Chapter 2

Quantization on compact Lie groups

In this chapter we briefly review the global quantization of operators and symbols on compact Lie groups following [RT13] and [RT10a] as well as more recent developments of this subject in this direction. Especially the monograph [RT10a] can serve as a companion for the material presented here, so we limit ourselves to explaining the main ideas only. This quantization yields full (finite dimensional) matrix-valued symbols for operators due to the fact that the unitary irreducible representations of compact Lie groups are all finite dimensional. Here, in order to motivate the developments on nilpotent groups, which is the main subject of the present monograph, we briefly review key elements of this theory referring to [RT10a] or to other sources for proofs and further details.

Technically, the machinery for such global quantization of operators on compact Lie groups appears to be simpler than that on graded Lie groups that we deal with in subsequent chapters. Indeed, since the symbols can be viewed as matrices (more precisely, as linear transformations of finite dimensional representation spaces), we do not have to worry about their domains of definitions, extensions, and other functional analytical properties arising in the nilpotent counterpart of the theory. Also, we have the Laplacian at our disposal, which is elliptic and bi-invariant, simplifying the analysis compared to the analysis based on, for example, the sub-Laplacian on the Heisenberg group, or more general Rockland operators on graded Lie groups. On the other hand, the theory on graded Lie groups is greatly assisted by the homogeneous structure, significantly simplifying the analysis of appearing difference operators and providing additional tools such as the naturally defined dilations on the group.

When we will be talking about the quantization on graded Lie groups in Chapter 5 we will be mostly concerned, at least in the first stage, about assigning an operator to a given symbol. In fact, it will be a small challenge by itself to make

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rigorous sense of a notion of a symbol there, but eventually we will show that the correspondence between symbols and operators is one-to-one. The situation on compact Lie groups is considerably simpler in this respect. Moreover, in (2.19) we will give a simple formula determining the symbol for a given operator. Thus, here we may talk about quantization of both symbols and operators, with the latter being often preferable from the point of view of applications, when we are concerned in establishing certain properties of a given operator and use its symbol as a tool for it.

Overall, this chapter is introductory, also serving as a motivation for the subsequent analysis, so we only sketch the ideas and refer for a thorough treatise with complete proofs to the monograph [RT10a] or to the papers that we point out in relevant places.

We do not discuss here all applications of this analysis in the compact setting. For example, we can refer to [DR14b] for applications of this analysis to Schatten classes, r-nuclearity, and trace formulae for operators on $L^2(G)$ and $L^p(G)$ for compact Lie groups G. For the functional calculus of matrix symbols and operators on G we refer to [RW14].

A related but different approach to the pseudo-differential calculus of [RT10a] has been also recently investigated in [Fis15]; there, a different notion of difference operators is defined intrinsically on each compact groups. This will not be discussed here.

2.1 Fourier analysis on compact Lie groups

Throughout this chapter G is always a compact Lie group. As in Chapter 1, we equip it with the uniquely determined probability Haar measure which is automatically bi-invariant by the compactness of G. We denote it by dx. We start by making a few remarks on the representation theory specific to compact Lie groups.

2.1.1 Characters and tensor products

An important first addition to Section 1.1 is that for a compact group G, every continuous irreducible unitary representation of G is finite dimensional. We denote by d_π the dimension of a finite dimensional representation π, $d_\pi = \dim \mathcal{H}_\pi$.

Another important property is the orthogonality of representation coefficients as follows. Let $\pi_1, \pi_2 \in \widehat{G}$ and let us choose some basis in the representation spaces so that we can view π_1, π_2 as matrices $\pi_1 = ((\pi_1)_{ij})_{i,j=1}^{d_{\pi_1}}$ and $\pi_2 = ((\pi_2)_{kl})_{k,l=1}^{d_{\pi_2}}$. Then:

- if $\pi_1 \neq \pi_2$, then $((\pi_1)_{ij}, (\pi_2)_{kl})_{L^2(G)} = 0$ for all i, j, k, l;
- if $\pi_1 = \pi_2$ but $(i, j) \neq (k, l)$, then $((\pi_1)_{ij}, (\pi_2)_{kl})_{L^2(G)} = 0$;

- if $\pi_1 = \pi_2$ and $(i,j) = (k,l)$, then
$$((\pi_1)_{ij}, (\pi_2)_{kl})_{L^2(G)} = \frac{1}{d_\pi}, \qquad \text{with } d_\pi = d_{\pi_1} = d_{\pi_2}.$$

For a finite dimensional continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ we denote
$$\chi_\pi(x) := \operatorname{Tr}(\pi(x)),$$
the *character* of the representation π. Characters have a number of fundamental properties most of which follow from properties of the trace:

- $\chi_\pi(e) = d_\pi$;
- $\pi_1 \sim \pi_2$ if and only if $\chi_{\pi_1} = \chi_{\pi_2}$;
- consequently, the character χ_π does not depend on the choice of the basis in the representation space $\mathcal{H}_\pi$;
- $\chi(yxy^{-1}) = \chi_\pi(x)$ for any $x, y \in G$;
- $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$;
- $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1}\chi_{\pi_2}$, with the tensor product $\pi_1 \otimes \pi_2$ defined in (2.1);
- a finite dimensional continuous unitary representation π of G is irreducible if and only if $\|\chi_\pi\|_{L^2(G)} = 1$.
- for $\pi_1, \pi_2 \in \widehat{G}$, $(\chi_{\pi_1}, \chi_{\pi_2})_{L^2(G)} = 1$ if $\pi_1 \sim \pi_2$, and $(\chi_{\pi_1}, \chi_{\pi_2})_{L^2(G)} = 0$ if $\pi_1 \not\sim \pi_2$;
- for any $f \in L^2(G)$, there is the decomposition
$$f = \sum_{\pi \in \widehat{G}} d_\pi f * \chi_\pi,$$
given by the projections (2.7).

If we take $\pi_1 \in \operatorname{Hom}(G, \mathcal{U}(\mathcal{H}_1))$ and $\pi_2 \in \operatorname{Hom}(G, \mathcal{U}(\mathcal{H}_2))$ two finite dimensional representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, their *tensor product* $\pi_1 \otimes \pi_2$ is the representation on $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\pi_1 \otimes \pi_2 \in \operatorname{Hom}(G, \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2))$, defined by
$$(\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2) := \pi_1(x)v_1 \otimes \pi_2(x)v_2. \tag{2.1}$$

Here the inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is induced from those on $\mathcal{H}_1$ and $\mathcal{H}_2$ by
$$(v_1 \otimes v_2, w_1 \otimes w_2)_{\mathcal{H}_1 \otimes \mathcal{H}_2} := (v_1, w_1)_{\mathcal{H}_1}(v_2, w_2)_{\mathcal{H}_2}.$$

In particular, it follows that
$$((\pi_1 \otimes \pi_2)(x)(v_1 \otimes v_2), w_1 \otimes w_2)_{\mathcal{H}_1 \otimes \mathcal{H}_2} = (\pi_1(x)v_1, w_1)_{\mathcal{H}_1}(\pi_2(x)v_2, w_2)_{\mathcal{H}_2}. \tag{2.2}$$

If $\pi_1, \pi_2 \in \widehat{G}$, the representation $\pi_1 \otimes \pi_2$ does not have to be irreducible, and we can decompose it into irreducible ones:

$$\pi_1 \otimes \pi_2 = \bigoplus_{\pi \in \widehat{G}} m_\pi \pi. \tag{2.3}$$

The constants $m_\pi = m_\pi(\pi_1, \pi_2)$ are called the *Clebsch-Gordan coefficients* and they determine the multiplicity of π in $\pi_1 \otimes \pi_2$,

$$m_\pi \pi \equiv \oplus_1^{m_\pi} \pi.$$

Also, we can observe that in view of the finite dimensionality only finitely many of m_π's are non-zero. Combining this with (2.2), we see that the product of any of the matrix coefficients of representations $\pi_1, \pi_2 \in \widehat{G}$ can be written as a finite linear combination of matrix coefficients of the representations from (2.3) with non-zero Clebsch-Gordan coefficients. In fact, this can be also seen on the level of characters providing more insight into the multiplicities m_π. First, for the tensor product of π_1 and π_2 we have $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$. Consequently, equality (2.3) implies

$$\chi_{\pi_1} \chi_{\pi_2} = \chi_{\pi_1 \otimes \pi_2} = \sum_{\pi \in \widehat{G}} m_\pi \chi_\pi \tag{2.4}$$

with

$$m_\pi = m_\pi(\pi_1, \pi_2) = (\chi_{\pi_1} \chi_{\pi_2}, \chi_\pi)_{L^2(G)}.$$

This equality can be now reduced to the maximal torus of G, for which we recall

Cartan's maximal torus theorem: Let $\mathbb{T}^l \hookrightarrow G$ be an injective group homomorphism with the largest possible l. Then two representations of G are equivalent if and only if their restrictions to $\mathbb{T}^l$ are equivalent. In particular, the restriction $\chi_\pi|_{\mathbb{T}^l}$ of χ_π to $\mathbb{T}^l$ determines the equivalence class $[\pi]$.

Now, coming back to (2.4), we can conclude that we have

$$\chi_{\pi_1}|_{\mathbb{T}^l} \ \chi_{\pi_2}|_{\mathbb{T}^l} = \sum_{\pi \in \widehat{G}} m_\pi \ \chi_\pi|_{\mathbb{T}^l}.$$

For a compact connected Lie group G, the maximal torus is also called the Cartan subgroup, and its dimension is denoted by $\operatorname{rank} G$, the *rank of* G.

Explicit formulae for representations and the Clebsch-Gordan coefficients on a number of compact groups have been presented by Vilenkin [Vil68] or Zhelobenko [Žel73], with further updates in [VK91, VK93] by Vilenkin and Klimyk.

2.1.2 Peter-Weyl theorem

As discussed in Section 1.3, the Casimir element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ can be viewed as an elliptic linear second order bi-invariant partial differential operator on G. If G is equipped with the uniquely determined (normalised) bi-invariant Riemannian metric, the Casimir element can be viewed as its (negative definite) Laplace-Beltrami operator, which we will denote by $\mathcal{L}_G$. Consequently, for any $D \in \mathfrak{U}(\mathfrak{g})$ we have

$$D\mathcal{L}_G = \mathcal{L}_G D.$$

The fundamental result on compact groups is the Peter-Weyl Theorem [PW27] giving a decomposition of $L^2(G)$ into eigenspaces of the Laplacian $\mathcal{L}_G$ on G, which we now sketch.

Theorem 2.1.1 (Peter-Weyl)**.** *The space $L^2(G)$ can be decomposed as the orthogonal direct sum of bi-invariant subspaces parametrised by $\widehat{G}$,*

$$L^2(G) = \bigoplus_{\pi \in \widehat{G}} V_\pi, \quad V_\pi = \{x \mapsto \mathrm{Tr}(A\pi(x)) : A \in \mathbb{C}^{d_\pi \times d_\pi}\},$$

the decomposition given by the Fourier series

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \, \mathrm{Tr}\left(\widehat{f}(\pi)\pi(x)\right). \tag{2.5}$$

After a choice of the orthonormal basis in each representation space $\mathcal{H}_\pi$, the set

$$\mathcal{B} := \left\{\sqrt{d_\pi}\ \pi_{ij} : \pi = (\pi_{ij})_{i,j=1}^{d_\pi}, \pi \in \widehat{G}\right\} \tag{2.6}$$

becomes an orthonormal basis for $L^2(G)$. For $f \in L^2(G)$, the convergence of the series in (2.5) holds for almost every $x \in G$, and also in $L^2(G)$.

One possible idea for the proof of the Peter-Weyl theorem is as follows. Let us take $\mathcal{B}$ as in (2.6). Finite linear combinations of elements of $\mathcal{B}$ are called the *trigonometric polynomials* on G, and we denote them by $\mathrm{span}(\mathcal{B})$. From the orthogonality of representations (see Section 2.1.1) we know that $\mathcal{B}$ is an orthonormal set in $L^2(G)$. It follows from (2.3) and the consequent discussion that $\mathrm{span}(\mathcal{B})$ is a subalgebra of $C(G)$, trivial representation is its identity, and it is involutive since $\pi^* \in \widehat{G}$ if $\pi \in \widehat{G}$. By invariance it is clear that $\mathcal{B}$ separates points of G. Consequently, by the Stone-Weierstrass theorem $\mathrm{span}(\mathcal{B})$ is dense in $C(G)$. Therefore, it is also dense in $L^2(G)$, giving the basis and implying the Peter-Weyl theorem.

For $f \in L^2(G)$, the decomposition

$$f = \sum_{\pi \in \widehat{G}} d_\pi f * \chi_\pi$$

given in Section 2.1.1 corresponds to the decomposition (2.5), the projections of $L^2(G)$ to V_π given by the convolution mappings

$$L^2(G) \ni f \mapsto f * \chi_\pi \in V_\pi. \tag{2.7}$$

The Peter-Weyl theorem can be also viewed as the decomposition of left or right regular representations of G on $L^2(G)$ into irreducible components. Indeed, from the homomorphism property of representations it follows that in the decomposition

$$L^2(G) = \bigoplus_{\pi\in\widehat{G}} \bigoplus_{j=1}^{d_\pi} \operatorname{span}\{\pi_{ij} : 1 \le i \le d_\pi\}, \tag{2.8}$$

the spans on the right hand side are π_L-invariant, and the restriction of π_L to each such space is equivalent to the representation π itself. This gives the decomposition of π_L into irreducible components as

$$\pi_L \sim \bigoplus_{\pi\in\widehat{G}} \bigoplus_{1}^{d_\pi} \pi.$$

The same is true for the decomposition of $L^2(G)$ into π_R-invariant subspaces $\operatorname{span}\{\pi_{ij} : 1 \le j \le d_\pi\}$, replacing the spans in (2.8).

It follows that the spaces V_π are bi-invariant subspaces of $L^2(G)$ and, therefore, they are eigenspaces of all bi-invariant operators. In particular, they are eigenspaces for the Laplacian $\mathcal{L}_G$ and, by varying the basis in the representation space $\mathcal{H}_\pi$, we see that V_π corresponds to the same eigenvalue of $\mathcal{L}_G$, which we denote by $-\lambda_\pi$, i.e.

$$-\mathcal{L}_G|_{V_\pi} = \lambda_\pi \mathrm{I}, \quad \lambda_\pi \ge 0. \tag{2.9}$$

It is useful to introduce also the quantity corresponding to the first order elliptic operator $(\mathrm{I} - \mathcal{L}_G)^{1/2}$,

$$\langle\pi\rangle := (1 + \lambda_\pi)^{1/2}, \tag{2.10}$$

so that we also have

$$(\mathrm{I} - \mathcal{L}_G)^{1/2}|_{V_\pi} = \langle\pi\rangle \mathrm{I}.$$

The quantity $\langle\pi\rangle$ and its powers become very useful in quantifying the growth/ decay of Fourier coefficients, and eventually of symbols of pseudo-differential operators.

Using the Fourier series expression (2.5) and the orthogonality of matrix coefficients of representations, one can readily show that the Plancherel identity takes the form

$$(f, g)_{L^2(G)} = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\widehat{f}(\pi)\widehat{g}(\pi)^*\right).$$

From this, it becomes natural to define the norm $\|\cdot\|_{\ell^2(\widehat{G})}$,

$$\|\widehat{f}\|_{\ell^2(\widehat{G})} = \left(\sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{\mathtt{HS}}^2\right)^{1/2}, \tag{2.11}$$

with

$$\|\widehat{f}(\pi)\|_{\mathtt{HS}} = \sqrt{\operatorname{Tr}\left(\widehat{f}(\pi)\widehat{f}(\pi)^*\right)}.$$

This norm defines the Hilbert space $\ell^2(\widehat{G})$ with the inner product

$$(\sigma,\tau)_{\ell^2(\widehat{G})} := \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}(\sigma(\pi)\tau(\pi)^*), \qquad \sigma,\tau\in\ell^2(\widehat{G}), \tag{2.12}$$

and

$$\|\sigma\|_{\ell^2(\widehat{G})} = (\sigma,\sigma)^{1/2}_{\ell^2(\widehat{G})} = \left(\sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|^2_{\mathtt{HS}}\right)^{1/2}, \quad \sigma\in\ell^2(\widehat{G}),$$

so that the Plancherel identity yields

$$\|f\|_{L^2(G)} = \|\widehat{f}\|_{\ell^2(\widehat{G})}. \tag{2.13}$$

We conclude the preliminary part by recording some useful relations between the dimensions d_π and the eigenvalues $\langle\pi\rangle$ for representations $\pi\in\widehat{G}$: there exists $C>0$ such that

$$d_\pi \le C\langle\pi\rangle^{\frac{\dim G}{2}} \quad \text{and, even stronger,} \quad d_\pi \le C\langle\pi\rangle^{\frac{\dim G-\operatorname{rank} G}{2}}. \tag{2.14}$$

The first estimate follows immediately from the Weyl asymptotic formula for the eigenvalue counting function for the first order elliptic operator $(\mathrm{I}-\mathcal{L}_G)^{1/2}$ on the compact manifold G recalling that d_π^2 is the multiplicity of the eigenvalue $\langle\pi\rangle$, and the second one follows with a little bit more work from the Weyl character formula, with $\operatorname{rank} G$ denoting the rank of G. There is also a simple convergence criterion

$$\sum_{\pi\in\widehat{G}} d_\pi^2\langle\pi\rangle^{-s} < \infty \quad \text{if and only if} \quad s>\dim G, \tag{2.15}$$

which follows from property (ii) in Section 2.1.3 applied to the delta-distribution δ_e at the unit element $e\in G$.

2.1.3 Spaces of functions and distributions on G

Different spaces of functions and distributions can be characterised in terms of the Fourier coefficients. For this, it is convenient to introduce the space of matrices taking into account the dimensions of representations. Thus, we set

$$\begin{aligned}\Sigma &:= \left\{\sigma=(\sigma(\pi))_{\pi\in\widehat{G}} : \sigma(\pi)\in\mathscr{L}(\mathcal{H}_\pi)\right\}\\ &\simeq \left\{\sigma=(\sigma(\pi))_{\pi\in\widehat{G}} : \sigma(\pi)\in\mathbb{C}^{d_\pi\times d_\pi}\right\},\end{aligned}$$

the second line valid after a choice of basis in $\mathcal{H}_\pi$, and we are interested in the images of function spaces on G in Σ under the Fourier transform.

As it will be pointed out in Remark 2.2.1, we should rather consider the quotient space $\Sigma/\sim$ as the space of Fourier coefficients, with the equivalence in Σ induced by the equivalence of representations. However, in order to simplify the exposition, we will keep the notation Σ as above.

The set Σ can be considered as a special case of the direct sum of Hilbert spaces described in (1.29), with the corresponding interpretation in terms of von Neumann algebras. However, a lot of the general machinery can be simplified in the present setting since the Fourier coefficients allow the interpretation of matrices indexed over the discrete set $\widehat{G}$, with the dimension of each matrix equal to the dimension of the corresponding representation.

Distributions

For any distribution $u \in \mathcal{D}'(G)$, its matrix Fourier coefficient at $\pi \in \widehat{G}$ is defined by

$$\widehat{u}(\pi) := \langle u, \pi^* \rangle.$$

These are well-defined since $\pi(x)$ are smooth (even analytic). This gives rise to the Fourier transform of distributions on G but we will come to this after stating a few properties of several function spaces.

The following equivalences are easy to obtain for spaces defined initially via their localisations to coordinate charts, in terms of the quantity $\langle \pi \rangle$ introduced in (2.10):

(i) as we have already seen, $f \in L^2(G)$ if and only if $\widehat{f} \in \ell^2(\widehat{G})$, i.e. if

$$\sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{\mathtt{HS}}^2 < \infty.$$

(ii) For any $s \in \mathbb{R}$, we have $f \in H^s(G)$ if and only if $\langle \pi \rangle^s \widehat{f} \in \ell^2(\widehat{G})$ if and only if

$$\sum_{\pi \in \widehat{G}} d_\pi \langle \pi \rangle^{2s} \|\widehat{f}(\pi)\|_{\mathtt{HS}}^2 < \infty.$$

(iii) $f \in C^\infty(G)$ if and only if for every $M > 0$ there exits $C_M > 0$ such that

$$\|\widehat{f}(\pi)\|_{\mathtt{HS}} \leq C_M \langle \pi \rangle^{-M}$$

holds for all $\pi \in \widehat{G}$.

(iv) $u \in \mathcal{D}'(G)$ if and only if there exist $M > 0$ and $C > 0$ such that

$$\|\widehat{u}(\pi)\|_{\mathtt{HS}} \leq C \langle \pi \rangle^{M}$$

holds for all $\pi \in \widehat{G}$.

The second characterisation (ii) follows from (i) if we observe that $f \in H^s(G)$ means that $(\mathrm{I} - \mathcal{L}_G)^{s/2} f \in L^2(G)$, and then pass to the Fourier transform side. The third characterisation (iii) follows if we observe that $\widehat{f}(\pi)$ must satisfy (ii) for all s and use estimates (2.14), and (iv) follows from (iii) by duality. The last two characterisations motivate to define spaces $\mathcal{S}(\widehat{G}), \mathcal{S}'(\widehat{G}) \subset \Sigma$ by

$$\mathcal{S}(\widehat{G}) := \left\{ \sigma \in \Sigma : \forall M > 0 \ \exists C_M > 0 \text{ such that } \|\sigma(\pi)\|_{\mathtt{HS}} \leq C_M \langle\pi\rangle^{-M} \right\}$$

and

$$\mathcal{S}'(\widehat{G}) := \left\{ \sigma \in \Sigma : \exists M > 0, C > 0 \text{ such that } \|\sigma(\pi)\|_{\mathtt{HS}} \leq C \langle\pi\rangle^{M} \right\},$$

with the seminormed topology on $\mathcal{S}(\widehat{G})$ defined by family

$$p_k(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \langle\pi\rangle^k \|\sigma(\pi)\|_{\mathtt{HS}},$$

and the dual topology on $\mathcal{S}'(\widehat{G})$. It follows that the Fourier inversion formula (2.5) can be extended to the following: the Fourier transform $\mathcal{F}_G$ in (1.2) and its inverse, defined by

$$(\mathcal{F}_G^{-1}\sigma)(x) := \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}(\sigma(\pi)\pi(x)), \tag{2.16}$$

are continuous as $\mathcal{F}_G : C^\infty(G) \to \mathcal{S}(\widehat{G})$, $\mathcal{F}_G^{-1} : \mathcal{S}(\widehat{G}) \to C^\infty(G)$, and are inverse to each other on $C^\infty(G)$ and $\mathcal{S}(\widehat{G})$. In particular, this implies that $\mathcal{S}(\widehat{G})$ is a nuclear Montel space. The distributional duality between $\mathcal{S}'(\widehat{G})$ and $\mathcal{S}(\widehat{G})$ is given by

$$\langle \sigma_1, \sigma_2 \rangle_{\widehat{G}} = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}(\sigma_1(\pi)\sigma_2(\pi)), \quad \sigma_1 \in \mathcal{S}'(\widehat{G}),\ \sigma_2 \in \mathcal{S}(\widehat{G}).$$

The Fourier transform can be then extended to the space of distributions $\mathcal{D}'(G)$. Thus, for $u \in \mathcal{D}'(G)$, we define $\mathcal{F}_G u \equiv \widehat{u} \in \mathcal{S}'(\widehat{G})$ by

$$\langle \mathcal{F}_G u, \tau \rangle_{\widehat{G}} := \left\langle u, \iota \circ \mathcal{F}_G^{-1}\tau \right\rangle_G, \quad \tau \in \mathcal{S}(\widehat{G}),$$

where $(\iota \circ \varphi)(x) = \varphi(x^{-1})$ and $\langle \cdot, \cdot \rangle_G$ is the distributional duality between $\mathcal{D}'(G)$ and $C^\infty(G)$. Analogously, its inverse is given by

$$\left\langle \mathcal{F}_G^{-1}\sigma, \varphi \right\rangle_G := \langle \sigma, \mathcal{F}_G(\iota \circ \varphi) \rangle_{\widehat{G}}, \quad \sigma \in \mathcal{S}'(\widehat{G}),\ \varphi \in C^\infty(G),$$

and these extended mappings are continuous between $\mathcal{D}'(G)$ and $\mathcal{S}'(\widehat{G})$ and are inverse to each other. It can be readily checked that they agree with their restrictions to spaces of test functions, explaining the appearance of the inversion mapping ι.

Gevrey spaces and ultradistributions

Recently, Gevrey spaces of ultradifferentiable functions as well as spaces of corresponding ultradistributions have been characterised as well. We say that a function $\phi \in C^\infty(G)$ is a *Gevrey-Roumieu ultradifferentiable function*, $\phi \in \gamma_s(G)$, if in every local coordinate chart, its local representative $\psi \in C^\infty(\mathbb{R}^n)$ belongs to $\gamma_s(\mathbb{R}^n)$, that is, satisfies the condition that there exist constants $A > 0$ and $C > 0$ such that

$$|\partial^\alpha \psi(x)| \leq CA^{|\alpha|}(\alpha!)^s$$

holds for all $x \in \mathbb{R}^n$ and all multi-indices α. For $s = 1$ we obtain the space of analytic functions on G. As with other spaces before, $\gamma_s(G)$ is thus defined as having its localisations in $\gamma_s(\mathbb{R}^n)$, and a question of its characterisation in terms of its Fourier coefficients arises.

Analogously, we say that ϕ is a *Gevrey-Beurling ultradifferentiable function*, $\phi \in \gamma_{(s)}(G)$, if its local representatives ψ satisfy the condition that for every $A > 0$ there exists $C_A > 0$ such that

$$|\partial^\alpha \psi(x)| \leq C_A A^{|\alpha|}(\alpha!)^s$$

holds for all $x \in \mathbb{R}^n$ and all multi-indices α. For $1 \leq s < \infty$, these spaces do not depend on the choice of local coordinates on G in the definition, and can be characterised as follows:

Proposition 2.1.2. *Let* $1 \leq s < \infty$.
(1) *We have* $\phi \in \gamma_s(G)$ *if and only if there exist* $B > 0$ *and* $K > 0$ *such that*

$$||\widehat{\phi}(\pi)||_{\mathtt{HS}} \leq Ke^{-B\langle\pi\rangle^{1/s}}$$

holds for all $\pi \in \widehat{G}$.
(2) *We have* $\phi \in \gamma_{(s)}(G)$ *if and only if for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$||\widehat{\phi}(\pi)||_{\mathtt{HS}} \leq K_B e^{-B\langle\pi\rangle^{1/s}}$$

holds for all $\pi \in \widehat{G}$.

The space of continuous linear functionals on $\gamma_s(G)$ $\left(\text{or } \gamma_{(s)}(G)\right)$ is called the space of *ultradistributions* and is denoted by $\gamma'_s(G)$ $\left(\text{or } \gamma'_{(s)}(G)\right)$, respectively.

For any $v \in \gamma'_s(G)$ $\left(\text{or } \gamma'_{(s)}(G)\right)$, we note that its Fourier coefficient at $\pi \in \widehat{G}$ can be defined analogously to the case of distributions by

$$\widehat{v}(\pi) := \langle v, \pi^* \rangle \equiv v(\pi^*).$$

These are well-defined since G is compact and hence $\pi(x)$ are analytic.

Proposition 2.1.3. *Let* $1 \leq s < \infty$.
(1) *We have* $v \in \gamma_s'(G)$ *if and only if for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$\|\widehat{v}(\pi)\|_{\mathtt{HS}} \leq K_B e^{B\langle\pi\rangle^{1/s}}$$

holds for all $\pi \in \widehat{G}$.
(2) *We have* $v \in \gamma_{(s)}'(G)$ *if and only if there exist* $B > 0$ *and* $K > 0$ *such that*

$$\|\widehat{v}(\pi)\|_{\mathtt{HS}} \leq K e^{B\langle\pi\rangle^{1/s}}$$

holds for all $\pi \in \widehat{G}$.

Proposition 2.1.2 can be actually extended to hold for any $0 < s < \infty$, and we refer to [DR14a] for proofs and further details. This can be viewed also from the point of view of general eigenfunction expansions of function of compact manifolds, see [DR16] for the treatment of more general Komatsu-type classes of ultradifferentiable functions and ultradistributions, building on an analogous description for analytic functions by Seeley [See69].

For a review of the representation theory of compact Lie groups and further constructions using the Littlewood-Paley decomposition based on the heat kernel we refer to Stein's book [Ste70b].

2.1.4 ℓ^p-spaces on the unitary dual $\widehat{G}$

For a general theory of non-commutative integration on locally compact unimodular groups we refer to Dixmier [Dix53] and Segal [Seg50, Seg53]. In this framework, the Hausdorff-Young inequality has been established (see Kunze [Kun58]) for a version of ℓ^p-spaces on the unitary dual $\widehat{G}$ based on the Schatten classes, namely, an inequality of the type

$$\left(\sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|^{p'}_{S^{p'}_{d_\pi}}\right)^{1/p'} \leq \|f\|_{L^p(G)} \text{ for } 1 < p \leq 2,$$

with an obvious modification for $p = 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and $S^{p'}_{d_\pi}$ is the $(d_\pi \times d_\pi)$-dimensional Schatten p'-class. While the theory of the above spaces is well-known (see e.g. Hewitt and Ross [HR70, Section 31] or Edwards [Edw72, Section 2.14]), here we describe and develop a little further another class of ℓ^p-spaces on $\widehat{G}$ which was considered in [RT10a, Section 10.3.3], to which we refer for details and proofs of statement that we do not prove here.

For $1 \leq p < \infty$, we define the space $\ell^p(\widehat{G}) \subset \Sigma$ by the condition

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi\in\widehat{G}} d_\pi^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \|\sigma(\pi)\|^p_{\mathtt{HS}}\right)^{1/p} < \infty.$$

For $p = \infty$, we define the space $\ell^\infty(\widehat{G}) \subset \Sigma$ by

$$\|\sigma\|_{\ell^\infty(\widehat{G})} := \sup_{\pi\in\widehat{G}} d_\pi^{-1/2} \|\sigma(\pi)\|_{\mathtt{HS}} < \infty.$$

For $p = 2$ we recover the space $\ell^2(\widehat{G})$ defined in (2.11), while the $\ell^1(\widehat{G})$-norm becomes

$$\|\sigma\|_{\ell^1(\widehat{G})} := \sum_{\pi\in\widehat{G}} d_\pi^{3/2} \|\sigma(\pi)\|_{\mathtt{HS}}.$$

This space and the Hausdorff-Young inequality for it become useful in, for example, proving Proposition 2.1.2. Also, it appears naturally in questions concerning the convergence of the Fourier series:

Remark 2.1.4. If $\sigma \in \ell^1(\widehat{G})$, then the (Fourier) series (2.16) converges absolutely and uniformly on G.

On the other hand, one can show that if $f \in C^k(G)$ with an even $k > \frac{1}{2}\dim G$, then $\widehat{f} \in \ell^1(\widehat{G})$ and the Fourier series (2.5) converges uniformly. Indeed, we can estimate

$$\begin{aligned}
\|\widehat{f}\|_{\ell^1(\widehat{G})} &= \sum_{\pi\in\widehat{G}} \frac{d_\pi^{3/2}}{\langle\pi\rangle^k} \|\pi((\mathrm{I}-\mathcal{L}_G)^{k/2} f)\|_{\mathtt{HS}} \\
&\le \left(\sum_{\pi\in\widehat{G}} d_\xi^2 \langle\pi\rangle^{-2k}\right)^{1/2} \left(\sum_{\pi\in\widehat{G}} d_\pi \|\pi((\mathrm{I}-\mathcal{L}_G)^{k/2} f)\|_{\mathtt{HS}}^2\right)^{1/2} \\
&\le C\|(\mathrm{I}-\mathcal{L}_G)^{k/2} f\|_{L^2(G)} < \infty,
\end{aligned}$$

in view of the Plancherel formula and (2.15), provided that $2k > \dim G$. In fact, the same argument shows the implication

$$f \in H^s(G), \quad s > \frac{1}{2}\dim G \qquad \Longrightarrow \qquad \widehat{f} \in \ell^1(\widehat{G}),$$

with the uniform convergence of the Fourier series (2.5) of f.

Regarding these $\ell^p(\widehat{G})$-spaces as weighted sequence spaces with weights given by powers of d_π, a general theory of interpolation spaces [BL76, Theorem 5.5.1] implies that they are interpolation spaces, namely, for any $1 \le p_0, p_1 < \infty$, we have

$$\left(\ell^{p_0}(\widehat{G}), \ell^{p_1}(\widehat{G})\right)_{\theta,p} = \ell^p(\widehat{G}),$$

where $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, see [RT10a, Proposition 10.3.40].

The Hausdorff-Young inequality holds for these spaces as well. Namely, if $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\|\widehat{f}\|_{\ell^{p'}(\widehat{G})} \leq \|f\|_{L^p(G)} \tag{2.17}$$

for all $f \in L^p(G)$, and

$$\|\mathcal{F}_G^{-1}\sigma\|_{L^{p'}(G)} \leq \|\sigma\|_{\ell^p(\widehat{G})}, \tag{2.18}$$

for all $\sigma \in \ell^p(\widehat{G})$.

We give a brief argument for these. To prove (2.18), on one hand we already have Plancherel's identity (2.13). On the other hand, from (2.16) we have

$$|(\mathcal{F}_G^{-1}\sigma)(x)| \leq \sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|_{\mathtt{HS}}\|\pi(x)\|_{\mathtt{HS}} = \sum_{\pi\in\widehat{G}} d_\pi^{3/2}\|\sigma(\pi)\|_{\mathtt{HS}} = \|\sigma\|_{\ell^1(\widehat{G})}.$$

Now the Stein-Weiss interpolation (see e.g. [BL76, Corollary 5.5.4]) implies (2.18). From this, (2.17) follows using the duality $\ell^p(\widehat{G})' = \ell^{p'}(\widehat{G})$, $1 \leq p < \infty$.

We remark that it is also possible to prove (2.17) directly by interpolation as well. However, one needs to employ an ℓ^∞-version of the interpolation theory with the change of measure, as e.g. in Lizorkin [Liz75].

Let us point out the continuous embeddings, similar to the usual ones:

Proposition 2.1.5. *We have*

$$\ell^p(\widehat{G}) \hookrightarrow \ell^q(\widehat{G}) \quad \textit{and} \quad \|\sigma\|_{\ell^q(\widehat{G})} \leq \|\sigma\|_{\ell^p(\widehat{G})} \qquad \forall\sigma \in \Sigma, \quad 1 \leq p \leq q \leq \infty.$$

Proof. We can assume $p < q$. Then, in the case $1 \leq p < \infty$ and $q = \infty$, we can estimate

$$\|\sigma\|^p_{\ell^\infty(\widehat{G})} = \left(\sup_{\pi\in\widehat{G}} d_\pi^{-\frac{1}{2}}\|\sigma(\pi)\|_{\mathtt{HS}}\right)^p \leq \sum_{\pi\in\widehat{G}} d_\pi^{2-\frac{p}{2}}\|\sigma(\pi)\|^p_{\mathtt{HS}} = \|\sigma\|^p_{\ell^p(\widehat{G})}.$$

Let now $1 \leq p < q < \infty$. Denoting $a_\pi := d_\pi^{\frac{2}{q}-\frac{1}{2}}\|\sigma(\pi)\|_{\mathtt{HS}}$, we get

$$\begin{aligned}\|\sigma\|_{\ell^q(\widehat{G})} &= \left(\sum_{\pi\in\widehat{G}} a_\pi^q\right)^{\frac{1}{q}} \leq \left(\sum_{\pi\in\widehat{G}} a_\pi^p\right)^{\frac{1}{p}} = \left(\sum_{\pi\in\widehat{G}} d_\pi^{p(\frac{2}{q}-\frac{1}{2})}\|\sigma(\pi)\|^p_{\mathtt{HS}}\right)^{\frac{1}{p}} \\ &\leq \|\sigma\|_{\ell^p(\widehat{G})},\end{aligned}$$

completing the proof. □

Finally, we establish a relation between the family $\ell^p(\widehat{G})$ and the corresponding Schatten family of ℓ^p-spaces, which we denote by $\ell^p_{sch}(\widehat{G})$, defined by the norms

$$\|\sigma\|_{\ell^p_{sch}(\widehat{G})} := \left(\sum_{\pi\in\widehat{G}} d_\pi\|\sigma(\pi)\|^p_{S^p}\right)^{1/p}, \quad \sigma \in \Sigma,\ 1 \leq p < \infty,$$

where $S^p = S^p_{d_\pi}$ is the $(d_\pi \times d_\pi)$-dimensional Schatten p-class, and

$$\|\sigma\|_{\ell^\infty_{sch}(\widehat{G})} := \sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}, \quad \sigma \in \Sigma.$$

We have the following relations:

Proposition 2.1.6. *For $1 \le p \le 2$, we have continuous embeddings as well as the estimates*

$$\ell^p(\widehat{G}) \hookrightarrow \ell^p_{sch}(\widehat{G}) \quad \text{and} \quad \|\sigma\|_{\ell^p_{sch}(\widehat{G})} \le \|\sigma\|_{\ell^p(\widehat{G})} \qquad \forall \sigma\in\Sigma, \quad 1\le p\le 2.$$

For $2 \le p \le \infty$, we have

$$\ell^p_{sch}(\widehat{G}) \hookrightarrow \ell^p(\widehat{G}) \quad \text{and} \quad \|\sigma\|_{\ell^p(\widehat{G})} \le \|\sigma\|_{\ell^p_{sch}(\widehat{G})} \qquad \forall \sigma\in\Sigma, \quad 2\le p\le \infty.$$

Proof. For $p = 2$, the norms coincide since $S^2 = \mathtt{HS}$. Let first $1 \le p < 2$. Since $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$, denoting by s_j its singular numbers, by the Hölder inequality we have

$$\|\sigma(\pi)\|^p_{S^p} = \sum_{j=1}^{d_\pi} s_j^p \le \left(\sum_{j=1}^{d_\pi} 1\right)^{\frac{2-p}{2}} \left(\sum_{j=1}^{d_\pi} s_j^{p\frac{2}{p}}\right)^{\frac{p}{2}} = d_\pi^{\frac{2-p}{2}} \|\sigma(\pi)\|^p_{\mathtt{HS}},$$

i.e.

$$\|\sigma(\pi)\|_{S^p} \le d_\pi^{\frac{2-p}{2p}} \|\sigma(\pi)\|_{\mathtt{HS}} \qquad (1\le p\le 2).$$

Consequently, it follows that

$$\|\sigma\|^p_{\ell^p_{sch}(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|^p_{S^p} \le \sum_{\pi\in\widehat{G}} d_\pi^{2-\frac{p}{2}} \|\sigma(\pi)\|^p_{\mathtt{HS}} = \|\sigma\|^p_{\ell^p(\widehat{G})},$$

proving the first claim. Conversely, for $2 < p < \infty$, we can estimate

$$\|\sigma(\pi)\|^2_{\mathtt{HS}} = \sum_{j=1}^{d_\pi} s_j^2 \le \left(\sum_{j=1}^{d_\pi} 1\right)^{\frac{p-2}{p}} \left(\sum_{j=1}^{d_\pi} s_j^{2\frac{p}{2}}\right)^{\frac{2}{p}} = d_\pi^{\frac{p-2}{p}} \|\sigma(\pi)\|^2_{S^p},$$

implying

$$\|\sigma(\pi)\|_{\mathtt{HS}} \le d_\pi^{\frac{p-2}{2p}} \|\sigma(\pi)\|_{S^p} \qquad (2<p<\infty).$$

It follows that

$$\|\sigma\|^p_{\ell^p(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi^{2-\frac{p}{2}} \|\sigma(\pi)\|^p_{\mathtt{HS}} \le \sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|^p_{S^p} = \|\sigma\|^p_{\ell^p_{sch}(\widehat{G})},$$

proving the second claim for $2 < p < \infty$. Finally, for $p = \infty$, the inequality

$$\|\sigma(\pi)\|_{\mathtt{HS}} \le d_\pi^{1/2} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}$$

implies

$$\|\sigma\|_{\ell^\infty(\widehat{G})} = \sup_{\pi\in\widehat{G}} d_\pi^{-1/2}\|\sigma(\pi)\|_{\mathtt{HS}} \leq \sup_{\pi\in\widehat{G}} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} = \|\sigma\|_{\ell^\infty_{sch}(\widehat{G})},$$

completing the proof. □

2.2 Pseudo-differential operators on compact Lie groups

In this section we look at linear continuous operators $A : C^\infty(G) \to \mathcal{D}'(G)$ and a global quantization of A yielding its full matrix-valued symbol. By the Schwartz kernel theorem (Theorem 1.4.1) there exists a unique distribution $K_A \in \mathcal{D}'(G\times G)$ such that

$$Af(x) = \int_G K_A(x,y)f(y)dy,$$

interpreted in the distributional sense. We can rewrite this as a right-convolution kernel operator

$$Af(x) = \int_G R_A(x, y^{-1}x)f(y)dy,$$

with

$$R_A(x,y) = K_A(x, xy^{-1}),$$

so that

$$Af(x) = (f * R_A(x,\cdot))(x).$$

2.2.1 Symbols and quantization

The idea for the following construction is that we define the symbol of A as the Fourier transform of its right convolution kernel in the second variable. However, for the presentation purposes we now take a different route and, instead, we define the mapping $\sigma_A : G \times \widehat{G} \to \Sigma$ by

$$\sigma_A(x,\pi) := \pi(x)^*(A\pi)(x), \tag{2.19}$$

with $(A\pi)(x) \in \mathscr{L}(\mathcal{H}_\pi)$ defined by

$$(A\pi(x)u, v)_{\mathcal{H}_\pi} := A(\pi(x)u, v)_{\mathcal{H}_\pi}$$

for all $u, v \in \mathcal{H}_\pi$. After choosing a basis in the representation space $\mathcal{H}_\pi$, we can interpret this as a matrix $\sigma_A(x,\pi) \in \mathbb{C}^{d_\pi\times d_\pi}$ and $(A\pi)_{ij} = A(\pi_{ij})$, i.e. the operator A acts on the matrix $\pi(x)$ componentwise, so that

$$\sigma_A(x,\pi)_{ij} = \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(x)} A\pi_{kj}(x).$$

We note that the symbol in (2.19) is well-defined since we can multiply the distribution $A\pi$ by a smooth (even analytic) matrix π.

Remark 2.2.1. We also observe that strictly speaking, the definition (2.19) depends on the choice of the representation π from its equivalence class $[\pi]$. Namely, if $\pi_1 \sim \pi_2$, so that

$$\pi_2(x) = U^{-1}\pi_1(x)U$$

for some unitary U and all $x \in G$, then

$$\widehat{f}(\pi_2) = U^{-1}\widehat{f}(\pi_1)U$$

and, therefore,

$$\sigma_A(x, \pi_2) = U^{-1}\sigma_A(x, \pi_1)U. \tag{2.20}$$

However, it can be readily checked that the quantization formula (2.22) below remains unchanged due to the presence of the trace. So, denoting by $\operatorname{Rep} G$ the set of all strongly continuous unitary irreducible representations of G, the symbol is well defined as a mapping

$$\sigma_A : G \times \operatorname{Rep} G \to \Sigma \qquad \text{or as} \qquad \sigma_A : G \times \widehat{G} \to \Sigma/\sim$$

where the equivalence on Σ is given by the equivalence of representations on $\operatorname{Rep} G$ inducing the equivalence on Σ by conjugations, as in formula (2.20). We will disregard this technicality in the current presentation to simplify the exposition, referring to [RT10a] for a more rigorous treatment. We note, however, that if $\pi_1 \sim \pi_2$, then

$$\operatorname{Tr}\left(\pi_1(x)\sigma_A(x, \pi_1)\widehat{f}(\pi_1)\right) = \operatorname{Tr}\left(\pi_2(x)\sigma_A(x, \pi_2)\widehat{f}(\pi_2)\right). \tag{2.21}$$

Using the symbol σ_A, it follows that the linear continuous operator $A : C^\infty(G) \to \mathcal{D}'(G)$ can be (de-)quantized as

$$Af(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x)\sigma_A(x, \pi)\widehat{f}(\pi)\right). \tag{2.22}$$

If the operator A maps $C^\infty(G)$ to itself and $f \in C^\infty(G)$, the formula (2.22) can be understood in the pointwise sense to hold for all $x \in G$, with the absolute convergence of the series. It can be shown that formulae (2.19) and (2.22) imply that σ_A is the Fourier transform of R_A, namely, we have

$$\sigma_A(x, \pi) = \int_G R_A(x, y)\pi(y)^* dy.$$

If the formula (2.22) holds, we will also write $A = \operatorname{Op}(\sigma_A)$.

In view of (2.21), the sum in (2.22) does not depend on the choice of a representation π from its equivalence class $[\pi]$.

Example 2.2.2. For the identity operator I we have its symbol

$$\sigma_{\mathrm{I}}(x,\pi) = \pi(x)^*\pi(x) = \mathrm{I}_{d_\pi}$$

is the identity matrix in $\mathbb{C}^{d_\pi \times d_\pi}$, by the unitarity of $\pi(x)$, so that (2.22) recovers the Fourier inversion formula (2.5) in this case. For the Laplacian $\mathcal{L}_G$ on G, we have

$$\sigma_{\mathcal{L}_G}(x,\pi) = \pi(x)^*\mathcal{L}_G\pi(x) = -\lambda_\pi \mathrm{I}_{d_\pi}$$

by the unitarity of π and (2.9), where $-\lambda_\pi$ are the eigenvalues of $\mathcal{L}_G$ corresponding to π. Consequently, we also have

$$\sigma_{(\mathrm{I}-\mathcal{L}_G)^{\mu/2}}(x,\pi) = \langle\pi\rangle^\mu \mathrm{I}_{d_\pi}.$$

Example 2.2.3. In the case of the torus $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and the representations $\{\pi_\xi\}_{\xi\in\mathbb{Z}^n}$ fixed as in Remark 1.1.4, we see that all $d_{\pi_\xi} = 1$. Hence

$$\sigma_A(x,\pi_\xi) \equiv \sigma_A(x,\xi) = e^{-2\pi i x\cdot\xi} A(e^{2\pi i x\cdot\xi}) \in \mathbb{C}, \quad (x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n,$$

with the quantization (2.22) becoming the *toroidal quantization*

$$Af(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\, \sigma_A(x,\xi)\, \widehat{f}(\xi),$$

for a thorough analysis of which we refer to [RT10b] and [RT10a, Section 4].

Example 2.2.4. With our choices of definitions, the symbols of left-invariant operators on G become independent of x. As shown in Section 1.5, if

$$Af = f * \kappa$$

for some $\kappa \in L^1(G)$, then it is left-invariant. Consequently, the right convolution kernel of A is $R_A(x,y) = \kappa(y)$ and, therefore, its Fourier transform is

$$\sigma_A(x,\pi) = \widehat{\kappa}(\pi).$$

On the other hand, if

$$Af = \kappa * f$$

for some $\kappa \in L^1(G)$, then it is right-invariant. In this case its right convolution kernel is $R_A(x,y) = \kappa(xyx^{-1})$ and, therefore, its Fourier transform in y gives

$$\sigma_A(x,\pi) = \pi(x)^*\widehat{\kappa}(\pi)\pi(x).$$

The notion of the symbol σ_A becomes already useful in stating a criterion for the L^2-boundedness for an operator A. We recall from Section 1.3 that X^α denotes the left-invariant partial differential operators of order $|\alpha|$ corresponding to a basis of left-invariant vector fields $X_1, \cdots, X_n$, $n = \dim G$, of the Lie algebra

$\mathfrak{g}$ of G. As the derivatives with respect to these vector fields in general do not commute, in principle we have to take into account their order in forming partial differential operators of higher degrees. However, we note that the subsequent statements remain valid if we restrict our choice to

$$X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

We will sometimes write X_x^α to emphasise that the derivatives are taken with respect to the variable x.

Theorem 2.2.5. *Let G be a compact Lie group and let $A : C^\infty(G) \to C^\infty(G)$ be a linear continuous operator. Let k be an integer such that $k > \frac{1}{2}\dim G$. Assume that there is a constant $C > 0$ such that*

$$\|X_x^\alpha \sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C$$

for all $(x,\pi) \in G \times \widehat{G}$, and all $|\alpha| \leq k$. Then A extends to a bounded operator from $L^2(G)$ to $L^2(G)$.

In this theorem and elsewhere, $\|\cdot\|_{\mathscr{L}(\mathcal{H}_\pi)}$ denotes the operator norm of $\sigma_A(x,\pi) \in \mathscr{L}(\mathcal{H}_\pi)$ or, after a choice of the basis, the operator norm of the matrix multiplication by the matrix $\sigma_A(x,\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$. The appearance of the operator norm is natural since for the convolution operators we have

$$\|f \mapsto f * h\|_{\mathscr{L}(L^2(G))} = \|f \mapsto h * f\|_{\mathscr{L}(L^2(G))} = \sup_{\pi \in \widehat{G}} \|\widehat{h}(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)}, \tag{2.23}$$

following from $\widehat{f * h}(\pi) = \widehat{h}(\pi)\widehat{f}(\pi)$ and Plancherel's theorem.

2.2.2 Difference operators and symbol classes

In order to describe the symbolic properties and to establish the symbolic calculus of operators we have to replace the derivatives in frequency, used in the symbolic calculus on $\mathbb{R}^n$, by suitable operations acting on the space Σ of Fourier coefficients. We call these operations *difference operators*. Roughly speaking, this corresponds to the idea that in the Calderón-Zygmund theory, the integral kernel K_A has singularities at the diagonal or, in other words, the right-convolution kernel $R_A(x,\cdot)$ has singularity at the unit element e of the group only. Therefore, if we form an operator with a new integral kernel $q(\cdot)R_A(x,\cdot)$ with a smooth $q \in C^\infty(G)$ satisfying $q(e) = 0$, the properties of this new operator should be better than those of the original operator A.

In [RT10a], the corresponding notion of difference operators has been introduced leading to the symbolic calculus of operators on G. However, we now follow the ideas of [RTW14] with a slightly more general treatment of difference operators.

Definition 2.2.6. Let $q \in C^\infty(G)$ vanish of order $k \in \mathbb{N}$ at the unit element $e \in G$, i.e. $(Dq)(e) = 0$ for all left-invariant differential operators $D \in \mathrm{Diff}^{k-1}(G)$ of order $k-1$. Then the *difference operator of order* k is an operator acting on the space Σ of Fourier coefficients by the formula

$$(\Delta_q \widehat{f})(\pi) := \widehat{qf}(\pi).$$

We denote the set of all difference operators of order k by $\mathrm{diff}^k(\widehat{G})$.

We now define families of first order difference operators replacing derivatives in the frequency variable in the Euclidean setting.

Definition 2.2.7. A collection of ℓ first order difference operators $\Delta_{q_1}, \ldots, \Delta_{q_\ell} \in \mathrm{diff}^1(\widehat{G})$ is called *admissible*, if the corresponding functions $q_1, \ldots, q_\ell \in C^\infty(G)$ satisfy

$$q_j(e) = 0, \;\; \mathrm{d}q_j(e) \neq 0, \;\; j = 1, \ldots, \ell,$$

and, moreover,

$$\mathrm{rank}(\mathrm{d}q_1(e), \ldots, \mathrm{d}q_\ell(e)) = \dim G.$$

It follows, in particular, that e is an isolated common zero of the family $\{q_j\}_{j=1}^{\ell}$. We call an admissible collection *strongly admissible*, if it is the only common zero, i.e. if

$$\bigcap_{j=1}^{\ell} \{x \in G : q_j(x) = 0\} = \{e\}.$$

We note that difference operators all commute with each other. For a given admissible collection of difference operators we use the multi-index notation

$$\Delta_\pi^\alpha := \Delta_{q_1}^{\alpha_1} \cdots \Delta_{q_\ell}^{\alpha_\ell} \;\; \text{and} \;\; q^\alpha(x) := q_1(x)^{\alpha_1} \cdots q_\ell(x)^{\alpha_\ell},$$

the dimension of the multi-index $\alpha \in \mathbb{N}_0^\ell$ depending on the number ℓ of difference operators in the collection. Consequently, there exist corresponding differential operators $X^{(\alpha)} \in \mathrm{Diff}^{|\alpha|}(G)$ such that the Taylor expansion formula

$$f(x) = \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} q^\alpha(x^{-1}) \, X^{(\alpha)} f(e) + \mathcal{O}(h(x)^N), \qquad h(x) \to 0, \tag{2.24}$$

holds true for any smooth function $f \in C^\infty(G)$ and any N, with $h(x)$ the geodesic distance from x to the identity element e. An explicit construction of operators $X^{(\alpha)}$ in terms of $q^\alpha(x)$ can be found in [RT10a, Section 10.6]. Operators X^α and $X^{(\alpha)}$ can be expressed in terms of each other.

Example 2.2.8. In the case of the torus, $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, let

$$q_j(x) = e^{-2\pi i x_j} - 1, \;\; j = 1, \ldots, n.$$

The collection $\{q_j\}_{j=1}^n$ is strongly admissible, and the corresponding difference operators take the form

$$(\Delta_{q_j}\sigma)(\pi_\xi) \equiv (\Delta_{q_j}\sigma)(\xi) = \sigma(\xi + e_j) - \sigma(\xi), \quad j = 1, \ldots, n,$$

with $\pi_\xi \in \widehat{\mathbb{T}^n}$ identified with $\xi \in \mathbb{Z}^n$, where e_j is the j^{th} unit vector in $\mathbb{Z}^n$. The periodic Taylor expansion takes the following form (see [RT10a, Theorem 3.4.4]): for any $\phi \in C^\infty(\mathbb{T}^n)$ we have

$$\phi(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!}(e^{2\pi i x} - 1)^\alpha X_z^{(\alpha)} \phi(z)|_{z=0} + \sum_{|\alpha|=N} \phi_\alpha(x)(e^{2\pi i x} - 1)^\alpha,$$

where $\phi_\alpha \in C^\infty(\mathbb{T}^n)$ and

$$(e^{2\pi i x} - 1)^\alpha := (e^{2\pi i x_1} - 1)^{\alpha_1} \cdots (e^{2\pi i x_n} - 1)^{\alpha_n}.$$

The operators $X_z^{(\alpha)}$ have the form

$$X_z^{(\alpha)} = X_{z_1}^{(\alpha_1)} \cdots X_{z_n}^{(\alpha_n)} \quad \text{with} \quad X_{z_k}^{(\alpha_k)} = \prod_{j=0}^{\alpha_k - 1} \left(\frac{1}{2\pi i} \frac{\partial}{\partial z_k} - j \right).$$

Example 2.2.9. For partial differential operators, it can be readily observed that the application of difference operators reduces the order of symbols. Thus, let

$$D = \sum_{|\alpha| \le N} c_\alpha(x) X_x^\alpha, \quad c_\alpha \in C^\infty(G).$$

Then it was shown in [RT10a, Proposition 10.7.4] that

$$\Delta_q \sigma_D(x,\pi) = \sum_{|\alpha| \le N} c_\alpha(x) \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} (X_x^\beta q)(e) \sigma_{X_x^{\alpha-\beta}}(x,\pi).$$

In particular, if q has zero of order M at $e \in G$ then $\mathrm{Op}(\Delta_q \sigma_D)$ is of order $N - M$.

Remark 2.2.10. We can estimate differences in terms of original symbols: assume that the symbol $\sigma \in \Sigma$ satisfies

$$\mu := \sup_\pi \langle \pi \rangle^{-m} \|\sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty$$

for some $m \in \mathbb{R}$. Then for any difference operator Δ_q defined in terms of a function $q \in C^\infty(G)$ we have the estimate

$$\|\Delta_q \sigma(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C\mu \|q\|_{C^{\varkappa + \lceil |m| \rceil}(G)} \langle \pi \rangle^m$$

with a constant C independent of σ and q, where $\varkappa = \lceil (\dim G)/2 \rceil$ is the smallest integer larger than half the dimension of G and $\lceil |m| \rceil$ is the smallest integer larger than $|m|$. We refer to [RW14, Lemma 7.1] for the proof. However, if q vanishes at the unit element e to some order, we can impose a much better behaviour.

The usual Hörmander classes $\Psi^m(G)$ of pseudo-differential operators on G viewed as a manifold can be characterised in terms of the matrix-valued symbols. Here we recall that $A \in \Psi^m(G)$ means that in every local coordinate chart $U \subset G$, the pullback of $A|_U$ to $\mathbb{R}^n$ is a pseudo-differential operator $A_U \in \Psi^m_{1,0}(\mathbb{R}^n)$, i.e. it can be written as

$$A_U f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)\widehat{f}(\xi)d\xi \quad \text{with} \quad \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx, \tag{2.25}$$

with symbol $a = a_U \in S^m_{1,0}(\mathbb{R}^n)$, i.e. satisfying

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

for all multi-indices α, β, and all $x, \xi \in \mathbb{R}^n$.

The following characterisation was partly proved in [RT10a, RT13] (namely (A)$\Longleftrightarrow$(C)) and completed in [RTW14] (namely (B)$\Longleftrightarrow$(C)$\Longleftrightarrow$(D)) .

Theorem 2.2.11. *Let G be a compact Lie group of dimension n. Let A be a linear continuous operator from $C^\infty(G)$ to $\mathcal{D}'(G)$. Then the following statements are equivalent:*

(A) $A \in \Psi^m(G)$.

(B) For every left-invariant differential operator $D \in \operatorname{Diff}^k(G)$ of order k and every difference operator $\Delta_q \in \operatorname{diff}^l(\widehat{G})$ of order l the symbol estimate

$$\|\Delta_q D\sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_{qD}\langle\pi\rangle^{m-l}$$

is valid.

(C) For an admissible collection $\Delta_1, \ldots, \Delta_\ell \in \operatorname{diff}^1(\widehat{G})$ we have

$$\|\Delta_\pi^\alpha X_x^\beta \sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_{\alpha\beta}\langle\pi\rangle^{m-|\alpha|}$$

for all multi-indices $\alpha \in \mathbb{N}_0^\ell$ and $\beta \in \mathbb{N}_0^n$. Moreover,

$$\operatorname{sing\,supp} R_A(x,\cdot) \subseteq \{e\}.$$

(D) For a strongly admissible collection $\Delta_1, \ldots, \Delta_\ell \in \operatorname{diff}^1(\widehat{G})$ we have

$$\|\Delta_\pi^\alpha X_x^\beta \sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_{\alpha\beta}\langle\pi\rangle^{m-|\alpha|}$$

for all multi-indices $\alpha \in \mathbb{N}_0^\ell$ and $\beta \in \mathbb{N}_0^n$.

Motivated by Theorem 2.2.11, (D), we may define symbol classes $S^m_{\rho,\delta}(G)$. Fixing a strongly admissible collection of difference operators

$$\Delta_1, \ldots, \Delta_\ell \in \operatorname{diff}^1(\widehat{G}),$$

we say that $\sigma_A \in S^m_{\rho,\delta}(G)$ if $\sigma_A(x,\cdot) \in \Sigma$ satisfies

$$\|\Delta_\pi^\alpha X_x^\beta \sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_{\alpha\beta}\langle\pi\rangle^{m-\rho|\alpha|+\delta|\beta|} \tag{2.26}$$

for all $(x,\pi) \in G \times \widehat{G}$ and for all multi-indices $\alpha \in \mathbb{N}_0^\ell$ and $\beta \in \mathbb{N}_0^n$. If $\rho > \delta$, this definition is independent of the choice of a strongly admissible collection of difference operators. The equivalence (A)$\Longleftrightarrow$(D) in Theorem 2.2.11 can be rephrased as

$$A \in \Psi^m(G) \Longleftrightarrow \sigma_A \in S^m_{1,0}(G).$$

For any $0 \le \delta < \rho \le 1$, the equivalence (B)$\Longleftrightarrow$(C)$\Longleftrightarrow$(D) in Theorem 2.2.11 remains valid for the symbol class $S^m_{\rho,\delta}(G)$ if we replace the symbolic conditions there by the condition (2.26). As we shall see later, the class $S^m_{\rho,\delta}(G)$ with different values of ρ and δ becomes useful in a number of applications.

Theorem 2.2.5 has analogue for (ρ,δ) classes:

Theorem 2.2.12. *Let $0 \le \delta < \rho \le 1$ and let A be an operator with symbol in $S^m_{\rho,\delta}(G)$. Then A is a bounded from $H^s(G)$ to $H^{s-m}(G)$ for any $s \in \mathbb{R}$.*

See [RW14, Theorem 5.1] for the proof.

2.2.3 Symbolic calculus, ellipticity, hypoellipticity

We now give elements of the symbolic calculus on the compact Lie group G. Here, we fix some strongly admissible collection of difference operators, with corresponding operators $X_x^{(\alpha)}$ coming from the Taylor expansion formula (2.24). We refer to [RT10a, Section 10.7.3] for proofs and other variants of the calculus below. We start with the composition.

Theorem 2.2.13. *Let $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta < \rho$. Let $A, B : C^\infty(G) \to C^\infty(G)$ be linear continuous operators with symbols $\sigma_A \in S^{m_1}_{\rho,\delta}(G)$ and $\sigma_B \in S^{m_1}_{\rho,\delta}(G)$. Then $\sigma_{AB} \in S^{m_1+m_2}_{\rho,\delta}(G)$ and we have*

$$\sigma_{AB} \sim \sum_{\alpha\ge 0} \frac{1}{\alpha!}(\Delta_\pi^\alpha \sigma_A)(X^{(\alpha)}\sigma_B),$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\sigma_{AB}(x,\pi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}(\Delta_\pi^\alpha \sigma_A)(x,\pi) X_x^{(\alpha)} \sigma_B(x,\pi) \in S^{m_1+m_2-(\rho-\delta)N}_{\rho,\delta}(G).$$

The composition formula together with Theorem 2.2.5 imply a criterion for the boundedness in L^2-Sobolev spaces.

Corollary 2.2.14. *Let G be a compact Lie group and let $A : C^\infty(G) \to C^\infty(G)$ be a linear continuous operator. Let $m \in \mathbb{R}$. Assume that the symbol σ_A satisfies*

$$\|X_x^\alpha \sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C_\alpha \langle \pi \rangle^m$$

for all $(x,\pi) \in G \times \widehat{G}$, and all multi-indices α. Then A extends to a bounded operator from $H^s(G)$ to $H^{s-m}(G)$, for all $s \in \mathbb{R}$.

Let us now present a construction of amplitude operators in our setting. Let $0 \leq \rho, \delta \leq 1$. We say that $a : G \times G \times \widehat{G} \to \Sigma$ is a *matrix-valued amplitude* in the class $\mathcal{A}^m_{\rho,\delta}(G)$ if for a strongly admissible collection of difference operators on $\widehat{G}$ we have the amplitude inequalities

$$\|\Delta_\pi^\alpha X_x^\beta X_y^\gamma a(x,y,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C_{\alpha\beta\gamma} \langle \pi \rangle^{m-\rho|\alpha|+\delta|\beta+\gamma|},$$

for all multi-indices α, β, γ and for all $(x,y,\pi) \in G \times G \times \widehat{G}$. The corresponding *amplitude operator* $\mathrm{Op}(a) : C^\infty(G) \to \mathcal{D}'(G)$ is defined by

$$\mathrm{Op}(a)f(x) := \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}\left(\pi(x) \int_G a(x,y,\eta) f(y) \pi(y)^* dy\right). \tag{2.27}$$

In the case $a(x,y,\pi) = \sigma_A(x,\pi)$ independent of y, we recover the quantization (2.22), namely, we have $\mathrm{Op}(a) = A$.

Theorem 2.2.15. *Let $a \in \mathcal{A}^m_{\rho,\delta}(G)$. If $0 \leq \delta < 1$ and $0 \leq \rho \leq 1$ then $\mathrm{Op}(a)$ is a continuous linear operator from $C^\infty(G)$ to $C^\infty(G)$. Moreover, if $0 \leq \delta < \rho \leq 1$, then $A = \mathrm{Op}(a)$ is a pseudo-differential operator with a matrix-valued symbol $\sigma_A \in S^m_{\rho,\delta}(G)$, which has the asymptotic expansion*

$$\sigma_A(x,\pi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_\pi^\alpha X_y^{(\alpha)} a(x,y,\pi)|_{y=x},$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\sigma_A(x,\pi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \Delta_\pi^\alpha X_y^{(\alpha)} a(x,y,\pi)|_{y=x} \in S^{m-(\rho-\delta)N}_{\rho,\delta}(G).$$

For the proof of this theorem we refer to [RT11]. Given the formula for the amplitude operators in Theorem 2.2.15, the symbol of the adjoint operator can be found as follows.

Theorem 2.2.16. *Let $m \in \mathbb{R}$ and $0 \leq \delta < \rho$. Let $A : C^\infty(G) \to C^\infty(G)$ be a linear continuous operator with symbol $\sigma_A \in S^m_{\rho,\delta}(G)$. Then the symbol σ_{A^*} of the adjoint operator A^* satisfies $\sigma_{A^*} \in S^m_{\rho,\delta}(G)$, and is given by*

$$\sigma_{A^*}(x,\pi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_\pi^\alpha X_x^{(\alpha)} \sigma_A(x,\pi)^*,$$

where $\sigma_A(x,\pi)^*$ is the adjoint matrix to $\sigma_A(x,\pi)$, and the asymptotic expansion means that for every $N \in \mathbb{N}$ we have

$$\sigma_{A^*}(x,\pi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}\Delta_\pi^\alpha X_x^{(\alpha)} \sigma_A(x,\pi)^* \in S_{\rho,\delta}^{m-(\rho-\delta)N}(G).$$

We recall that the operator $A \in \Psi^m(G)$ on G viewed as a manifold is elliptic if all of its localisations to coordinate charts are (locally) elliptic. This can be characterised in terms of the matrix-valued symbols. A combination of [RTW14, Theorem 4.1] and [RT10a, Theorem 10.9.10] yields

Theorem 2.2.17. *An operator $A \in \Psi^m(G)$ is elliptic if and only if its symbol $\sigma_A(x,\pi)$ is invertible for all but finitely many $\pi \in \widehat{G}$, and for all such π satisfies*

$$\|\sigma_A(x,\pi)^{-1}\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C\langle\pi\rangle^{-m}$$

for all $x \in G$. Furthermore, in this case, assume that

$$\sigma_A \sim \sum_{j=0}^{\infty} \sigma_{A_j}, \quad A_j \in \Psi^{m-j}(G).$$

Let $\sigma_B \sim \sum_{k=0}^{\infty} \sigma_{B_k}$, where

$$\sigma_{B_0}(x,\pi) = \sigma_{A_0}(x,\pi)^{-1}$$

for large $\langle\pi\rangle$, and the symbols σ_{B_k} are defined recursively by

$$\sigma_{B_N} = -\sigma_{B_0} \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{|\gamma|=N-j-k} \frac{1}{\gamma!} (\Delta_\pi^\gamma \sigma_{B_k})(X_x^{(\gamma)} \sigma_{A_j}).$$

Then $\mathrm{Op}(\sigma_{B_k}) \in \Psi^{-m-k}(G)$, $B = \mathrm{Op}(\sigma_B) \in \Psi^{-m}(G)$, and the operators $AB - I$ and $BA - I$ are in $\Psi^{-\infty}(G)$.

One can also provide a criterion for the hypoellipticity in terms of matrix-valued symbols ([RTW14]), in analogy to the one on $\mathbb{R}^n$ given by Hörmander ([Hör67b]).

Theorem 2.2.18. *Let $m \ge m_0$ and $0 \le \delta < \rho \le 1$. Let $A \in \mathrm{Op}(S_{\rho,\delta}^m(G))$ be a pseudo-differential operator with symbol $\sigma_A \in S_{\rho,\delta}^m(G)$ which is invertible for all but finitely many $\pi \in \widehat{G}$, and for all such π satisfies*

$$\|\sigma_A(x,\pi)^{-1}\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C\langle\pi\rangle^{-m_0}$$

for all $x \in G$. Assume also that (for a strongly admissible collection of difference operators) we have

$$\|\sigma_A(x,\pi)^{-1}\left[\Delta_\pi^\alpha X_x^\beta \sigma_A(x,\pi)\right]\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C\langle\pi\rangle^{-\rho|\alpha|+\delta|\beta|}$$

for all multi-indices α, β, all $x \in G$, and all but finitely many $\pi \in \widehat{G}$. Then there exists an operator $B \in \mathrm{Op}(S^{-m_0}_{\rho,\delta}(G))$ such that $AB - I$ and $BA - I$ belong to $\Psi^{-\infty}(G)$. Consequently, we have

$$\textit{sing supp } Au = \textit{sing supp } u$$

for all $u \in \mathcal{D}'(G)$.

We finish this section with several results that are usually expected from the calculus. The following asymptotic expansion formula was established in [RW14].

Proposition 2.2.19. *Let $\sigma_j \in S^{m_j}_{\rho,\delta}(G)$, $j \in \mathbb{N}_0$, $0 \leq \delta < \rho \leq 1$, be a family of symbols with $m_j \searrow -\infty$. Then there exists a symbol $\sigma \in S^{m_0}_{\rho,\delta}(G)$ such that*

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in S^{m_N}_{\rho,\delta}(G)$$

for all $N \in \mathbb{N}_0$.

The functional calculus of matrix valued symbols and its operator counterpart have been also developed in [RW14]. A notable corollary of such functional calculus is the following

Corollary 2.2.20. *Let $0 \leq \delta < \rho \leq 1$ and let $m \geq 0$. Assume $\sigma_A \in S^{2m}_{\rho,\delta}(G)$ satisfies $\sigma_A(x,\pi) > 0$ and*

$$\|\sigma_A(x,\pi)^{-1}\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C\langle\pi\rangle^{-2m}$$

for all x and π. Then the square root

$$\sigma_B(x,\pi) = \sqrt{\sigma_A(x,\pi)}$$

in the sense of positive matrices is a symbol satisfying $\sigma_B \in S^m_{\rho,\delta}(G)$.

This is the corollary of the following more general result:

Theorem 2.2.21. *Let $0 \leq \delta \leq 1$ and $0 < \rho \leq 1$. Assume $\sigma_A \in S^m_{\rho,\delta}(G)$, $m \geq 0$, is positive definite, invertible, and satisfies*

$$\|\sigma_A(x,\pi)^{-1}\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C\langle\pi\rangle^{-m}$$

for all x and for all but finitely many π. Then for any number $s \in \mathbb{C}$,

$$\sigma_B(x,\pi) := \sigma_A(x,\pi)^s = \exp(s \log \sigma_A(x,\pi))$$

defines a symbol $\sigma_B \in S^{m'}_{\rho,\delta}(G)$, with $m' = \mathrm{Re}\,(ms)$.

In fact, the assumptions of Theorem 2.2.21 imply something stronger, namely, that the symbol $\sigma_A(x,\pi)$ is parameter-elliptic with respect to $\mathbb{R}_-$; we refer to [RW14] for the definition of parameter-ellipticity in this setting, and for a more general exposition and statements of the functional calculus on compact Lie groups.

2.2.4 Fourier multipliers and L^p-boundedness

Here we give an overview of the L^p-estimates for the Fourier multipliers and for non-invariant operators on compact Lie groups following [RW13, RW15]. We set aside the case of bi-invariant operators (or spectral multipliers) noting that there exist many results in this direction (see e.g. N. Weiss [Wei72], Coifman and G. Weiss [CW74], Stein [Ste70b], Cowling [Cow83], Alexopoulos [Ale94], to refer the reader to only a few). Instead, we concentrate on the case of left-invariant operators (or Fourier multipliers). To the best of our knowledge the literature in this case is much smaller, with a notable exception of a multiplier theorem for left-invariant operators on the group SU(2) treated by Coifman and Weiss [CW71b], Coifman and de Guzmán [CdG71], and appearing in more detail in the monograph by Coifman and Weiss [CW71a]. The conditions there are formulated using specific explicit expressions involving Clebsch-Gordan coefficients on SU(2), but they can be recast in a much shorter form using the concept of difference operators. It also allows one to treat the case of general compact Lie groups. Finally we note that there exist also results for the spectral multipliers in the sub-Laplacian, also on SU(2), for which we refer to Cowling and Sikora [CS01].

First, we discuss left-invariant operators $A : C^\infty(G) \to \mathcal{D}'(G)$, so that the matrix-valued symbol $\sigma_A(x,\pi) = \sigma_A(\pi)$ is independent of x and can be given as

$$\sigma_A(\pi) = \pi(x)^*(A\pi)(x) = (A\pi)(e).$$

The multiplier theorems that we will present can be said to be of *Mihlin-Hörmander type* in the sense that they provide analogues of famous multiplier theorems on $\mathbb{R}^n$ by Mihlin [Mih56, Mih57] and Hörmander [Hör60].

In order to formulate the results, we need to fix a particular collection of first order difference operators associated to the elements of the unitary dual $\widehat{G}$. Thus, for a fixed representation $\pi_0 \in \widehat{G}$, we notice that the $(d_{\pi_0} \times d_{\pi_0})$-matrix $\pi_0(x) - \mathrm{I}_{d_{\pi_0}}$ vanishes at $x = e$. Consequently, we define the difference operators ${}_{\pi_0}\mathbb{D} = ({}_{\pi_0}\mathbb{D}_{ij})_{i,j=1}^{d_{\pi_0}}$ associated with its elements,

$${}_{\pi_0}\mathbb{D}_{ij} := \Delta_{(\pi_0)_{ij} - \delta_{i,j}},$$

where $\delta_{i,j}$ is the Kronecker delta. For a family of difference operators of this type,

$$\mathbb{D}_1 =_{\pi_1} \mathbb{D}_{i_1 j_1},\ \mathbb{D}_2 =_{\pi_2} \mathbb{D}_{i_2 j_2},\ \ldots, \mathbb{D}_m =_{\pi_m} \mathbb{D}_{i_m j_m}, \tag{2.28}$$

with $\pi_k \in \widehat{G}$, $1 \le i_k, j_k \le d_{\pi_k}$, $1 \le k \le m$, we define

$$\mathbb{D}^\alpha := \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_m^{\alpha_m}. \tag{2.29}$$

The described difference operators ${}_{\pi_0}\mathbb{D}$ have a number of useful properties. For example, they satisfy the *finite Leibniz formula* (while general difference operators satisfy only an asymptotic Leibniz formula, see [RT10a, Section 10.7.4]). Namely, for any fixed π_0, they satisfy

$$\mathbb{D}_{ij}(\sigma\tau) = (\mathbb{D}_{ij}\sigma)\tau + \sigma(\mathbb{D}_{ij}\tau) + \sum_{k=1}^{d_{\pi_0}} (\mathbb{D}_{ik}\sigma)(\mathbb{D}_{kj}\tau). \tag{2.30}$$

The collection of difference operators

$$\{{}_{\pi_0}\mathbb{D}_{ij} : \pi_0 \in \widehat{G}, 1 \le i,j \le d_{\pi_0}\}$$

is strongly admissible. Moreover, it has a finite strongly admissible sub-collection. Indeed, a homomorphic embedding of G into $\mathcal{U}(N)$ for some N is itself a representation of G. Decomposing it into irreducible components gives the desired finite family of π_0's.

We now formulate the first result on the L^p-boundedness of left-invariant operators.

Theorem 2.2.22. *Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a left-invariant linear continuos operator on a compact Lie group G, and let k denote the smallest even integer such that $k > \frac{1}{2}\dim G$. Assume that the symbol σ_A of A satisfies*

$$\|\mathbb{D}^\alpha \sigma_A(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_\alpha \langle \pi \rangle^{-|\alpha|} \tag{2.31}$$

for all multi-indices $|\alpha| \le k$ and all $\pi \in \widehat{G}$. Then the operator A is of weak type (1,1) and is bounded on $L^p(G)$ for all $1 < p < \infty$.

We note that by Theorem 2.2.11, imposing conditions (2.31) for all multi-indices α would imply that A is a left-invariant pseudo-differential operator in Hörmander's class, $A \in \Psi^0(G)$, for which the L^p-boundedness would follow from the corresponding L^p-boundedness in $\mathbb{R}^n$ for its localisations. However, imposing conditions (2.31) for multi-indices $|\alpha| \le k$ still assures that the operator A is of Calderón-Zygmund type (in the sense of Coifman and Weiss, see Section A.4). The proof of the L^p-boundedness for $1 < p \le 2$ follows by Marcinkiewicz interpolation theorem (see Proposition 1.5.1) from the L^2-boundedness (and hence also weak (2,2) type) in Theorem 2.2.5, and from weak (1,1) type, which becomes, therefore, the main task.

For $2 < p < \infty$, the result follows by duality. Before we give an idea behind the proof of the weak (1,1) type, let us formulate several corollaries from Theorem 2.2.22. We recall that the Sobolev space $W^{p,s}(G)$ on G is the usual Sobolev space on G as a manifold defined by requiring all the localisations to belong to the Euclidean space $W^{p,s}(\mathbb{R}^n) = (\mathrm{I} - \mathcal{L}_{\mathbb{R}^n})^{-s/2} L^p(\mathbb{R}^n)$, where $\mathcal{L}_{\mathbb{R}^n}$ is the Laplacian on $\mathbb{R}^n$ and $s \in \mathbb{R}$.

Corollary 2.2.23. *Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a left-invariant linear continuous operator on a compact Lie group G. Let $0 \le \rho \le 1$ and let k denote the smallest* *even integer such that* $k > \frac{1}{2}\dim G$. *Assume that the symbol* σ_A *of* A *satisfies*

$$\|\mathbb{D}^\alpha \sigma_A(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_\alpha \langle\pi\rangle^{-\rho|\alpha|}$$

for all multi-indices $|\alpha| \le k$ *and all* $\pi \in \widehat{G}$. *Then the operator* A *extends to a bounded operator from the Sobolev space* $W^{p,r}(G)$ *to* $L^p(G)$ *for any* $1 < p < \infty$, *with*

$$r = k(1-\rho)|\frac{1}{p} - \frac{1}{2}|.$$

Example 2.2.24. Let

$$\mathcal{L}_{sub} = X^2 + Y^2$$

be a sub-Laplacian on SU(2). Then it was shown in [RTW14] that it has a parametrix with the matrix-valued symbol in the class $S^{-1}_{\frac{1}{2},0}(\mathrm{SU}(2))$. Consequently, for any $1 < p < \infty$, Corollary 2.2.23 implies the subelliptic estimate

$$\|f\|_{W^{p,s+1-|\frac{1}{p}-\frac{1}{2}|}(\mathrm{SU}(2))} \le C_p \|\mathcal{L}_{sub} f\|_{W^{p,s}(\mathrm{SU}(2))},$$

where the estimate is extended from $s = 0$ to any $s \in \mathbb{R}$ by the calculus. We refer to [RTW14] for the construction and discussion of parametrices for other operators, including the heat and the wave operator, d'Alambertian, and some higher order operators, on SU(2) and on $\mathbb{S}^3$, and to [RW13, RW15] for the corresponding L^p-estimates.

Example 2.2.25. Let (ϕ, θ, ψ) be the standard Euler angles on SU(2), see e.g. [RT10a, Chapter 11] for a detailed treatment of SU(2). Thus, we have $0 \le \phi < 2\pi$, $0 \le \theta \le \pi$, and $-2\pi \le \psi < 2\pi$, and every element

$$u = u(\phi,\theta,\psi) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$$

is parametrised in such a way that

$$2a\bar{a} = 1 + \cos\theta,\ 2ab = ie^{i\phi}\sin\theta,\ -2a\bar{b} = ie^{i\psi}\sin\theta.$$

Conversely, we can also write

$$u(\phi,\theta,\psi) = \begin{pmatrix} \cos(\frac{\theta}{2})e^{i(\phi+\psi)/2} & i\sin(\frac{\theta}{2})e^{i(\phi-\psi)/2} \\ i\sin(\frac{\theta}{2})e^{-i(\phi-\psi)/2} & \cos(\frac{\theta}{2})e^{-i(\phi+\psi)/2} \end{pmatrix} \in \mathrm{SU}(2).$$

Let X be a left-invariant vector field on G normalised in such a way that $\|X\| = \|\partial/\partial\psi\|$ with respect to the Killing form. It was shown in [RTW14] that for $\gamma \in \mathbb{C}$,

$$\text{the operator } X + \gamma \text{ is invertible if and only if } i\gamma \notin \frac{1}{2}\mathbb{Z},$$

and, moreover, for such γ, the inverse $(X+\gamma)^{-1}$ has its matrix-valued symbol in the class $S^0_{0,0}(\mathrm{SU}(2))$. The same conclusion remains true if we replace SU(2) by $\mathbb{S}^3$, with the corresponding selection of Euler's angles. Then, Corollary 2.2.23 and the calculus imply the subelliptic estimate

$$\|f\|_{W^{p,s}(\mathbb{S}^3)} \le C_p \|(X+\gamma)f\|_{W^{p,s+2|\frac{1}{p}-\frac{1}{2}|}(\mathbb{S}^3)}, \quad 1<p<\infty,\ s\in\mathbb{R}.$$

There is an analogue of this estimate on arbitrary compact Lie groups, see [RW15].

Let us briefly indicate an idea behind the proof of Theorem 2.2.22. In order to use the theory of singular integral operators (according to Coifman and Weiss, see Section A.4), we first define a suitable quasi-distance on G.

Let $\mathrm{Ad}: G \to \mathcal{U}(\mathfrak{g})$ be the adjoint representation of G. Then by the Peter-Weyl theorem it can be decomposed as a direct sum of irreducible representations,

$$\mathrm{Ad} = (\dim Z(G))1 \oplus \bigoplus_{\pi\in\Theta_0} \pi,$$

where $Z(G)$ is the centre of G, 1 is the trivial representation, and Θ_0 is an index set for the representations entering in this decomposition. Then we define a smooth non-negative function

$$r^2(x) := \dim G - \mathrm{Tr}\,\mathrm{Ad}(x) = \sum_{\pi\in\Theta_0} (d_\pi - \chi_\pi(x)), \tag{2.32}$$

which is central, non-degenerate, and vanishes of the second order at the unit element $e\in G$. It can be then checked that the function

$$d(x,y) := r(x^{-1}y)$$

is the quasi-distance in the sense of Section A.4. Consequently, one can check that the operator A satisfies Calderón-Zygmund conditions of spaces of homogeneous type, in terms of the quasi-distance above. Such a verification relies heavily on the developed symbolic calculus, Leibniz rules for difference operators, and criteria for the weak (1,1) type in terms of suitably defined mollifiers. We refer to [RW15] for further details of this construction.

Using the function $r(x)$, one can refine the statement of Theorem 2.2.22. Thus, let us define the difference operator associated with $r^2(x)$, namely,

$$\mathbin{\triangle\!\!\!\triangle} := \Delta_{r^2} = \mathcal{F}_G\, r^2(x)\, \mathcal{F}_G^{-1},$$

and we have that $\mathbin{\triangle\!\!\!\triangle} \in \mathrm{diff}^2(\widehat{G})$ is the second order difference operator.

Theorem 2.2.26. *Let $A: C^\infty(G)\to\mathcal{D}'(G)$ be a left-invariant linear continuous operator on a compact Lie group G, and let k denote the smallest even integer such that $k>\frac{1}{2}\dim G$. Assume that the symbol σ_A of A satisfies*

$$\|\mathbin{\triangle\!\!\!\triangle}^{k/2}\sigma_A(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C\langle\pi\rangle^{-k} \tag{2.33}$$

as well as

$$\|\mathbb{D}^\alpha \sigma_A(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C_\alpha \langle \pi \rangle^{-|\alpha|} \tag{2.34}$$

for all multi-indices $|\alpha| \leq k-1$ *and all* $\pi \in \widehat{G}$. *Then the operator* A *is of weak type (1,1) and is bounded on* $L^p(G)$ *for all* $1 < p < \infty$.

We note that, comparing (2.33) to the condition (2.31) in Theorem 2.2.22, only a single difference condition of order k is required in Theorem 2.2.26. This has interesting consequences, already in the case of the torus, as we will show in Example 2.2.27.

Moreover, the assumption (2.34) can be refined further: namely, to form a strongly admissible family of first order difference operators giving $\mathbb{D}^\alpha$ in (2.28) and (2.29), it is enough to take only $\pi_k \in \Theta_0$, the set of the irreducible components of the adjoint representation.

In all the theorems of this section an assumption that k is an even integer is present. This seems to be related to the technical part of the argument, namely, to the usage of the second order difference operator $\triangle\!\!\!\!\triangle$ that is naturally related to the quasi-metric on G as well as satisfies the finite Leibniz formula. The latter can be derived from (2.30) using the decomposition

$$\triangle\!\!\!\!\triangle = -\sum_{\pi \in \Theta_0} \sum_{i=1}^{d_\pi} {}_\pi\mathbb{D}_{ii},$$

which follows from the definition of $r^2(x)$ in (2.32). Thus, it satisfies

$$\triangle\!\!\!\!\triangle(\sigma\tau) = (\triangle\!\!\!\!\triangle\sigma)\tau + \sigma(\triangle\!\!\!\!\triangle\tau) - \sum_{\pi \in \Theta_0} \sum_{i,j=1}^{d_\pi} ({}_\pi\mathbb{D}_{ij}\sigma)({}_\pi\mathbb{D}_{ji}\tau),$$

and becomes instrumental in establishing the relation between assumption (2.33) and properties of the integral kernel of A in terms of the quasi-metric. However, we note also that the condition on the even number of analogous expressions appears already in the multiplier theorem for bi-invariant operators, established by rather different methods by N. Weiss [Wei72].

Example 2.2.27. Let us consider now the case of the torus, $G = \mathbb{T}^n$. In this case, the left-invariant operators take the form

$$Af(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i x \cdot \xi} \sigma(\xi) \widehat{f}(\xi) \quad \text{with} \quad \widehat{f}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

or, in other words,

$$\widehat{Af}(\xi) = \sigma(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{Z}^n.$$

We take

$$r^2(x) = 2n - \sum_{j=1}^{n} (e^{2\pi i x_j} + e^{-2\pi i x_j}),$$

so that

$$\mathbb{A}\sigma(\xi) = 2n\sigma(\xi) - \sum_{j=1}^{n} (\sigma(\xi + e_j) + \sigma(\xi - e_j)),$$

where $\xi \in \mathbb{Z}^n$ and e_j is the j^{th} unit vector in $\mathbb{Z}^n$. The appearing operator $\mathbb{A}$ is rather curious since it replaces the assumptions usually imposed on all highest order difference conditions as, for example, in the suitably modified toroidal version of Hörmander's multiplier theorem [Hör60] (where one would need to make assumptions on all differences of order $\left[\frac{n}{2}\right] + 1$), or in Marcienkiewicz' version of multiplier theorem of Nikolskii [Nik77, Section 1.5.3] (where one imposes difference conditions up to order n). To clarify the nature of the operator $\mathbb{A}$, we give the examples for $\mathbb{T}^2$ and $\mathbb{T}^3$. As a consequence of Theorem 2.2.26 we get the following statements. Let $1 < p < \infty$. Assume that

$$|\sigma(\xi)| \le C \text{ and } |\xi| \; |\sigma(\xi + e_j) - \sigma(\xi)| \le C,$$

for all $\xi \in \mathbb{Z}^2$ and $j = 1, 2$, or $\xi \in \mathbb{Z}^3$ and $j = 1, 2, 3$, respectively. Furthermore, assume that

$$|\xi|^2 \; |\sigma(\xi) - \frac{1}{4}\sum_{j=1}^{2} (\sigma(\xi + e_j) + \sigma(\xi - e_j))| \le C \quad \text{for } \mathbb{T}^2,$$

or

$$|\xi|^2 |\sigma(\xi) - \frac{1}{6}\sum_{j=1}^{3} (\sigma(\xi + e_j) + \sigma(\xi - e_j))| \le C \quad \text{for } \mathbb{T}^3,$$

respectively. Then the operator A is bounded on $L^p(\mathbb{T}^2)$ or $L^p(\mathbb{T}^3)$, respectively.

We now drop the assumption of left-invariance and consider general linear continuous operators from $C^\infty(G)$ to $\mathcal{D}'(G)$. Then we can assure the L^p-boundedness provided we complement the differences in π with derivatives with respect to x.

Theorem 2.2.28. *Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a linear continuous operator on a compact Lie group G, and let k denote the smallest even integer such that $k > \frac{1}{2}\dim G$. Let $1 < p < \infty$ and let $l > \frac{\dim G}{p}$ be an integer. Assume that the symbol σ_A of A satisfies*

$$\|X_x^\beta \mathbb{A}^{k/2} \sigma_A(x, \pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C \langle \pi \rangle^{-k} \tag{2.35}$$

as well as

$$\|X_x^\beta \mathbb{D}^\alpha \sigma_A(x, \pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \le C_\alpha \langle \pi \rangle^{-|\alpha|} \tag{2.36}$$

for all $\pi \in \widehat{G}$ and for all multi-indices α and β with $|\alpha| \le k - 1$ and $|\beta| \le l$. Then the operator A is bounded on $L^p(G)$.

We refer to [RW15] for the detailed proofs of all the results in this section.

2.2.5 Sharp Gårding inequality

The sharp Gårding inequality on $\mathbb{R}^n$ is an important lower bound for operators with positive symbols, finding many applications in the theory of partial differential equations of elliptic, parabolic and hyperbolic types. The original Gårding inequality for elliptic operators has been established by Gårding in [Går53]. It says that if $p \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$, is a symbol satisfying

$$\operatorname{Re} p(x,\xi) \geq c|\xi|^m,$$

$c > 0$, for all $x \in \mathbb{R}^n$ and ξ large enough, then the corresponding pseudo-differential operator

$$p(x,D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} p(x,\xi)\widehat{f}(\xi)d\xi$$

satisfies the following lower bound: for every $s \in \mathbb{R}$ and every compact set $K \subset \mathbb{R}^n$ there exist some constants c_0, c_1 such that

$$\operatorname{Re}\,(p(x,D)f,f)_{L^2(\mathbb{R}^n)} \geq c_0\|f\|^2_{H^{m/2}(\mathbb{R}^n)} - c_1\|f\|^2_{H^s(\mathbb{R}^n)} \tag{2.37}$$

holds for all $f \in \mathcal{D}(K)$. Its improvement, the so-called *sharp Gårding inequality* was obtained by Hörmander in [Hör66]. It says that if $p \in S^m_{\rho,\delta}(\mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$, is a non-negative symbol, $p(x,\xi) \geq 0$ for all $x, \xi \in \mathbb{R}^n$, then the corresponding pseudo-differential operator satisfies the lower bound

$$\operatorname{Re}\,(p(x,D)f,f)_{L^2(\mathbb{R}^n)} \geq -c\|f\|^2_{H^{(m-(\rho-\delta))/2}(\mathbb{R}^n)} \tag{2.38}$$

for all $f \in \mathcal{D}(K)$. This inequality was further generalised to systems by Lax and Nirenberg [LN66], Kumano-go [Kg81], and Vaillancourt [Vai70]. It has been also extended to regain two derivatives for the class $S^2_{1,0}(\mathbb{R}^n)$ by Fefferman and Phong [FP78]. For expositions concerning sharp Gårding inequalities with different proofs we refer to monographs of Kumano-go [Kg81], Taylor [Tay81], Lerner [Ler10], or Friedrichs' notes [Fri70]. There is also an approach based on constructions in space variables rather than in frequency one, developed by Nagase [Nag77].

The situation with Gårding inequalities on manifolds is more complicated. The main problem is that the assumption that the symbol of a pseudo-differential operator is non-negative is harder to formulate since the full symbol is not invariantly defined. For second order pseudo-differential operators, under the non-negativity assumption on the principal symbol and certain non-degeneracy assumptions on the sub-principal symbol, a lower bound now known as Melin-Hörmander inequality has been obtained by Melin [Mel71] and Hörmander [Hör77]. The non-degeneracy conditions on the sub-principal symbol can be somehow relaxed, see [MPP07].

Nevertheless, in our setting we are assisted by the fact that the algebraic structure of a Lie group gives us the notion of the full symbol in (2.19). This symbol, however, is not needed for the standard Gårding inequality (2.37) since the ellipticity is determined by the principal symbol only. Thus, the standard Gårding inequality (2.37) on compact Lie groups has been established in [BGJR89] using Langlands' results for semi-groups on Lie groups [Lan60].

Let us first look at a possible assumption for the positivity of an operator in the invariant situation. If an operator A is given by the convolution $Af = \kappa * f$, we obtain

$$(Af, f)_{L^2(G)} = (\kappa * f, f)_{L^2(G)} = (\widehat{f}\,\widehat{\kappa}, \widehat{f})_{\ell^2(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\widehat{f}(\pi)\ \widehat{\kappa}(\pi)\ \widehat{f}(\pi)^*\right),$$

where we used the Plancherel identity (2.13). On the other hand, according to Section 1.5, A is right-invariant, and according to Example 2.2.4 its symbol is $\sigma_A(x,\pi) = \pi(x)^*\widehat{\kappa}(\xi)\pi(x)$. Thus, we get that A is a positive operator if and only if the matrix $\widehat{\kappa}(\pi)$ is positive for all $\pi \in \widehat{G}$, i.e. when $(\widehat{\kappa}(\pi)v, v)_{\mathcal{H}_\pi} \geq 0$ for all $v \in \mathcal{H}_\pi$. But this means that the symbol σ_A is positive, $\sigma_A(x,\pi) \geq 0$ for all $(x,\pi) \in G \times \widehat{G}$. Analogously, for left-invariant operators $Af = f * \kappa$, one sees that

$$(Af, f)_{L^2(G)} = (f * \kappa, f)_{L^2(G)} = (\widehat{\kappa}\,\widehat{f}, \widehat{f})_{\ell^2(\widehat{G})} = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\widehat{f}(\pi)^*\ \widehat{\kappa}(\pi)\ \widehat{f}(\pi)\right).$$

So again, A is a positive operator if and only if its symbol $\sigma_A(\pi) = \widehat{\kappa}(\pi)$ is positive.

This motivates a hypothesis that the positivity of the matrix-valued symbol on G would be an analogue of the positivity of the Kohn-Nirenberg symbol on $\mathbb{R}^n$. Indeed, we have the following criterion, which for non-invariant operators becomes a sufficient condition:

Theorem 2.2.29. *Let $A \in \Psi^m(G)$ be such that its matrix-valued symbol σ_A is positive, i.e.*

$$\sigma_A(x,\pi) \geq 0 \ \ \textit{for all} \ \ (x,\pi) \in G \times \widehat{G}.$$

Then there exists a constant c such that

$$\operatorname{Re}(Af, f)_{L^2(G)} \geq -c\|f\|^2_{H^{(m-1)/2}(G)}$$

for all $f \in C^\infty(G)$.

The usual proofs of the sharp Gårding inequality on $\mathbb{R}^n$ (that is, the proofs not relying on the anti-Wick quantization) make use of a positive approximation of a pseudo-differential operator, the so-called Friedrichs symmetrisation, approximating an operator with non-negative symbol of order m by a positive operator modulo an error of order $m-1$. This construction, indeed, allows one to gain one derivative needed for the sharp Gårding inequality. Unfortunately, such an approximation in the frequency variable seems to be less useful on a Lie group G because the unitary dual $\widehat{G}$ is not well adapted for such purpose. However, one can carry out, instead, a symmetrisation in the space variables using the symbolic calculus of operators for the construction. In particular, it relies heavily on dealing with the symbol class $S^m_{1,\frac{1}{2}}(G)$ defined in Section 2.2.3.

As in the case of operators on $\mathbb{R}^n$, the sharp Gårding inequality leads to several further conclusions concerning the L^2-boundedness of operators. For example, pseudo-differential operators of the first order are bounded on $L^2(\mathbb{R}^n)$ provided their matrix-valued symbols are bounded:

Corollary 2.2.30. *Let $A \in \Psi^1(G)$ be such that its matrix-valued symbol σ_A satisfies*

$$\|\sigma_A(x,\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq C$$

for all $(x,\pi) \in G \times \widehat{G}$. Then the operator A is bounded from $L^2(G)$ to $L^2(G)$.

It can be also used to determine constants as bounds for operator norms of mappings between L^2-Sobolev spaces. For the proofs of the statements in this section, as well as for further details we refer the reader to [RT11].

In the above, we concentrated on symbol classes $S^m_{1,0}(G)$ of type $(1,0)$. However, certain conclusions can be made also for operators with symbols of type (ρ,δ).

Proposition 2.2.31 (Gårding's inequality on G)**.** *Let $0 \leq \delta < \rho \leq 1$ and $m > 0$. Let $A \in \operatorname{Op} S^{2m}_{\rho,\delta}(G)$ be elliptic and such that $\sigma_A(x,\xi) \geq 0$ for all x and co-finitely many ξ. Then there are constants $c_1, c_2 > 0$ such that for any function $f \in H^m(G)$ the inequality*

$$\operatorname{Re}(Af,f)_{L^2} \geq c_1\|f\|^2_{H^m} - c_2\|f\|^2_{L^2}$$

holds true.

The statement follows by the calculus from its special case $m = \rho - \delta$. We refer to [RW14, Corollary 6.2] for the proofs.