on locally compact groups with emphasis on the decomposition of reducible representations in continuous Hilbert sums, see also Bruhat [Bru68]. A more extensive discussion of this subject is given in Appendix B.2, more precisely in Section B.2.5. An abstract version of the Plancherel theorem is also given in the appendix in Theorem B.2.32.

Our framework

The representation theory of a general locally compact group may be very wild. However, in favourable cases most of the traditional Fourier analysis on compact Lie groups (described in Section 2.1) remains valid under natural modifications; for instance, the sum over the discrete dual in the compact case is replaced by an integral. By favourable cases we mean the following hypothesis:

(H) The group G is separable locally compact, unimodular, and of type I.

(See e.g. Dixmier [Dix77]). For our purpose, it suffices to know that any Lie group which is either compact or nilpotent satisfies (H). Its unitary dual $\widehat{G}$ is a standard Borel space.

We will now present the abstract Plancherel theorem as obtained by Dixmier in [Dix77, §18.8] and stated in Theorem B.2.32. Here, we will formulate it neither in its logical order with the viewpoint of proving its statement nor in its full generality since this would require introducing a lot of additional notation. Instead, we present its consequences applicable to our setting, starting with the existence of the Plancherel measure.

The Plancherel formula

We start by describing the part of the Plancherel theorem dealing with the Plancherel formula. First if $\phi \in C_c(G)$ and $\pi \in \operatorname{Rep} G$, then $\widehat{\phi}(\pi)$ is a bounded operator on $\mathcal{H}_\pi$ (as the group Fourier transform of an integrable function) and one checks easily that its Hilbert-Schmidt norm is constant on the class of $\pi \in \operatorname{Rep} G$ in $\widehat{G}$. Hence $\|\widehat{\phi}(\pi)\|_{\mathtt{HS}(\mathcal{H}_\pi)}$ may be viewed as depending on $\pi \in \widehat{G}$. The Plancherel formula states that there exists a unique positive σ-finite measure μ, called the *Plancherel measure*, such that for any $\phi \in C_c(G)$ we have

$$\int_G |\phi(x)|^2 dx = \int_{\widehat{G}} \left\| \widehat{\phi}(\pi) \right\|^2_{\mathtt{HS}(\mathcal{H}_\pi)} d\mu(\pi). \tag{1.28}$$

In the compact or nilpotent case, the Plancherel measure can be described explicitly via the Peter-Weyl Theorem (see Theorem 2.1.1) or the orbit method (see Theorem 1.8.5), respectively.

The Plancherel formula in (1.28) may be reformulated in the following (more precise) way. The group Fourier transform is an isometry from $C_c(G)$ endowed with the $L^2(G)$-norm to the Hilbert space

$$L^2(\widehat{G}) := \int_{\widehat{G}}^{\oplus} \mathtt{HS}(\mathcal{H}_\pi) d\mu(\pi). \tag{1.29}$$

Hence the space $L^2(\widehat{G})$ is defined (see Section B.1 or, e.g., [Dix81, Part II ch. I]) as the space of μ-measurable fields of Hilbert-Schmidt operators $\{\sigma_\pi \in \mathtt{HS}(\mathcal{H}_\pi) : \pi \in \widehat{G}\}$ which are square integrable in the sense that

$$\|\sigma\|^2_{L^2(\widehat{G})} := \int_{\widehat{G}} \|\sigma_\pi\|^2_{\mathtt{HS}} d\mu(\pi) < \infty.$$

Here we use the usual identifications of a strongly continuous irreducible unitary representation from $\operatorname{Rep} G$ with its equivalence class in $\widehat{G}$, and of a field of operators on $\widehat{G}$ with its equivalence class with respect to the Plancherel measure μ. One can check that indeed, the properties above do not depend on a particular representative of π and of the field of operators. The Plancherel formula implies that $\mathcal{F}_G$ extends to an isometry on $L^2(G)$. We keep the same notation $\mathcal{F}_G$ for this map, allowing us to consider the Fourier transform of a square integrable function. The abstract Plancherel theorem states moreover that the isometry $\mathcal{F}_G : L^2(G) \to L^2(\widehat{G})$ is surjective. In other words, $\mathcal{F}_G$ maps $L^2(G)$ onto $L^2(\widehat{G})$ isometrically.

Note that for any $\phi, \psi \in L^2(G)$, the operator $\pi(\phi)\ \pi(\psi)^*$ is trace class on $\mathcal{H}_\pi$ for almost all $\pi \in \operatorname{Rep} G$ with

$$\operatorname{Tr}|\pi(\phi)\ \pi(\psi)^*| \leq \|\pi(\phi)\|_{\mathtt{HS}(\mathcal{H}_\pi)}\|\pi(\psi)^*\|_{\mathtt{HS}(\mathcal{H}_\pi)} = \|\pi(\phi)\|_{\mathtt{HS}(\mathcal{H}_\pi)}\|\pi(\psi)\|_{\mathtt{HS}(\mathcal{H}_\pi)},$$

and that $\operatorname{Tr}|\pi(\phi)\ \pi(\psi)^*|$ and $\operatorname{Tr}(\pi(\phi)\ \pi(\psi)^*)$ are constant on the class of $\pi \in \operatorname{Rep} G$ in $\widehat{G}$. Thus these traces can be viewed as being parametrised by $\pi \in \widehat{G}$. The bilinearisation of the Plancherel formula yields

$$\int_G \phi(x)\overline{\psi(x)}dx = \int_{\widehat{G}} \operatorname{Tr}(\pi(\phi)\ \pi(\psi)^*)\, d\mu(\pi). \tag{1.30}$$

One also checks easily, for example by density of $C_c(G)$ in $L^2(G)$, that Formula (1.17), that is,

$$\widehat{f * g}(\pi) = \widehat{g}(\pi)\widehat{f}(\pi) \tag{1.31}$$

or, in the other notation,

$$\pi(f * g) = \pi(g)\pi(f),$$

remains valid for $f \in L^1(G)$ and $g \in L^2(G)$ and also for $f \in L^2(G)$ and $g \in L^1(G)$.

We now present the parts of the Plancherel theorem (relevant for our subsequent analysis) regarding the description of the group von Neumann algebra.

Group von Neumann algebra

In this monograph, we realise the von Neumann algebra of a group G as the algebra denoted by $\mathscr{L}_L(L^2(G))$ and defined as follows.

Definition 1.8.7. Let $\mathscr{L}(L^2(G))$ denote the set of bounded linear operators $L^2(G) \to L^2(G)$, and let $\mathscr{L}_L(L^2(G))$ be the subset formed by the operators in $\mathscr{L}(L^2(G))$ which are left-invariant (in the sense of Definition 1.1.3).

Endowed with the operator norm and composition of operators, one checks easily that $\mathscr{L}_L(L^2(G))$ is a von Neumann algebra, see Section B.2.5 for the exposition of its general ideas.

Given a μ-measurable field of uniformly bounded operators $\sigma = \{\sigma_\pi\}$, the operator $T_\sigma \in \mathscr{L}_L(L^2(G))$ defined via

$$\widehat{T_\sigma \phi}(\pi) = \sigma_\pi \widehat{\phi}(\pi), \quad \phi \in L^2(G), \tag{1.32}$$

is in $\mathscr{L}_L(L^2(G))$. Using (1.30), this yields that the operator $T_\sigma : \mathcal{S}(G) \to \mathcal{S}'(G)$ can also be defined by

$$(T_\sigma \phi, \psi)_{L^2(G)} = \int_{\widehat{G}} \mathrm{Tr}\,(\sigma_\pi \; \pi(\phi) \; \pi(\psi)^*)\, d\mu(\pi), \quad \phi, \psi \in L^2(G). \tag{1.33}$$

This defines a map $\sigma \mapsto T_\sigma$ from $L^\infty(\widehat{G})$ to $\mathscr{L}_L(L^2(G))$ where the space $L^\infty(\widehat{G})$ is defined by

Definition 1.8.8. Let $L^\infty(\widehat{G})$ denote the space of μ-measurable fields on $\widehat{G}$ of uniformly bounded operators $\sigma = \{\sigma_\pi \in \mathscr{L}(\mathcal{H}_\pi),\; \pi \in \widehat{G}\}$, that is,

$$\sup_{\pi \in \widehat{G}} \|\sigma_\pi\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty. \tag{1.34}$$

Here we use the usual identifications of a strongly continuous irreducible unitary representation from $\mathrm{Rep}\, G$ with its equivalence class in $\widehat{G}$, and of a field of operators on $\widehat{G}$ with its equivalence class with respect to the Plancherel measure μ. One can check that indeed, being in $L^\infty(\widehat{G})$ does not depend on a particular representative of π and of the field of operators. In (1.34), the supremum is to be understood as the essential supremum with respect to the Plancherel measure μ.

We endow $L^\infty(\widehat{G})$ with the pointwise composition given by

$$\sigma\tau := \{\sigma_\pi \tau_\pi,\; \pi \in \widehat{G}\}, \quad \text{for} \quad \sigma = \{\sigma_\pi,\; \pi \in \widehat{G}\}, \tau = \{\tau_\pi,\; \pi \in \widehat{G}\} \in L^\infty(\widehat{G}),$$

and the essential supremum norm

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma_\pi\|_{\mathscr{L}(\mathcal{H}_\pi)}. \tag{1.35}$$

We may sometimes abuse the notation and write $\|\sigma_\pi\|_{L^\infty(\widehat{G})}$ when no confusion is possible.

One checks easily that $L^\infty(\widehat{G})$ is a von Neumann algebra and that the map

$$L^\infty(\widehat{G}) \ni \sigma \longmapsto T_\sigma \in \mathscr{L}_L(L^2(G)),$$

is a morphism of von Neumann algebras. The Plancherel theorem implies that this map is in fact a bijection and an isometry, and hence a von Neumann algebra isomorphism. More precisely it yields that for any $T \in \mathscr{L}_L(L^2(G))$, there exists a μ-measurable field of uniformly bounded operators $\{\sigma_\pi^{(T)}\}$ such that for any $\phi \in L^2(G)$ the Hilbert-Schmidt operators $\widehat{T\phi}(\pi)$ and $\sigma_\pi^{(T)} \widehat{f}(\pi)$ are equal μ-almost everywhere; the field $\{\sigma_\pi^{(T)}\}$ is unique up to a μ-negligible set.

Note that by the Schwartz kernel theorem (see Corollary 3.2.1), an operator $T \in \mathscr{L}_L(L^2(G))$ is of convolution type with kernel $\kappa \in \mathcal{D}'(G)$,

$$Tf = f * \kappa, \quad f \in \mathcal{D}(G).$$

If $\kappa \in \mathcal{D}'(G)$ is such that the corresponding convolution operator $\mathcal{D}(G) \ni f \mapsto f*\kappa$ extends to a bounded operator T_κ on $L^2(G)$ then $T_\kappa \in \mathscr{L}_L(L^2(G))$ and we extend the definition of the group Fourier transform by setting

$$\sigma_\pi^{(T)} := \pi(\kappa) \equiv \widehat{\kappa}(\pi). \tag{1.36}$$

We denote by $\mathcal{K}(G)$ the set of such distributions κ:

Definition 1.8.9. Let $\mathcal{K}(G)$ denote the space of distributions $\kappa \in \mathcal{D}'(G)$ such that the corresponding convolution operator

$$\mathcal{D}(G) \ni f \mapsto f * \kappa$$

extends to a bounded operator on $L^2(G)$.

If G is a connected simply connected nilpotent Lie group, the Schwartz kernel theorem (see Corollary 3.2.1), implies in fact that the distributions in $\mathcal{K}(G)$ are tempered, i.e. $\mathcal{K}(G) \subset \mathcal{S}'(G)$.

If $\kappa \in \mathcal{K}(G)$, then κ^* defined via $\kappa^*(x) = \bar{\kappa}(x^{-1})$ is also in $\mathcal{K}(G)$. If $\kappa_1, \kappa_2 \in \mathcal{K}(G)$ and $T_{\kappa_1}, T_{\kappa_2} \in \mathscr{L}_L(L^2(G))$ denote the associated right-convolution operator, then $T_{\kappa_1}T_{\kappa_2} \in \mathscr{L}_L(L^2(G))$ and we denote by $\kappa_2 * \kappa_1$ its convolution kernel. One checks easily that this convolution product coincides or extends the already defined convolution products in Section 1.5. Furthermore $\mathcal{K}(G)$ equipped with this convolution product, the $*$-adjoint and the operator norm

$$\|\kappa\|_{\mathcal{K}(G)} := \|f \mapsto f * \kappa\|_{\mathscr{L}(L^2(G))} \tag{1.37}$$

is a von Neumann algebra. It is naturally isomorphic to $\mathscr{L}_L(L^2(G))$.

The part of the Plancherel theorem that we have already presented implies that the space $\mathcal{K}(G)$ is a von Neumann algebra isomorphic to $\mathscr{L}_L(L^2(G))$ and to $L^\infty(\widehat{G})$. Moreover, the group Fourier transform defined on $\mathcal{K}(G)$ gives the isomorphism between $\mathcal{K}(G)$ and $L^\infty(\widehat{G})$.

Naturally, $L^1(G)$ is embedded in $\mathcal{K}(G)$ since if $\kappa \in L^1(G)$, then the operator $\phi \mapsto \phi * \kappa$ is in $\mathscr{L}_L(L^2(G))$. Note that Young's inequality (see Proposition 1.5.2) implies

$$\|\widehat{\kappa}\|_{L^\infty(\widehat{G})} = \|\kappa\|_{\mathcal{K}} \leq \|\kappa\|_{L^1(G)}. \tag{1.38}$$

Furthermore, as $\mathcal{F}_G(\phi * \kappa) = \widehat{\kappa}\widehat{\phi}$ (see e.g. (1.31)), there is no conflict of notation between the group Fourier transforms defined first on $L^1(G)$ via (1.2) and then on $\mathcal{K}(G)$ in (1.36) as these group Fourier transforms coincide, since the field of operators associated to an operator in $\mathscr{L}_L(L^2(G))$ is unique.

More generally, the proof of Example 1.8.10 below shows that the space of complex Borel measures $M(G)$ (which contains $L^1(G)$) is contained in $\mathcal{K}(G)$, that is,

$$L^1(G) \subset M(G) \subset \mathcal{K}(G).$$

Moreover, their group Fourier transform may be defined directly via (1.39) below or as of an element of $\mathcal{K}(G)$ via Definition 1.36.

Example 1.8.10 (Complex Borel measures). Any complex Borel measure η on G is in $\mathcal{K}(G)$ and

$$\|\eta\|_{\mathcal{K}} \leq \|\eta\|_{M(G)},$$

where $\|\eta\|_{M(G)}$ denotes the total mass of η.

The group Fourier transform of a complex Borel measure η is given in the sense of Bochner by the integral

$$\mathcal{F}_G(\eta)(\pi) \equiv \widehat{\eta}(\pi) \equiv \pi(\eta) := \int_G \pi(x)^* d\eta(x). \tag{1.39}$$

In particular, the group Fourier transform of the Dirac measure δ_e at the neutral element is the identity operator

$$\widehat{\delta_e}(\pi) \equiv \pi(\delta_e) = \mathrm{I}_{\mathcal{H}_\pi}$$

on the representation space $\mathcal{H}_\pi$. More generally, the group Fourier transform of the Dirac measure δ_{x_o} at the element $x_o \in G$ is

$$\widehat{\delta_{x_o}}(\pi) = \pi(x_o).$$

Proof of Example 1.8.10. By Jensen's inequality, for $p = 1$ and 2 (in fact for any $p \in [1, \infty)$), the operator $T_\eta : \mathcal{D}(G) \ni \phi \mapsto \phi * \eta$ extends to an L^p-bounded operator with norm $\|\eta\|$.

If $\phi \in C_c(G)$, then $\phi * \eta \in L^1(G)$ (see Example 1.8.10) and we have in the sense of Bochner, using the change of variable $y = xz^{-1}$,

$$\begin{aligned}
\pi(\phi * \eta) &= \int_{G\times G} \phi(xz^{-1})\pi(x)^* dx d\eta(z) = \int_{G\times G} \phi(y)\pi(yz)^* dy d\eta(z) \\
&= \int_{G\times G} \phi(y)\pi(z)^*\pi(y)^* dy d\eta(z) = \int_G \pi(z)^* d\eta(z) \int_G \phi(y)\pi(y)^* dy \\
&= \pi(\eta)\,\pi(\phi),
\end{aligned}$$

confirming the formula for $\pi(\eta)$. Since the field of operators associated to an operator in $\mathscr{L}_L(L^2(G))$ is unique, the group Fourier transform of η as an element of $\mathcal{K}(G)$ is $\{\pi(\eta), \pi \in \widehat{G}\}$ defined in (1.39). □

The abstract Plancherel theorem

We now summarise the consequences of Dixmier's abstract Plancherel theorem, see Theorem B.2.32, that we will use:

Theorem 1.8.11 (Abstract Plancherel theorem)**.** *Let G be a Lie group satisfying hypothesis (H). We denote by μ its Plancherel measure.*

The Fourier transform $\mathcal{F}_G$ extends to an isometry from $L^2(G)$ onto

$$L^2(\widehat{G}) := \int_{\widehat{G}}^{\oplus} \mathtt{HS}(\mathcal{H}_\pi) d\mu(\pi).$$

The Fourier transform of an element f of $\mathcal{K}(G)$, i.e. $f \in \mathcal{D}'(G)$ such that the operator $\mathcal{D}(G) \ni \phi \mapsto \phi * f$ extends boundedly to $L^2(G)$, has a meaning as a field of uniformly (μ-essentially) bounded operators

$$\{\widehat{f}(\pi) \equiv \pi(f) \,:\, \pi \in \widehat{G}\} \in L^\infty(\widehat{G})$$

satisfying

$$\pi(\phi * f) = \pi(f)\pi(\phi)$$

for any $\phi \in \mathcal{D}(G)$ and $\pi \in \widehat{G}$. Conversely, any field in $L^\infty(\widehat{G})$ leads to an element of $\mathcal{K}(G)$. Furthermore

$$\|f\|_{\mathcal{K}} = \|\phi \mapsto \phi * f\|_{\mathscr{L}(L^2(G))} = \sup_{\pi\in\widehat{G}} \|\widehat{f}(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}. \tag{1.40}$$

The Fourier transform is a von Neumann algebra isomorphism from $\mathcal{K}(G)$ onto $L^\infty(\widehat{G})$. In particular, it is a bijection from $\mathcal{K}(G)$ onto $L^\infty(\widehat{G})$ and satisfies

$$\forall f_1, f_2, f \in \mathcal{K}(G) \quad \mathcal{F}_G(f_1 * f_2) = \mathcal{F}_G(f_2)\mathcal{F}_G(f_1) \quad \textit{and} \quad \mathcal{F}_G(f^*) = \mathcal{F}_G(f)^*,$$

if $f^*(x) = \bar{f}(x^{-1})$. Moreover

$$\|\widehat{f}\|_{L^\infty(\widehat{G})} = \|f\|_{\mathcal{K}(G)}.$$

If G is a connected simply connected nilpotent Lie group, the elements of $\mathcal{K}(G)$ are tempered distributions.

Naturally the various definitions of group Fourier transforms on $L^1(G)$ or on the space $M(G)$ of regular complex measures on G, on $L^2(G)$ or on $\mathcal{K}(G)$, coincide on any intersection of these subspaces of $\mathcal{D}'(G)$. This can be seen easily using the abstract Plancherel theorem, especially the bijections $\mathcal{F}_G : L^2(G) \to L^2(\widehat{G})$ and $\mathcal{F}_G : \mathcal{K}(G) \to L^\infty(\widehat{G})$, together with the properties of the convolution and of the representations, especially (1.31).

1.8.3 Fields of operators acting on smooth vectors

Let us assume that the group G satisfies hypothesis (H) as in the previous section and is also a Lie group. This means that G is a unimodular Lie group of type I, for instance a compact or nilpotent Lie group.

In our subsequent analysis, we will need to consider fields of operators parametrised by $\widehat{G}$ but not necessarily bounded, for instance the fields given by the $\pi(X)^\alpha$'s.

The definition of fields of smooth vectors or of operators defined on smooth vectors will be a consequence of the following lemma. For a more general setting for measurable fields of operators see Section B.1.5.

Lemma 1.8.12. *Let $\pi_1, \pi_2 \in \mathrm{Rep}\, G$ with $\pi_1 \sim_T \pi_2$, that is, we assume that π_1 and π_2 are intertwined by the unitary operator T, i.e. $T\pi_1 = \pi_2 T$. Then T maps $\mathcal{H}^\infty_{\pi_1}$ onto $\mathcal{H}^\infty_{\pi_2}$ bijectively.*

Proof. This is an easy consequence of the Dixmier-Malliavin theorem, see Theorem 1.7.8. □

Lemma 1.8.12 allows us to define fields of operators not necessarily bounded but just defined on smooth vectors:

Definition 1.8.13. A *$\widehat{G}$-field of operators defined on smooth vectors* is a family of classes of operators $\{\sigma_\pi, \pi \in \widehat{G}\}$ where

$$\sigma_\pi := \{\sigma_{\pi_1} : \mathcal{H}^\infty_{\pi_1} \to \mathcal{H}_{\pi_1}, \pi_1 \in \pi\}$$

for each $\pi \in \widehat{G}$ viewed as a subset of $\mathrm{Rep}\, G$, satisfying for any two elements σ_{π_1} and σ_{π_2} in σ_π:

$$\pi_1 \sim_T \pi_2 \Longrightarrow \sigma_{\pi_2} T = T \sigma_{\pi_1}.$$

It is measurable when for one (and then any) choice of realisation π_1 and any vector $x_{\pi_1} \in \mathcal{H}^\infty_{\pi_1}$, as π runs over $\widehat{G}$, the resulting field $\{\sigma_{\pi_1} x_{\pi_1}, \pi \in \widehat{G}\}$ is μ-measurable whenever $\int_{\widehat{G}} \|x_{\pi_1}\|^2_{\mathcal{H}_{\pi_1}} d\mu(\pi) < \infty$.

We will allow ourselves the shorthand notation

$$\sigma = \{\sigma_\pi : \mathcal{H}^\infty_\pi \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$$

to indicate that the $\widehat{G}$-field of operators is defined on smooth vectors. Unless otherwise stated, all the $\widehat{G}$-fields of operators are assumed to be measurable and with operators defined on smooth vectors. We may allow ourselves to write $\sigma = \{\sigma_\pi, \pi \in \widehat{G}\}$. Note that we do not require the domain of each operator to be the whole representation space $\mathcal{H}_{\pi_1}$ but just the space of smooth vectors.

The next definition would allow us to compose such fields of operators.

Definition 1.8.14. A *measurable $\widehat{G}$-field of operators acting on the smooth vectors* is a measurable $\widehat{G}$-field of operators $\sigma = \{\sigma_\pi : \mathcal{H}^\infty_\pi \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ such that for any $\pi_1 \in \mathrm{Rep}\, G$, we have

$$\sigma_{\pi_1}(\mathcal{H}^\infty_{\pi_1}) \subset \mathcal{H}^\infty_{\pi_1}.$$

We will often abuse the notation and write

$$\{\sigma_\pi : \mathcal{H}^\infty_\pi \to \mathcal{H}^\infty_\pi, \pi \in \widehat{G}\}$$

to express the fact that the measurable $\widehat{G}$-field of operators act on smooth vectors.

Remark 1.8.15. Let $\sigma = \{\sigma_\pi : \mathcal{H}^\infty_\pi \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ be a $\widehat{G}$-field. If $\pi_1 \sim_T \pi_2$ that is, we assume that π_1 and π_2 are intertwined by the unitary operator T, then T maps $\sigma_{\pi_1}(\mathcal{H}^\infty_{\pi_1})$ onto $\sigma_{\pi_2}(\mathcal{H}^\infty_{\pi_2})$ bijectively. Thus the range $\sigma_\pi(\mathcal{H}^\infty_\pi)$ makes sense as the collection of the equivariant ranges $\sigma_{\pi_1}(\mathcal{H}^\infty_\pi)$ for $\pi_1 \in \pi \subset \mathrm{Rep}\, G$.

Consequently, in Definition 1.8.14, it suffices that $\sigma_{\pi_1}(\mathcal{H}^\infty_{\pi_1}) \subset \mathcal{H}^\infty_{\pi_1}$ for one representation $\pi_1 \in \pi$ for each $\pi \in \widehat{G}$.

Remark 1.8.16. We will often consider measurable field of operators $\sigma_{\pi,s}$ acting on smooth vectors and parametrised not only by $\widehat{G}$ but also by another set S. When this set S is a subset of some $\mathbb{R}^n$, we say that this parametrisation is smooth whenever the map appearing in Definition 1.8.14 above is not only measurable with respect to $\widehat{G}$ but also smooth with respect to the S-variable. Note that this hypothesis yields the existence of the fields of operators given by $D_s\sigma_{\pi,s}$ where D_s is a (smooth) differential operator on S.

It is clear that one can sum two fields $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ and $\tau = \{\tau_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ defined on smooth vectors. We may then write

$$\sigma + \tau = \{\sigma_\pi + \tau_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$$

for the resulting field. If σ and τ act on smooth vectors, then so does $\sigma + \tau$.

It is also clear that one can compose two fields $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ and $\tau = \{\tau_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, \pi \in \widehat{G}\}$ defined on smooth vectors if the first one acts on smooth vectors. We may then write

$$\sigma\tau = \{\sigma_\pi\tau_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$$

for the resulting field which is then defined on smooth vectors. Note that $\sigma\tau$ is not obtained as the composition of two unbounded operators on $\mathcal{H}_\pi$ as in Definition A.3.2 but as the composition of two operators acting on the same space $\mathcal{H}_\pi^\infty$.

Almost by definition of smooth vectors, we have the following example of measurable fields of operators acting on smooth vectors:

Example 1.8.17. If $T \in \mathfrak{U}(\mathfrak{g})$ then $\{\pi(T), \pi \in \widehat{G}\}$ yields a measurable field of operators acting on smooth vectors and parametrised by $\widehat{G}$ (see also Proposition 1.7.3).

If $T_1, T_2 \in \mathfrak{U}(\mathfrak{g})$ then the composition of $\{\pi(T_1), \pi \in \widehat{G}\}$ with $\{\pi(T_2), \pi \in \widehat{G}\}$ as field of operators acting on smooth vectors is $\{\pi(T_1T_2), \pi \in \widehat{G}\}$.

The definition of Fourier transform and Proposition 1.7.6 (iv) easily imply the next example of measurable fields of operators acting on smooth vectors:

Example 1.8.18. If $\phi \in \mathcal{D}(G)$, then $\widehat{\phi} = \{\pi(\phi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty, \pi \in \widehat{G}\}$ is a measurable $\widehat{G}$-field of operators acting on smooth vectors.

If $\phi_1, \phi_2 \in \mathcal{D}(G)$, then the composition of $\widehat{\phi}_1$ with $\widehat{\phi}_2$ as fields of operators acting on smooth vectors is $\widehat{\phi_2 * \phi_1}$.

If G is simply connected and nilpotent, the properties above also hold for Schwartz functions.

A field $\sigma = \{\sigma_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ always gives by restriction operators that are defined on smooth vectors. If we start from a field of operators $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ defined on smooth vectors, we can not always extend it to operators defined on every $\mathcal{H}_\pi$. However, since the space $\mathcal{H}_\pi^\infty$ of smooth vectors is dense in $\mathcal{H}_\pi$ (see Proposition 1.7.7), each operator $\sigma_{\pi_1} : \mathcal{H}_{\pi_1}^\infty \to \mathcal{H}_{\pi_1}$, $\pi_1 \in \operatorname{Rep} G$, has a unique extension to a bounded operator on $\mathcal{H}_{\pi_1}$ provided that such an extension exists. In this case, σ_{π_2} would have the same property if $\pi_1 \sim \pi_2$, and the operator norm $\|\sigma_{\pi_1}\|_{\mathscr{L}(\mathcal{H}_{\pi_1})}$ or the Hilbert-Schmidt norm $\|\sigma_{\pi_1}\|_{\mathtt{HS}(\mathcal{H}_{\pi_1})}$ of σ_{π_1} are constant (maybe infinite) for $\pi_1 \in \pi$. Hence we may regard these norms as being parametrised by $\pi \in \widehat{G}$. Furthermore, if $\|\sigma\|_{L^\infty(\widehat{G})}$ or $\|\sigma\|_{L^2(\widehat{G})}$ are finite, then the field of bounded operators in $L^\infty(\widehat{G})$ or $L^2(\widehat{G})$ (resp.) is unique and extends σ.

On a compact Lie group, any $\widehat{G}$-field of operators is measurable and the operators act on smooth vectors. This is because in this case $\widehat{G}$ is discrete and countable, and all the strongly continuous irreducible representations are finite dimensional and these have only smooth vectors, see the Peter-Weyl theorem in Theorem 2.1.1.

However on a non-compact Lie group, we can not restrict ourselves to the case of $\widehat{G}$-fields acting on smooth vectors in general since a non-compact Lie group may have infinite dimensional (strongly continuous irreducible) representations with non-smooth vectors and we then can find fields in $L^2(\widehat{G})$ which do not act on smooth vectors. Indeed, in this case, we can find a measurable field $\{v_\pi, \pi \in \widehat{G}\}$ of non-smooth vectors satisfying $\int_{\widehat{G}} \|v_\pi\|^2_{\mathtt{HS}(\mathcal{H}_\pi)} d\mu(\pi) < \infty$, and then construct the field of operators $\{v_\pi \otimes v_\pi^*, \pi \in \widehat{G}\}$ in $L^2(\widehat{G})$ which does not act on smooth vectors. Such field of vectors $\{v_\pi\}$ are easy to find for instance on the Heisenberg group $\mathbb{H}_n$ whose case is detailed in Chapter 6: in this case, almost all the representations in $\widehat{\mathbb{H}}_n$ may be realised on $L^2(\mathbb{R}^n)$ and the space of smooth vectors then coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, see Section 6.2.1.

We can give a sufficient condition for a field to act on smooth vectors:

Lemma 1.8.19. *Let $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi\}$ be a field defined on smooth vectors. If for each $\phi \in \mathcal{D}(G)$, $\sigma\widehat{\phi}$ is a field of operators acting on smooth vectors, that is,*

$$\sigma\widehat{\phi} = \{\sigma_\pi \pi(\phi) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty\},$$

then σ acts on smooth vectors.

Proof. Let us assume that $\sigma\widehat{\phi}$ is a field of operators acting on smooth vectors for every $\phi \in \mathcal{D}(G)$. Then, for each $\pi \in \widehat{G}$ realised as a representation and each smooth vector $v \in \mathcal{H}_\pi^\infty$, $\sigma_\pi \widehat{\phi}(\pi) v$ is smooth. By the Dixmier-Malliavin Theorem, see Theorem 1.7.8. the finite linear combination of the vectors of the form $\phi(\pi)v$ form $\mathcal{H}_\pi^\infty$. Therefore $\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$, and the statement is proved. ☐

As an application of Lemma 1.8.19, we see that the field $\widehat{\delta_{x_o}}$ given at the end of Example 1.8.10 acts on smooth vectors:

Example 1.8.20. For any $x_o \in G$, the field $\widehat{\delta_{x_o}} = \{\pi(x_o) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty\} \in L^\infty(\widehat{G})$ acts on smooth vectors.

Proof. Let $x_o \in G$. If $\phi \in \mathcal{D}(G)$, then by (1.4), $\pi(x_o)\pi(\phi) = \widehat{\phi(\cdot\, x_o)}(\pi)$ and $\phi(\cdot\, x_o) \in \mathcal{D}(G)$. Thus for any $v \in \mathcal{H}_\pi^\infty$, $\pi(x_o)\pi(\phi)v$ is smooth. We conclude using Lemma 1.8.19. □

To summarise, we will identify measurable $\widehat{G}$-fields $\sigma = \{\sigma_\pi : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi, \pi \in \widehat{G}\}$ defined on smooth vectors with their possible extensions whenever possible. If the group is non-compact, we can not restrict ourselves to fields acting on smooth vectors.