# Chapter 3

# Homogeneous Lie groups

By definition a homogeneous Lie group is a Lie group equipped with a family of dilations compatible with the group law. The abelian group $(\mathbb{R}^n, +)$ is the very first example of homogeneous Lie group. Homogeneous Lie groups have proved to be a natural setting to generalise many questions of Euclidean harmonic analysis. Indeed, having both the group and dilation structures allows one to introduce many notions coming from the Euclidean harmonic analysis. There are several important differences between the Euclidean setting and the one of homogeneous Lie groups. For instance the operators appearing in the latter setting are usually more singular than their Euclidean counterparts. However it is possible to adapt the technique in harmonic analysis to still treat many questions in this more abstract setting.

As explained in the introduction (see also Chapter 4), we will in fact study operators on a subclass of the homogeneous Lie group, more precisely on graded Lie groups. A graded Lie group is a Lie group whose Lie algebra admits a $(\mathbb{N})$-gradation. Graded Lie groups are homogeneous and in fact the relevant structure for the analysis of graded Lie groups is their natural homogeneous structure and this justifies presenting the general setting of homogeneous Lie groups. From the point of view of applications, the class of graded Lie groups contains many interesting examples, in fact all the ones given in the introduction. Indeed these groups appear naturally in the geometry of certain symmetric domains and in some subelliptic partial differential equations. Moreover, they serve as local models for contact manifolds and CR manifolds, or for more general Heisenberg manifolds, see the discussion in the Introduction.

The references for this chapter of the monograph are [FS82, ch. I] and [Goo76], as well as Fulvio Ricci's lecture notes [Ric]. However, our conventions and notation do not always follow the ones of these references. The treatment in this chapter is, overall, more general than that in the above literature since we also consider distributions and kernels of complex homogeneous degrees and adapt our analysis for subsequent applications to Sobolev spaces and to the op-

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erator quantization developed in the following chapters. Especially, our study of complex homogeneities allows us to deal with complex powers of operators (e.g. in Section 4.3.2).

# 3.1 Graded and homogeneous Lie groups

In this section we present the definition and the first properties of graded Lie groups. Since many of their properties can be explained in the more general setting of homogeneous Lie groups, we will also present these groups.

## 3.1.1 Definition and examples of graded Lie groups

We start with definitions and examples of graded and stratified Lie groups.

**Definition 3.1.1.** (i) A Lie algebra $\mathfrak{g}$ is *graded* when it is endowed with a vector space decomposition (where all but finitely many of the $V_j$'s are $\{0\}$):

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} V_j \quad \text{such that} \quad [V_i, V_j] \subset V_{i+j}.$$

(ii) A Lie group is *graded* when it is a connected simply connected Lie group whose Lie algebra is graded.

The condition that the group is connected and simply connected is technical but important to ensure that the exponential mapping is a global diffeomorphism between the group and its Lie algebra.

The classical examples of graded Lie groups and algebras are the following.

*Example* 3.1.2 (Abelian case). The abelian group $(\mathbb{R}^n, +)$ is graded: its Lie algebra $\mathbb{R}^n$ is trivially graded, i.e. $V_1 = \mathbb{R}^n$.

*Example* 3.1.3 (Heisenberg group). The Heisenberg group $\mathbb{H}_{n_o}$ given in Example 1.6.4 is graded: its Lie algebra $\mathfrak{h}_{n_o}$ can be decomposed as

$$\mathfrak{h}_{n_o} = V_1 \oplus V_2 \quad \text{where} \quad V_1 = \oplus_{i=1}^{n_o} \mathbb{R}X_i \oplus \mathbb{R}Y_i \quad \text{and} \quad V_2 = \mathbb{R}T.$$

(For the notation, see Example 1.6.4 in Section 1.6.)

*Example* 3.1.4 (Upper triangular matrices). The group $T_{n_o}$ of $n_o \times n_o$ matrices which are upper triangular with 1 on the diagonal is graded: its Lie algebra $\mathfrak{t}_{n_o}$ of $n_o \times n_o$ upper triangular matrices with 0 on the diagonal is graded by

$$\mathfrak{t}_{n_o} = V_1 \oplus \ldots \oplus V_{n_o - 1} \quad \text{where} \quad V_j = \oplus_{i=1}^{n_o - j} \mathbb{R}E_{i,i+j}.$$

(For the notation, see Example 1.6.5 in Section 1.6.) The vector space $V_j$ is formed by the matrices with only non-zero coefficients on the $j$-th upper off-diagonal.

As we will show in Proposition 3.1.10, a graded Lie algebra (hence possessing a natural dilation structure) must be nilpotent. The converse is not true, see Remark 3.1.6, Part 2.

Examples 3.1.2–3.1.4 are stratified in the following sense:

**Definition 3.1.5.** (i) A Lie algebra $\mathfrak{g}$ is *stratified* when $\mathfrak{g}$ is graded, $\mathfrak{g} = \oplus_{j=1}^{\infty} V_j$, and the first stratum $V_1$ generates $\mathfrak{g}$ as an algebra. This means that every element of $\mathfrak{g}$ can be written as a linear combination of iterated Lie brackets of various elements of $V_1$.

(ii) A Lie group is *stratified* when it is a connected simply connected Lie group whose Lie algebra is stratified.

*Remark* 3.1.6. Let us make the following comments on existence and uniqueness of gradations.

1. A gradation over a Lie algebra is not unique: the same Lie algebra may admit different gradations. For example, any vector space decomposition of $\mathbb{R}^n$ yields a graded structure on the group $(\mathbb{R}^n, +)$. More convincingly, we can decompose the 3 dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ as

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{3} V_j \quad \text{with} \quad V_1 = \mathbb{R}X_1,\ V_2 = \mathbb{R}Y_1,\ V_3 = \mathbb{R}T.$$

This last example can be easily generalised to find several gradations on the Heisenberg groups $\mathbb{H}_{n_o}$, $n_o = 2, 3, \ldots$, which are not the classical ones given in Example 3.1.3. Another example would be

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{8} V_j \quad \text{with} \quad V_3 = \mathbb{R}X_1,\ V_5 = \mathbb{R}Y_1,\ V_8 = \mathbb{R}T, \tag{3.1}$$

and all the other $V_j = \{0\}$.

2. A gradation may not even exist. The first obstruction is that the existence of a gradation implies nilpotency; in other words, a graded Lie group or a graded Lie algebra are nilpotent, as we shall see in the sequel (see Proposition 3.1.10). Even then, a gradation of a nilpotent Lie algebra may not exist. As a curiosity, let us mention that the (dimensionally) lowest nilpotent Lie algebra which is not graded is the seven dimensional Lie algebra given by the following commutator relations:

$$[X_1, X_j] = X_{j+1} \quad \text{for } j = 2, \ldots, 6, \qquad [X_2, X_3] = X_6,$$
$$[X_2, X_4] = [X_5, X_2] = [X_3, X_4] = X_7.$$

They define a seven dimensional nilpotent Lie algebra of step 6 (with basis $\{X_1, \ldots, X_7\}$). It is the (dimensionally) lowest nilpotent Lie algebra which is not graded. See, more generally, [Goo76, ch.I §3.2].

3. To go back to the problem of uniqueness, different gradations may lead to 'morally equivalent' decompositions. For instance, if a Lie algebra $\mathfrak{g}$ is graded by $\mathfrak{g} = \oplus_{j=1}^{\infty} V_j$ then it is also graded by $\mathfrak{g} = \oplus_{j=1}^{\infty} W_j$ where $W_{2j'+1} = \{0\}$ and $W_{2j'} = V_{j'}$. This last example motivates the presentation of homogeneous Lie groups: indeed graded Lie groups are homogeneous and the natural homogeneous structure for the graded Lie algebra

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j = \oplus_{j=1}^{\infty} W_j$$

is the same for the two gradations.

Moreover, the relevant structure for the analysis of graded Lie groups is their natural homogeneous structure.

4. There are plenty of graded Lie groups which are not stratified, simply because the first vector subspace of the gradation may not generate the whole Lie algebra (it may be $\{0\}$ for example). This can also be seen in terms of dilations defined in Section 3.1.2. Moreover, a direct product of two stratified Lie groups is graded but may be not stratified as their stratification structures may not 'match'. We refer to Remark 3.1.13 for further comments on this topic.

### 3.1.2 Definition and examples of homogeneous Lie groups

We now deal with a more general subclass of Lie groups, namely the class of homogeneous Lie groups.

**Definition 3.1.7.** (i) A family of *dilations* of a Lie algebra $\mathfrak{g}$ is a family of linear mappings

$$\{D_r,\, r>0\}$$

from $\mathfrak{g}$ to itself which satisfies:

- the mappings are of the form

$$D_r = \operatorname{Exp}(A \ln r) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\ln(r) A)^{\ell},$$

where $A$ is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, Exp denotes the exponential of matrices and $\ln(r)$ the natural logarithm of $r>0$,

- each $D_r$ is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects the Lie bracket:

$$\forall X, Y \in \mathfrak{g},\ r>0 \qquad [D_r X, D_r Y] = D_r[X,Y].$$

(ii) A *homogeneous* Lie group is a connected simply connected Lie group whose Lie algebra is equipped with dilations.

(iii) We call the eigenvalues of $A$ the *dilations' weights* or weights. The set of dilations' weights, or in other worlds, the set of eigenvalues of $A$ is denoted by $\mathcal{W}_A$.

We can realise the mappings $A$ and $D_r$ in a basis of $A$-eigenvectors as the diagonal matrices

$$A \equiv \begin{pmatrix} \upsilon_1 & & & \\ & \upsilon_2 & & \\ & & \ddots & \\ & & & \upsilon_n \end{pmatrix} \quad \text{and} \quad D_r \equiv \begin{pmatrix} r^{\upsilon_1} & & & \\ & r^{\upsilon_2} & & \\ & & \ddots & \\ & & & r^{\upsilon_n} \end{pmatrix}.$$

The dilations' weights are $\upsilon_1, \ldots, \upsilon_n$.

*Remark* 3.1.8. Note that if $\{D_r\}$ is a family of dilations of the Lie algebra $\mathfrak{g}$, then $\tilde{D}_r := D_{r^\alpha} := \mathrm{Exp}(\alpha A \ln r)$ defines a new family of dilations $\{\tilde{D}_r, \, r > 0\}$ for any $\alpha > 0$. By adjusting $\alpha$ if necessary, we may assume that the dilations' weights satisfy certain properties in order to compare different families of dilations and in order to fix one of such families. For example in [FS82], it is assumed that the minimum eigenvalue is 1.

Graded Lie algebras are naturally equipped with dilations: if the Lie algebra $\mathfrak{g}$ is graded by

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j,$$

then we define the dilations

$$D_r := \mathrm{Exp}(A \ln r)$$

where $A$ is the operator defined by $AX = jX$ for $X \in V_j$.

The converse is true:

**Lemma 3.1.9.** *If a Lie algebra $\mathfrak{g}$ has a family of dilations such that the weights are all rational, then $\mathfrak{g}$ has a natural gradation.*

*Proof.* By adjusting the weights (see Remark 3.1.8), we may assume that all the eigenvalues are positive integers. Then the decomposition in eigenspaces gives the the gradation of the Lie algebra. □

Before discussing the dilations in the examples given in Section 3.1.1 and other examples of homogeneous Lie groups, let us state the following crucial property.

**Proposition 3.1.10.** *The following holds:*

*(i) A Lie algebra equipped with a family of dilations is nilpotent.*

*(ii) A homogeneous Lie group is a nilpotent Lie group.*

*Proof of Proposition 3.1.10.* Let $\{D_r = \text{Exp}(A \ln r)\}$ be the family of dilations. By Remark 3.1.8, we may assume that the smallest weight is 1. For $\upsilon \in \mathcal{W}_A$ let $W_\upsilon \subset \mathfrak{g}$ be the corresponding eigenspace of $A$. If $\upsilon \in \mathbb{R}$ but $\upsilon \notin \mathcal{W}_A$ then we set $W_\upsilon := \{0\}$.

Thus $D_r X = r^\upsilon X$ for $X \in W_\upsilon$. Moreover, if $X \in W_\upsilon$ and $Y \in W_{\upsilon'}$ then

$$D_r[X,Y] = [D_r X, D_r Y] = r^{\upsilon+\upsilon'}[X,Y]$$

and hence

$$[W_\upsilon, W_{\upsilon'}] \subset W_{\upsilon+\upsilon'}.$$

In particular, since $\upsilon \geq 1$ for $\upsilon \in \mathcal{W}_A$, we see that the ideals in the lower series of $\mathfrak{g}$ (see (1.18)) satisfy

$$\mathfrak{g}_{(j)} \subset \oplus_{a \geq j} W_a.$$

Since the set $\mathcal{W}_A$ is finite, it follows that $\mathfrak{g}_{(j)} = \{0\}$ for $j$ sufficiently large. Consequently the Lie algebra $\mathfrak{g}$ and its corresponding Lie group $G$ are nilpotent. $\square$

Let $G$ be a homogeneous Lie group with Lie algebra $\mathfrak{g}$ endowed with dilations $\{D_r\}_{r>0}$. By Proposition 3.1.10, the connected simply connected Lie group $G$ is nilpotent. We can transport the dilations to the group using the exponential mapping $\exp_G = \exp$ of $G$ (see Proposition 1.6.6 (a)) in the following way: the maps

$$\exp_G \circ D_r \circ \exp_G^{-1}, \quad r > 0,$$

are automorphisms of the group $G$; we shall denote them also by $D_r$ and call them *dilations on* $G$. This explains why homogeneous Lie groups are often presented as Lie groups endowed with dilations.

We may write

$$rx := D_r(x) \quad \text{for } r > 0 \text{ and } x \in G.$$

The dilations on the group or on the Lie algebra satisfy

$$D_{rs} = D_r D_s, \quad r, s > 0.$$

As explained above, Examples 3.1.2, 3.1.3 and, 3.1.4 are naturally homogeneous Lie groups:

*In Example 3.1.2:* The abelian group $(\mathbb{R}^n, +)$ is homogeneous when equipped with the usual dilations $D_r x = rx$, $r > 0$, $x \in \mathbb{R}^n$.

*In Example 3.1.3:* The Heisenberg group $\mathbb{H}_{n_o}$ is homogeneous when equipped with the dilations

$$rh = (rx, ry, r^2 t), \quad h = (x, y, t) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_o} \times \mathbb{R}.$$

The corresponding dilations on the Heisenberg Lie algebra $\mathfrak{h}_{n_o}$ are given by

$$D_r(X_j) = rX_j, \; D_r(Y_j) = rY_j, \; j = 1, \ldots, n_o, \text{ and } D_r(T) = r^2 T.$$

*In Example 3.1.4:* The group $T_{n_o}$ is homogeneous when equipped with the dilations defined by

$$[D_r(M)]_{i,j} = r^{j-i}[M]_{i,j} \quad 1 \le i < j \le n_o,\ M \in T_{n_o}.$$

The corresponding dilations on the Lie algebra $\mathfrak{t}_{n_o}$ are given by

$$D_r(E_{i,j}) = r^{j-i}E_{i,j} \quad 1 \le i < j \le n_o.$$

As already seen for the graded Lie groups, the same homogeneous Lie group may admit various homogeneous structures, that is, a nilpotent Lie group or algebra may admit different families of dilations, even after renormalisation of the eigenvalues (see Remark 3.1.8). This can already be seen from the examples in the graded case (see Remark 3.1.6 part 1). These examples can be generalised as follows.

*Example* 3.1.11. On $\mathbb{R}^n$ we can define

$$D_r(x_1, \ldots, x_n) = (r^{\upsilon_1}x_1, \ldots, r^{\upsilon_n}x_n),$$

where $0 < \upsilon_1 \le \ldots \le \upsilon_n$, and on $\mathbb{H}_{n_o}$ we can define

$$D_r(x_1, \ldots, x_{n_o}, y_1, \ldots, y_{n_o}, t) = (r^{\upsilon_1}x_1, \ldots, r^{\upsilon_{n_o}}x_{n_o}, r^{\upsilon_1'}y_1, \ldots, r^{\upsilon_{n_o}'}y_{n_o}, r^{\upsilon''}t),$$

where $\upsilon_j > 0$, $\upsilon_j' > 0$ and $\upsilon_j + \upsilon_j' = \upsilon''$ for all $j = 1, \ldots, n_o$.

These families of dilations give graded structures whenever the weights $\upsilon_j$ for $\mathbb{R}^n$ and $\upsilon_j, \upsilon_j', \upsilon''$ for $\mathbb{H}_{n_o}$ are all rational or, more generally, all in $\alpha\mathbb{Q}^+$ for a fixed $\alpha \in \mathbb{R}_+$. From this remark it is not difficult to construct a homogeneous non-graded structure: on $\mathbb{R}^3$, consider the diagonal $3 \times 3$ matrix $A$ with entries, e.g., 1 and $\pi$ and $1 + \pi$.

*Example* 3.1.12. Continuing the example above, choosing the $\upsilon_j$ and $\upsilon_j'$'s rational in a certain way, it is also possible to find a homogeneous structure for $\mathbb{H}_{n_o}$ such that the corresponding gradation of $\mathfrak{h}_{n_o} = \oplus_{j=1}^{\infty} V_j$ does exist but is necessarily such that $V_1 = \{0\}$: we choose $\upsilon_j, \upsilon_j'$ positive integers different from 1 but with 1 as greatest common divisor (for instance for $n_o = 2$, take $\upsilon_1 = 3, \upsilon_2 = 2, \upsilon_1' = 5, \upsilon_2' = 6$ and $\upsilon'' = 8$). As an illustration for Corollary 4.1.10 in the sequel, with this example, the homogeneous dimension is $Q = 3 + 2 + 5 + 6 + 8 = 24$ while the least common multiple is $\nu_o = 2 \times 3 \times 5 = 30$, so we have here $Q < \nu_o$.

If nothing is specified, we assume that the groups $(\mathbb{R}^n, +)$ and $\mathbb{H}_{n_o}$ are endowed with their classical structure of graded Lie groups as described in Examples 3.1.2 and 3.1.3.

*Remark* 3.1.13. We continue with several comments following those given in Remark 3.1.6.

1. The converse of Proposition 3.1.10 does not hold, namely, not every nilpotent Lie algebra or group admits a family of dilations. An example of a nine dimensional nilpotent Lie algebra which does not admit any family of dilations is due to Dyer [Dye70].

2. A direct product of two stratified Lie groups is graded but may be not stratified as their stratification structures may not 'match'. This can be also seen on the level of dilations defined in Section 3.1.2. Jumping ahead and using the notion of homogeneous operators, we see that this remark may be an advantage for example when considering the sub-Laplacian $\mathcal{L} = X^2 + Y^2$ on the Heisenberg group $\mathbb{H}_1$. Then the operator
$$-\mathcal{L} + \partial_t^k$$
for $k \in \mathbb{N}$ odd, becomes homogeneous on the direct product $\mathbb{H}_1 \times \mathbb{R}$ when it is equipped with the dilation structure which is not the one of a stratified Lie group, see Lemma 4.2.11 or, more generally, Remark 4.2.12.

3. In our definition of a homogeneous structure we started with dilations defined on the Lie algebra inducing dilations on the Lie group. If we start with a Lie group the situation may become slightly more involved. For example, $\mathbb{R}^3$ with the group law
$$xy = (\operatorname{arcsinh}(\sinh(x_1) + \sinh(y_1)), x_2 + y_2 + \sinh(x_1)y_3, x_3 + y_3)$$
is a 2-step nilpotent stratified Lie group, the first stratum given by
$$X = \cosh(x_1)^{-1}\partial_{x_1}, \quad Y = \sinh(x_1)\partial_{x_2} + \partial_{x_3},$$
and their commutator is
$$T = [X, Y] = \partial_{x_2}.$$
It may seem like there is no obvious homogeneous structure on this group but we can see it going to its Lie algebra which is isomorphic to the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group $\mathbb{H}_1$. Consequently, the above group itself is isomorphic to $\mathbb{H}_1$ with the corresponding dilation structure.

4. In fact, the same argument as above shows that if we defined a stratified Lie group by saying that there is a collection of vector fields on it stratified with respect to their commutation relations, then for every such stratified Lie group there always exists a homogeneous stratified Lie group isomorphic to it. Indeed, since the Lie algebra is stratified and has a natural dilation structure with integer weights, we obtain the required homogeneous Lie group by exponentiating this Lie algebra. We refer to e.g. [BLU07, Theorem 2.2.18] for a detailed proof of this.

Refining the proof of Proposition 3.1.10, we can obtain the following technical result which gives the existence of an 'adapted' basis of eigenvectors for the dilations.

**Lemma 3.1.14.** *Let $\mathfrak{g}$ be a Lie algebra endowed with a family of dilations $\{D_r, r > 0\}$. Then there exists a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, positive numbers $\upsilon_1, \ldots, \upsilon_n > 0$, and an integer $n'$ with $1 \le n' \le n$ such that*

$$\forall t > 0 \quad \forall j = 1, \ldots, n \qquad D_t(X_j) = t^{\upsilon_j} X_j, \tag{3.2}$$

*and*

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathbb{R}X_{n'+1} \oplus \ldots \oplus \mathbb{R}X_n. \tag{3.3}$$

*Moreover, $X_1, \ldots, X_{n'}$ generate the algebra $\mathfrak{g}$, that is, any element of $\mathfrak{g}$ can be written as a linear combination of these vectors together with all their iterated Lie brackets.*

This result and its proof are due to ter Elst and Robinson (see [tER97, Lemma 2.2]). Condition (3.2) says that $\{X_j\}_{j=1}^n$ is a basis of eigenvectors for the mapping $A$ given by

$$D_r = \mathrm{Exp}(A \ln r).$$

Condition (3.3) says that this basis can be chosen so that the first $n'$ vectors of this basis generate the whole Lie algebra and the others span (linearly) the derived algebra $[\mathfrak{g}, \mathfrak{g}]$.

*Proof of Lemma 3.1.14.* We continue with the notation of the proof of Proposition 3.1.10. For each weight $\upsilon \in \mathcal{W}_A$, we choose a basis

$$\{Y_{\upsilon,1}, \ldots, Y_{\upsilon,d'_\upsilon}, Y_{\upsilon,d'_\upsilon+1}, \ldots, Y_{\upsilon,d_\upsilon}\} \text{ of } W_\upsilon$$

such that $\{Y_{\upsilon,d'_\upsilon+1}, \ldots, Y_{\upsilon,d_\upsilon}\}$ is a basis of the subspace

$$W_\upsilon \bigcap \left( \mathrm{Span} \bigcup_{\upsilon'+\upsilon''=\upsilon} [W_{\upsilon'}, W_{\upsilon''}] \right).$$

Since $\mathfrak{g} = \oplus_{\upsilon \in \mathcal{W}_A} W_\upsilon$, we have by construction that

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Span}\{Y_{\upsilon,j} \ : \ \upsilon \in \mathcal{W}_A, \ d'_\upsilon + 1 \le j \le d_\upsilon\}.$$

Let $\mathfrak{h}$ be the Lie algebra generated by

$$\{Y_{\upsilon,j} \ : \ \upsilon \in \mathcal{W}_A, \ 1 \le j \le d'_\upsilon\}. \tag{3.4}$$

We now label and order the weights, that is, we write

$$\mathcal{W}_A = \{\upsilon_1, \ldots, \upsilon_m\}$$

with $1 \le \upsilon_1 < \ldots < \upsilon_m$. It follows by induction on $N = 1, 2 \ldots, m$ that $\oplus_{j=1}^N W_{\upsilon_j}$ is contained in $\mathfrak{h}$ and hence $\mathfrak{h} = \mathfrak{g}$ and the set (3.4) generate (algebraically) $\mathfrak{g}$.

A basis with the required property is given by

$$Y_{\upsilon_1,1}, \ldots, Y_{\upsilon_1,d'_{\upsilon_1}}, \ldots, Y_{\upsilon_m,1}, \ldots, Y_{\upsilon_m,d'_{\upsilon_m}} \quad \text{for } X_1, \ldots, X_{n'},$$

and

$$Y_{\upsilon_1,d'_{\upsilon_1}+1}, \ldots, Y_{\upsilon_1,d_{\upsilon_1}}, \ldots, Y_{\upsilon_m,d'_{\upsilon_m}+1}, \ldots, Y_{\upsilon_m,d_{\upsilon_m}} \quad \text{for } X_{n'+1}, \ldots, X_n.$$

$\square$

### 3.1.3 Homogeneous structure

In this section, we shall be working on a fixed homogeneous Lie group $G$ of dimension $n$ with dilations

$$\{D_r = \text{Exp}(A \ln r)\}.$$

We denote by $\upsilon_1, \ldots, \upsilon_n$ the weights, listed in increasing order and with each value listed as many times as its multiplicity, and we assume without loss of generality (see Remark 3.1.8) that $\upsilon_1 \geq 1$. Thus,

$$1 \leq \upsilon_1 \leq \upsilon_2 \leq \ldots \leq \upsilon_n. \tag{3.5}$$

If the group $G$ is graded, then the weights are also assumed to be integers with one as their greatest common divisor (again see Remark 3.1.8).

By Proposition 3.1.10 the Lie group $G$ is nilpotent connected simply connected. Thus it may be identified with $\mathbb{R}^n$ equipped with a polynomial law, using the exponential mapping $\exp_G$ of the group (see Section 1.6). With this identification its unit element is $0 \in \mathbb{R}^n$ and it may also be denoted by $0_G$ or simply by $0$.

We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that

$$AX_j = \upsilon_j X_j$$

for each $j$. This yields a Lebesgue measure on $\mathfrak{g}$ and a Haar measure on $G$ by Proposition 1.6.6. If $x$ or $g$ denotes a point in $G$ the Haar measure is denoted by $dx$ or $dg$. The Haar measure of a measurable subset $S$ of $G$ is denoted by $|S|$.

We easily check that

$$|D_r(S)| = r^Q |S|, \quad \int_G f(rx)dx = r^{-Q} \int_G f(x)dx, \tag{3.6}$$

where

$$Q = \upsilon_1 + \ldots + \upsilon_n = \text{Tr}A. \tag{3.7}$$

The number $Q$ is larger (or equal) than the usual dimension of the group:

$$n = \dim G \leq Q,$$

and may replace it for certain questions of analysis. For this reason the number $Q$ is called the *homogeneous dimension* of $G$.

#### Homogeneity

Any function defined on $G$ or on $G\backslash\{0\}$ can be composed with the dilations $D_r$. Using property (3.6) of the Haar measure and the dilations, we have for any measurable functions $f$ and $\phi$ on $G$, provided that the integrals exist,

$$\int_G (f \circ D_r)(x)\ \phi(x)\ dx = r^{-Q} \int_G f(x)\ (\phi \circ D_{\frac{1}{r}})(x)\ dx. \tag{3.8}$$

Therefore, we can extend the map $f \mapsto f \circ D_r$ to distributions via

$$\langle f \circ D_r, \phi \rangle := r^{-Q} \langle f, \phi \circ D_{\frac{1}{r}} \rangle, \quad f \in \mathcal{D}'(G), \ \phi \in \mathcal{D}(G). \tag{3.9}$$

We can now define the homogeneity of a function or a distribution in the same way:

**Definition 3.1.15.** Let $\nu \in \mathbb{C}$.

(i) A function $f$ on $G \backslash \{0\}$ or a distribution $f \in \mathcal{D}'(G)$ is *homogeneous of degree* $\nu \in \mathbb{C}$ (or $\nu$-homogeneous) when

$$f \circ D_r = r^\nu f \quad \text{for any } r > 0.$$

(ii) A linear operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is *homogeneous of degree* $\nu \in \mathbb{C}$ (or $\nu$-homogeneous) when

$$T(\phi \circ D_r) = r^\nu (T\phi) \circ D_r \quad \text{for any } \phi \in \mathcal{D}(G),\, r > 0.$$

*Remark* 3.1.16. We will also say that a linear operator $T : E \to F$, where $E$ is a Fréchet space containing $\mathcal{D}(G)$ as a dense subset, and $F$ is a Fréchet space included in $\mathcal{D}'(G)$, is homogeneous of degree $\nu \in \mathbb{C}$ when its restriction as an operator from $\mathcal{D}(G)$ to $\mathcal{D}'(G)$ is. For example, it will apply to the situation when $T$ is a linear operator from $L^p(G)$ to some $L^q(G)$.

*Example* 3.1.17 (Coordinate function). The coordinate function $x_j = [x]_j$ given by

$$G \ni x = (x_1, \ldots, x_n) \longmapsto x_j = [x]_j, \tag{3.10}$$

is homogeneous of degree $\upsilon_j$.

*Example* 3.1.18 (Koranyi norm). The function defined on the Heisenberg group $\mathbb{H}_{n_o}$ by

$$\mathbb{H}_{n_o} \ni (x, y, t) \longmapsto \left( \left( |x|^2 + |y|^2 \right)^2 + t^2 \right)^{1/4},$$

where $|x|$ and $|y|$ denote the canonical norms of $x$ and $y$ in $\mathbb{R}^{n_o}$, is homogeneous of degree 1. It is sometimes called the Koranyi norm.

*Example* 3.1.19 (Haar measure). Equality (3.8) shows that the Haar measure, viewed as a tempered distribution, is a homogeneous distribution of degree $Q$ (see (3.7)). We can write this informally as

$$d(rx) = r^Q dx,$$

see (3.6).

*Example* 3.1.20 (Dirac measure at 0). The Dirac measure at 0 is the probability measure $\delta_0$ given by

$$\int_G f d\delta_0 = f(0).$$

It is homogeneous of degree $-Q$ since for any $\phi \in \mathcal{D}(G)$ and $r > 0$, we have

$$\langle \delta_0 \circ D_r, \phi \rangle = r^{-Q} \langle \delta_0, \phi \circ D_{\frac{1}{r}} \rangle = r^{-Q} \phi(\frac{1}{r} 0) = r^{-Q} \phi(0) = \langle r^{-Q} \delta_0, \phi \rangle.$$

*Example* 3.1.21 (Invariant vector fields). Let $X \in \mathfrak{g}$ be viewed as a left-invariant vector field $X$ or a right-invariant vector field $\tilde{X}$ (cf. Section 1.3). We assume that $X$ is in the $\upsilon_j$-eigenspace of $A$. Then the left and right-invariant differential operators $X$ and $\tilde{X}$ are homogeneous of degree $\upsilon_j$. Indeed,

$$\begin{aligned} X(f \circ D_r)(x) &= \partial_{t=0} \{ f \circ D_r (x \exp_G(tX)) \} = \partial_{t=0} \{ f (rx \exp_G(r^{\upsilon_j} tX)) \} \\ &= r^{\upsilon_j} \partial_{t'=0} \{ f (rx \exp_G(t'X)) \} = r^{\upsilon_j} (Xf)(rx), \end{aligned}$$

and similarly for $\tilde{X}$.

The following properties are very easy to check:

**Lemma 3.1.22.** *(i) Whenever it makes sense, the product of two functions, distributions or operators of degrees $\nu_1$ and $\nu_2$ is homogeneous of degree $\nu_1 \nu_2$.*

*(ii) Let $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ be a $\nu$-homogeneous operator. Then its formal adjoint and transpose $T^*$ and $T^t$, given by*

$$\int_G (Tf) \bar{g} = \int_G f(\overline{T^* g}), \quad \int_G (Tf) g = \int_G f(T^t g), \quad f, g \in \mathcal{D}(G),$$

*are also homogeneous with degree $\bar{\nu}$ and $\nu$ respectively.*

Consequently for any non-zero multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \backslash \{0\}$, the function

$$x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \tag{3.11}$$

and the operators

$$\left( \frac{\partial}{\partial x} \right)^\alpha := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \ldots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}, \; X^\alpha := X_1^{\alpha_1} \ldots X_n^{\alpha_n} \text{ and } \tilde{X}^\alpha := \tilde{X}_1^{\alpha_1} \ldots \tilde{X}_n^{\alpha_n},$$

are homogeneous of degree

$$[\alpha] := \upsilon_1 \alpha_1 + \ldots + \upsilon_n \alpha_n. \tag{3.12}$$

Formula (3.12) defines the *homogeneous degree* of the multi-index $\alpha$. It is usually different from the length of $\alpha$ given by

$$|\alpha| := \alpha_1 + \ldots + \alpha_n.$$

For $\alpha = 0$, the function $x^\alpha$ and the operators $(\frac{\partial}{\partial x})^\alpha$, $X^\alpha$, $\tilde{X}^\alpha$ are defined to be equal, respectively, to the constant function 1 and the identity operator I, which are of degree $[\alpha] := 0$.

With this convention for each $\alpha \in \mathbb{N}_0^n$, the differential operators $(\frac{\partial}{\partial x})^\alpha$, $X^\alpha$ and $\tilde{X}^\alpha$ are of order $|\alpha|$ but of homogeneous degree $[\alpha]$.

One easily checks for $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ that

$$[\alpha_1] + [\alpha_2] = [\alpha_1 + \alpha_2], \quad |\alpha_1| + |\alpha_2| = |\alpha_1 + \alpha_2|.$$

**Proposition 3.1.23.** *Let the operator $T$ be homogeneous of degree $\nu_T$ and let $f$ be a function or a distribution homogeneous of degree $\nu_f$. Then, whenever $Tf$ makes sense, the distribution $Tf$ is homogeneous of degree $\nu_f - \nu_T$.*

*In particular, if $f \in \mathcal{D}'(G)$ is homogeneous of degree $\nu$, then*

$$X^\alpha f, \tilde{X}^\alpha f, \partial^\alpha f$$

*are homogeneous of degree $\nu - [\alpha]$.*

*Proof.* The first claim follows from the formal calculation

$$(Tf) \circ D_r = r^{-\nu_T} T(f \circ D_r) = r^{-\nu_T} T(r^{\nu_f} f) = r^{-\nu_T + \nu_f} Tf.$$

The second claim follows from the first one since $X^\alpha$, $\tilde{X}^\alpha f$ and $\partial^\alpha f$ are well defined on distributions and are homogeneous of the same degree $[\alpha]$ given by (3.12). □

### 3.1.4 Polynomials

By Propositions 3.1.10 and 1.6.6 we already know that the group law is polynomial. This means that each $[xy]_j$ is a polynomial in the coordinates of $x$ and of $y$. The homogeneous structure implies certain additional properties of this polynomial.

**Proposition 3.1.24.** *For any $j = 1, \ldots, n$, we have*

$$[xy]_j = x_j + y_j + \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^n \setminus \{0\} \\ [\alpha]+[\beta]=\upsilon_j}} c_{j,\alpha,\beta} x^\alpha y^\beta.$$

*In particular, this sum over $[\alpha]$ and $[\beta]$ can involve only coordinates in $x$ or $y$ with degrees of homogeneity strictly less than $\upsilon_j$.*

For example,

$$\begin{array}{llll}
\text{for } \upsilon_1 : & [xy]_1 & = & x_1 + y_1, \\
\text{for } \upsilon_2 : & [xy]_2 & = & x_2 + y_2 + \displaystyle\sum_{[\alpha]=[\beta]=\upsilon_1} c_{2,\alpha,\beta} x^\alpha y^\beta, \\
\text{for } \upsilon_3 : & [xy]_3 & = & x_3 + y_3 + \displaystyle\sum_{\substack{[\alpha]=\upsilon_1,\, [\beta]=\upsilon_2 \\ \text{or } [\alpha]=\upsilon_2,\, [\beta]=\upsilon_1}} c_{3,\alpha,\beta} x^\alpha y^\beta,
\end{array}$$

and so on.

*Proof.* Let $j = 1, \ldots, n$. From the Baker-Campbell-Hausdorff formula (see Theorem 1.3.2) applied to the two vectors $X = x_1 X_1 + \ldots + x_n X_n$ and $Y = y_1 X_1 + \ldots + y_n X_n$ of $\mathfrak{g}$, we have with our notation that

$$[xy]_j = x_j + y_j + R_j(x, y)$$

where $R_j(x, y)$ is a polynomial in $x_1, y_1, \ldots, x_n, y_n$. Moreover, $R_j$ must be a finite linear combination of monomials $x^\alpha y^\beta$ with $|\alpha| + |\beta| \geq 2$:

$$R_j(x, y) = \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ |\alpha|+|\beta| \geq 2}} c_{j,\alpha,\beta} x^\alpha y^\beta.$$

We now use the dilations. Since the function $x_j$ is homogeneous of degree $\upsilon_j$, we easily check

$$R_j(rx, ry) = r^{\upsilon_j} R_j(x, y)$$

for any $r > 0$ and this forces all the coefficients $c_{j,\alpha,\beta}$ with $[\alpha] + [\beta] \neq \upsilon_j$ to be zero. The formula follows. $\square$

Recursively using Proposition 3.1.24, we obtain for any $\alpha \in \mathbb{N}_0^n \backslash \{0\}$:

$$(xy)^\alpha = [xy]_1^{\alpha_1} \ldots [xy]_n^{\alpha_n} = \sum_{\substack{\beta_1,\beta_2 \in \mathbb{N}_0^n \\ [\beta_1]+[\beta_2]=[\alpha]}} c_{\beta_1,\beta_2}(\alpha) x^{\beta_1} y^{\beta_2}, \tag{3.13}$$

with

$$c_{\beta_1,0}(\alpha) = \begin{cases} 0 & \text{if } \beta_1 \neq \alpha \\ 1 & \text{if } \beta_1 = \alpha \end{cases} \quad \text{and} \quad c_{0,\beta_2}(\alpha) = \begin{cases} 0 & \text{if } \beta_2 \neq \alpha \\ 1 & \text{if } \beta_2 = \alpha \end{cases}. \tag{3.14}$$

**Definition 3.1.25.** A function $P$ on $G$ is a *polynomial* if $P \circ \exp_G$ is a polynomial on $\mathfrak{g}$.

For example the coordinate functions $x_1, \ldots, x_n$ defined in (3.10) or, more generally, the monomials $x^\alpha$ defined in (3.11) are (homogeneous) polynomials on $G$.

It is clear that every polynomial $P$ on $G$ can be written as a unique finite linear combination of the monomials $x^\alpha$, that is,

$$P = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha, \tag{3.15}$$

where all but finitely many of the coefficients $c_\alpha \in \mathbb{C}$ vanish. The *homogeneous degree* of a polynomial $P$ written as (3.15) is

$$D^\circ P := \max\{[\alpha] : \alpha \in \mathbb{N}_0^n \text{ with } c_\alpha \neq 0\},$$

which is often different from its *isotropic degree*:

$$d^\circ P := \max\{|\alpha| : \alpha \in \mathbb{N}_0^n \text{ with } c_\alpha \neq 0\}.$$

For example on $\mathbb{H}_{n_o}$, $1+t$ is a polynomial of homogeneous degree 2 but isotropic degree 1.

**Definition 3.1.26.** We denote by $\mathcal{P}(G)$ the set of all polynomials on $G$. For any $M \geq 0$ we denote by $\mathcal{P}_{\leq M}$ the set of polynomials $P$ on $G$ such that $D^\circ P \leq M$ and by $\mathcal{P}^{iso}_{\leq M}$ the set of polynomials on $G$ such that $d^\circ P \leq M$. We also define in the same way $\mathcal{P}_{<M}$, $\mathcal{P}_{=M}$, $\mathcal{P}_{\geq M}$ and so on, and similarly for $\mathcal{P}^{iso}$.

It is clear that $\mathcal{P}(G)$ is an algebra, for pointwise multiplication, which is generated by the $x_j$'s.

It is not difficult to see:

**Lemma 3.1.27.** *The subspaces $\mathcal{P}_{\leq M}$ and $\mathcal{P}^{iso}_{\leq M}$ of $\mathcal{P}$ are finite dimensional with bases $\{x^\alpha : \alpha \in \mathbb{N}_0^n, [\alpha] \leq M\}$ and $\{x^\alpha : \alpha \in \mathbb{N}_0^n, |\alpha| \leq M\}$, respectively. Furthermore,*

$$\forall M \geq 0 \qquad \mathcal{P}_{\leq M} \subset \mathcal{P}^{iso}_{\leq M} \subset \mathcal{P}_{\leq \upsilon_n M}.$$

*Proof.* The first part of the lemma is clear. For the second, because of (3.5), we have

$$\forall \alpha \in \mathbb{N}_0^n \qquad |\alpha| \leq [\alpha] \leq \upsilon_n |\alpha|. \tag{3.16}$$

Therefore,

$$\forall P \in \mathcal{P} \qquad d^\circ P \leq D^\circ P \leq \upsilon_n d^\circ P,$$

and the inclusions follow. $\square$

By Proposition 3.1.24, $[xy]_j$ is in $\mathcal{P}_{\leq \upsilon_j}$ as a function of $x$ for each $y$, and also as a function of $y$ for each $x$. Hence each subspace $\mathcal{P}_{\leq M}$ is invariant under left and right translation. This is not the case for $\mathcal{P}^{iso}_{\leq M}$ (unless $\mathcal{P}^{iso}_{\leq M} \sim \mathbb{C}$ or $G = (\mathbb{R}^n, +)$); consequently, it will not be of much use to us.

### 3.1.5 Invariant differential operators on homogeneous Lie groups

We now investigate expressions for left- and right-invariant operators on homogeneous Lie groups.

**Proposition 3.1.28.** *The left and right-invariant vector fields $X_j$ and $\tilde{X}_j$, for any $j = 1, \ldots, n$, can be written as*

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} P_{j,k}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_j} + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} \frac{\partial}{\partial x_k} P_{j,k} \\ \tilde{X}_j &= \frac{\partial}{\partial x_j} + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} Q_{j,k}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_j} + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} \frac{\partial}{\partial x_k} Q_{j,k}, \end{aligned}$$

*where $P_{j,k}$ and $Q_{j,k}$ are homogeneous polynomials on $G$ of homogeneous degree $\upsilon_k - \upsilon_j > 0$.*

*Proof.* For any $x \in G$, we denote by $L_x : G \to G$ the left-translation, i.e. $L_x(y) = xy$. Let $j = 1, \ldots, n$. Recall that $X_j$ is the differential operator invariant under left-translation which agrees with $\frac{\partial}{\partial x_j}$ at 0, that is, for any $f \in C^\infty(G)$ and $x_o \in G$, we have

$$(X_j f) \circ L_{x_o}(0) = X_j(f \circ L_{x_o})(0) \quad \text{and} \quad X_j(f)(0) = \frac{\partial f}{\partial x_j}(0).$$

Thus

$$\begin{aligned}
(X_j f)(x_o) &= (X_j f) \circ L_{x_o}(0) = X_j(f \circ L_{x_o})(0) = \frac{\partial}{\partial x_j}(f \circ L_{x_o})(0) \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x_o) \frac{\partial [x_o x]_k}{\partial x_j}(0),
\end{aligned}$$

by the chain rule. But by Proposition 3.1.24,

$$\begin{aligned}
\frac{\partial [x_o x]_k}{\partial x_j}(0) &= \frac{\partial}{\partial x_j} \left\{ [x_o]_k + x_k + \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^n \setminus \{0\} \\ [\alpha]+[\beta]=\upsilon_k}} c_{k,\alpha,\beta} x_o^\alpha x^\beta \right\}(0) \\
&= \delta_{j,k} + \sum_{\substack{\beta = e_j,\, \alpha \in \mathbb{N}_0^n \setminus \{0\} \\ [\alpha]+[\beta]=\upsilon_k}} c_{k,\alpha,\beta} x_o^\alpha,
\end{aligned}$$

where $e_j$ is the multi-index with 1 in the $j$-th place and zeros elsewhere, and $\delta_{j,k}$ is the Kronecker delta. The assertion for $X_j$ now follows immediately, and the assertion for $\tilde{X}_j$ is proved in the same way using right translations. □

Proposition 3.1.28 gives, in particular,

$$\begin{aligned}
\text{for } \upsilon_n : \quad X_n &= \frac{\partial}{\partial x_n}, \\
\text{for } \upsilon_{n-1} : \quad X_{n-1} &= \frac{\partial}{\partial x_{n-1}} + P_{n-1,n} \frac{\partial}{\partial x_n}, \\
\text{for } \upsilon_{n-2} : \quad X_{n-2} &= \frac{\partial}{\partial x_{n-2}} + P_{n-2,n-1} \frac{\partial}{\partial x_{n-1}} + P_{n-2,n} \frac{\partial}{\partial x_n},
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial}{\partial x_n} &= X_n, \\
\frac{\partial}{\partial x_{n-1}} &= X_{n-1} - P_{n-1,n} X_n, \\
\frac{\partial}{\partial x_{n-2}} &= X_{n-2} - P_{n-2,n-1}\left(X_{n-1} - P_{n-1,n} X_n\right) - P_{n-2,n} X_n,
\end{aligned}$$

and so forth, with similar formulae for the right-invariant vector fields. This shows that there are formulas for the $\frac{\partial}{\partial x_j}$'s of the same sort as for the $X_j$'s and $\tilde{X}_j$'s, that is,

$$\frac{\partial}{\partial x_j} = X_j + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} p_{j,k} X_k = \tilde{X}_j + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} q_{j,k} \tilde{X}_k, \tag{3.17}$$

where $p_{j,k}$ and $q_{j,k}$ are homogeneous polynomials on $G$ of homogeneous degree $\upsilon_k - \upsilon_j > 0$.

*Remark* 3.1.29.

1. Given the formulae above and the condition on the degree, it is not difficult to see that the $P_{j,k}$ and $Q_{j,k}$ in Proposition 3.1.28 and the $p_{j,k}$ and $q_{j,k}$ in (3.17), with $\upsilon_k > \upsilon_j$, are polynomials in $(x_1, \ldots, x_{k-1})$ and commute with $X_k$, $\tilde{X}_k$ and $\frac{\partial}{\partial x_k}$ respectively.
2. The first part of Proposition 3.1.28 and its proof are valid for any nilpotent Lie group (see Remark 1.6.7, part (1)). In our setting here, the homogeneous structure implies the additional property that the $P_{j,k}$ and $Q_{j,k}$ are homogeneous.

**Corollary 3.1.30.** *For any* $\alpha \in \mathbb{N}_0^n\backslash\{0\}$,

$$\begin{aligned} X^\alpha &= \sum_{\substack{\beta\in\mathbb{N}_0^n,\,|\beta|\leq|\alpha|\\ [\beta]\geq[\alpha]}} P_{\alpha,\beta}\tilde{X}^\beta = \sum_{\substack{\beta\in\mathbb{N}_0^n,\,|\beta|\leq|\alpha|\\ [\beta]\geq[\alpha]}} \tilde{X}^\beta p_{\alpha,\beta}, \\ \tilde{X}^\alpha &= \sum_{\substack{\beta\in\mathbb{N}_0^n,\,|\beta|\leq|\alpha|\\ [\beta]\geq[\alpha]}} Q_{\alpha,\beta} X^\beta = \sum_{\substack{\beta\in\mathbb{N}_0^n,\,|\beta|\leq|\alpha|\\ [\beta]\geq[\alpha]}} X^\beta q_{\alpha,\beta}, \end{aligned}$$

*where* $P_{\alpha,\beta}, p_{\alpha,\beta}, Q_{\alpha,\beta}, q_{\alpha,\beta}$ *are homogeneous polynomials of homogeneous degree* $[\beta]-[\alpha]$.

*Proof.* By Proposition 3.1.28 we obtain recursively for any $\alpha \in \mathbb{N}_0^n\backslash\{0\}$ that

$$X^\alpha = \sum_{\substack{\beta\in\mathbb{N}_0^n,\,|\beta|\leq|\alpha|\\ [\beta]\geq[\alpha]}} P_{\alpha,\beta}\left(\frac{\partial}{\partial x}\right)^\beta, \tag{3.18}$$

with $P_{\alpha,\beta}$ homogeneous polynomial of degree $[\beta]-[\alpha]$. Similar formulae yield $\tilde{X}^\alpha$ in terms of the $\left(\frac{\partial}{\partial x}\right)^\beta$'s.

Recursively from (3.17), we also obtain similar formulae for $\left(\frac{\partial}{\partial x}\right)^\alpha$ in terms of the $X^\beta$ or $\tilde{X}^\beta$.

The assertion comes form combining these formulae, with a similar argument for $p_{\alpha,\beta}$ and $q_{\alpha,\beta}$. □

**Corollary 3.1.31.** *For any $M \geq 0$, the maps*

$$\begin{array}{rll} (i) & P & \longmapsto \left\{ \left(\frac{\partial}{\partial x}\right)^{\alpha} P(0) \right\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] \leq M}, \\ (ii) & P & \longmapsto \{X^{\alpha} P(0)\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] \leq M}, \\ (iii) & P & \longmapsto \left\{ \tilde{X}^{\alpha} P(0) \right\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] \leq M}, \end{array}$$

*are linear isomorphisms from $\mathcal{P}_{\leq M}$ to $\mathbb{C}^{\dim \mathcal{P}_{\leq M}}$. Also, the maps*

$$\begin{array}{rll} (i) & P & \longmapsto \left\{ \left(\frac{\partial}{\partial x}\right)^{\alpha} P(0) \right\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] = M}, \\ (ii) & P & \longmapsto \{X^{\alpha} P(0)\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] = M}, \\ (iii) & P & \longmapsto \left\{ \tilde{X}^{\alpha} P(0) \right\}_{\alpha \in \mathbb{N}_0^n,\, [\alpha] = M}, \end{array}$$

*are linear isomorphisms from $\mathcal{P}_{=M}$ to $\mathbb{C}^{\dim \mathcal{P}_{=M}}$.*

*Proof.* By Lemma 3.1.27, the vector subspace $\mathcal{P}_{\leq M}$ of $\mathcal{P}$ is finite dimensional, with basis $\{x^{\alpha} : \alpha \in \mathbb{N}_0^n,\, [\alpha] \leq M\}$. Hence case (i) is a simple consequence of Taylor's Theorem on $\mathbb{R}^n$.

Note that in the formula (3.18), $P_{\alpha,\beta}$ is a constant function when $[\alpha] = [\beta]$ and $P_{\alpha,\beta}(0) = 0$ when $[\alpha] > [\beta]$. Hence

$$X^{\alpha}|_0 = \sum_{\substack{\beta \in \mathbb{N}_0^n,\, |\beta| \leq |\alpha| \\ [\beta] = [\alpha]}} P_{\alpha,\beta} \left(\frac{\partial}{\partial x}\right)^{\beta} \Big|_0 .$$

We have similar result from the other formulae relating $X^{\alpha}$, $\tilde{X}^{\alpha}$ and $\left(\frac{\partial}{\partial x}\right)^{\alpha}$.

Cases (ii) and (iii) follow from these observations together with case (i). The case of the homogeneous polynomials of order $M$ is similar. □

We may use the following property without referring to it.

**Corollary 3.1.32.** *Let $\alpha, \beta \in \mathbb{N}_0^n$. The differential operator $X^{\alpha} X^{\beta}$ is a linear combination of $X^{\gamma}$ with $[\gamma] \in \mathbb{N}_0^n$, $[\gamma] = [\alpha] + [\beta]$:*

$$X^{\alpha} X^{\beta} = \sum_{\substack{\gamma \in \mathbb{N}_0^n,\, |\gamma| \leq |\alpha| + |\beta| \\ [\gamma] = [\alpha] + [\beta]}} c'_{\alpha,\beta,\gamma} X^{\gamma}. \tag{3.19}$$

*The differential operator $\tilde{X}^{\alpha} \tilde{X}^{\beta}$ is a linear combination of $\tilde{X}^{\gamma}$ with $[\gamma] \in \mathbb{N}_0^n$, $|\gamma| \leq |\alpha| + |\beta|$ and $[\gamma] = [\alpha] + [\beta]$.*

*Proof.* The differential operator $X^\alpha X^\beta$ is a left-invariant differential operator of order $|\alpha| + |\beta|$ by (3.18), and it is a linear combination of $X^\gamma$, $|\gamma| \leq |\alpha| + |\beta|$ (see Section 1.3),

$$X^\alpha X^\beta = \sum_{\gamma \in \mathbb{N}_0^n,\, |\gamma| \leq |\alpha|+|\beta|} c'_{\alpha,\beta,\gamma} X^\gamma.$$

By homogeneity, for any $r > 0$ and any function $f \in C^\infty(G)$, we have on one hand,

$$X^\alpha X^\beta (f \circ D_r) = r^{[\alpha]+[\beta]} (X^\alpha X^\beta f) \circ D_r,$$

and on the other hand,

$$\begin{aligned} X^\alpha X^\beta (f \circ D_r) &= \sum_{\gamma \in \mathbb{N}_0^n,\, |\gamma| \leq |\alpha|+|\beta|} c'_{\alpha,\beta,\gamma} X^\gamma (f \circ D_r) \\ &= \sum_{\gamma \in \mathbb{N}_0^n,\, |\gamma| \leq |\alpha|+|\beta|} c'_{\alpha,\beta,\gamma} r^{[\gamma]} (X^\gamma f) \circ D_r. \end{aligned}$$

Choosing $f$ suitably (for example $f$ being polynomials of homogeneous degree at most $[\alpha] + [\beta]$, see Corollary 3.1.31), this implies that if $[\alpha] + [\beta] \neq [\gamma]$ then $c'_{\alpha,\beta,\gamma} = 0$, showing (3.19).

The property for the right-invariant vector fields is similar. □

### 3.1.6 Homogeneous quasi-norms

We can define an Euclidean norm $|\cdot|_E$ on $\mathfrak{g}$ by declaring the $X_j$'s to be orthonormal. We may also regard this norm as a function on $G$ via the exponential mapping, that is,

$$|x|_E = |\exp_G^{-1} x|_E.$$

However, this norm is of limited use for our purposes, since it does not interact in a simple fashion with dilations. We therefore define:

**Definition 3.1.33.** A *homogeneous quasi-norm* is a continuous non-negative function

$$G \ni x \longmapsto |x| \in [0, \infty),$$

satisfying

(i) (symmetric) $|x^{-1}| = |x|$ for all $x \in G$,

(ii) (1-homogeneous) $|rx| = r|x|$ for all $x \in G$ and $r > 0$,

(iii) (definite) $|x| = 0$ if and only if $x = 0$.

The $|\cdot|$-*ball centred at* $x \in G$ *with radius* $R > 0$ is defined by

$$B(x, R) := \{y \in G : \ |x^{-1}y| < R\}.$$

*Remark* 3.1.34. With such definition, we have for any $x, x_o \in G$, $R > 0$,

$$x_o B(x, R) = B(x_o x, R), \tag{3.20}$$

since

$$z \in x_o B(x, R) \Longleftrightarrow x_o^{-1} z \in B(x, R) \Longleftrightarrow |x^{-1} x_o^{-1} z| < R \Longleftrightarrow z \in B(x_o x, R).$$

In particular, we see that

$$B(x, r) = x B(0, r).$$

It is also easy to check that

$$B(0, r) = D_r(B(0, 1)).$$

Note that in our definition of quasi-balls, we choose to privilege the left translations. Indeed, the set $\{y \in G \ : \ |yx^{-1}| < R\}$ may also be defined as a quasi-ball but one would have to use the right translation instead of the left $x_o$-translation to have a similar property to (3.20).

An important example of a quasi-norm is given by Example 3.1.18 on the Heisenberg group $\mathbb{H}_{n_o}$. More generally, on any homogeneous Lie group, the following functions are homogeneous quasi-norms:

$$|(x_1, \ldots, x_n)|_p = \left( \sum_{j=1}^{n} |x_j|^{\frac{p}{\upsilon_j}} \right)^{\frac{1}{p}}, \tag{3.21}$$

for $0 < p < \infty$, and for $p = \infty$:

$$|(x_1, \ldots, x_n)|_\infty = \max_{1 \le j \le n} |x_j|^{\frac{1}{\upsilon_j}}. \tag{3.22}$$

In Definition 3.1.33 we do not require a homogeneous quasi-norm to be smooth away from the origin but some authors do. Quasi-norms with added regularity always exist as well but, in fact, a distinction between different quasi-norms is usually irrelevant for many questions of analysis because of the following property:

**Proposition 3.1.35.** *(i) Every homogeneous Lie group $G$ admits a homogeneous quasi-norm that is smooth away from the unit element.*

*(ii) Any two homogeneous quasi-norms $|\cdot|$ and $|\cdot|'$ on $G$ are mutually equivalent:*

$$\|\cdot\| \asymp \|\cdot\|' \quad \textit{in the sense that} \quad \exists a, b > 0 \quad \forall x \in G \qquad a|x|' \le |x| \le b|x|'.$$

*Proof.* Let us consider the function

$$\Psi(r,x) = |D_r x|_E^2 = \sum_{j=1}^n r^{2\upsilon_j} x_j^2.$$

Let us fix $x \neq 0$. The function $\Psi(r,x)$ is continuous, strictly increasing in $r$ and satisfies

$$\Psi(r,x) \underset{r\to 0}{\longrightarrow} 0 \quad \text{and} \quad \Psi(r,x) \underset{r\to+\infty}{\longrightarrow} +\infty.$$

Therefore, there is a unique $r>0$ such that $|D_r x|_E = 1$. We set $|x|_o := r^{-1}$.

Hence we have defined a map

$$G\backslash\{0\} \ni x \mapsto |x|_o^{-1} \in (0,\infty)$$

which is the implicit function for $\Psi(r,x) = 1$. This map is smooth since the function $\Psi(r,x)$ is smooth from $(0,+\infty)\times G\backslash\{0\}$ to $(0,\infty)$ and $\partial_r\Psi(r,x)$ is always different from zero. Setting $|0_G|_o := 0$, the map $|\cdot|_o$ clearly satisfies the properties of Definition 3.1.33. This shows part (i).

For Part (ii), it is sufficient to prove that any homogeneous quasi-norm is equivalent to $|\cdot|_o$ constructed above. Before doing so, we observe that the unit spheres in the Euclidean norm and the homogeneous quasi-norm $|\cdot|_o$ coincide, that is,

$$\mathfrak{S} := \{x\in G \,:\, |x|_E = 1\} = \{x \in G \,:\, |x|_o = 1\}.$$

Let $|\cdot|$ be any other homogeneous norm. Since it is a definite function (see (iii) of Definition 3.1.33) its restriction to $\mathfrak{S}$ is never zero. By compactness of $\mathfrak{S}$ and continuity of $|\cdot|$, there are constants $a,b>0$ such that

$$\forall x\in\mathfrak{S} \qquad a\leq |x| \leq b.$$

For any $x\in G\backslash\{0\}$, let $t>0$ be given by $t^{-1} = |x|_o$. We have $D_t x\in\mathfrak{S}$, and thus

$$a\leq |D_t x|\leq b \quad \text{and} \quad a|x|_o = t^{-1}a \leq |x| \leq t^{-1}b = b|x|_o.$$

The conclusion of Part (ii) follows. □

*Remark* 3.1.36. If $G$ is graded, the formula (3.21) for $p = 2\upsilon_1\ldots\upsilon_n$ gives another concrete example of a homogeneous quasi-norm smooth away from the origin since $x\mapsto |x|_p^p$ is then a polynomial in the coordinate functions $\{x_j\}$.

Proposition 3.1.35 and our examples of homogeneous quasi-norms show that the usual Euclidean topology coincides with the topology associated with any homogeneous quasi-norm:

**Proposition 3.1.37.** *If $|\cdot|$ is a homogeneous quasi-norm on $G \sim \mathbb{R}^n$, the topology induced by the $|\cdot|$-balls*

$$B(x,R) := \{y \in G : \ |x^{-1}y| < R\},$$

*$x \in G$ and $R > 0$, coincides with the Euclidean topology of $\mathbb{R}^n$.*

*Any closed ball or sphere for any homogeneous quasi-norm is compact. It is also bounded with respect to any norm of the vector space $\mathbb{R}^n$ or any other homogeneous quasi-norm on $G$.*

*Proof of Proposition 3.1.37.* It is a routine exercise of topology to check that the equivalence of norm given in Proposition 3.1.35 implies that the topology induced by the balls of two different homogeneous quasi-norms coincide. Hence we can choose the norm $|\cdot|_\infty$ given by (3.22) and the corresponding balls

$$B_\infty(x,R) := \{y \in G : \ |x^{-1}y|_\infty < R\}.$$

We also consider the supremum Euclidean norm given by

$$|(x_1,\ldots,x_n)|_{E,\infty} = \max_{1\leq j\leq n} |x_j|,$$

and its corresponding balls

$$B_{E,\infty}(x,R) := \{y \in G : \ |-x+y|_{E,\infty} < R\}.$$

That the topologies induced by the two families of balls

$$\{B_\infty(x,R)\}_{x\in G, R>0} \quad \text{and} \quad \{B_{E,\infty}(x,R)\}_{x\in G, R>0}$$

must coincide follows from the following two observations. Firstly it is easy to check for any $R \in (0,1)$

$$B_\infty(0, R^{\frac{1}{\upsilon_1}}) \subset B_{E,\infty}(0,R) \subset B_\infty(0, R^{\frac{1}{\upsilon_n}}).$$

Secondly for each $x \in G$, the mappings $\Psi_x : y \mapsto x^{-1}y$ and $\Psi_{E,x} : y \mapsto -x+y$ are two smooth diffeomorphisms of $\mathbb{R}^n$. Hence these mappings are continuous with continuous inverses (with respect to the Euclidean topology). Furthermore, by Remark 3.1.34, we have

$$\Psi_x(B_\infty(x,R)) = B_\infty(0,R) \quad \text{and} \quad \Psi_{E,x}(B_{E,\infty}(x,R)) = B_{E,\infty}(0,R).$$

The second part of the statement follows from the first and from the continuity of homogeneous quasi-norms. $\square$

The next proposition justifies the terminology of ‘quasi-norm’ by stating that every homogeneous quasi-norm satisfies the triangle inequality up to a constant, the other properties of a norm being already satisfied.

**Proposition 3.1.38.** *If $|\cdot|$ is a homogeneous quasi-norm on $G$, there is a constant $C > 0$ such that*

$$|xy| \le C\,(|x| + |y|) \qquad \forall x, y \in G.$$

*Proof.* Let $|\cdot|$ be a quasi-norm on $G$. Let $\bar{B} := \{x \,:\, |x| \le 1\}$ be its associated closed unit ball. By Proposition 3.1.37, $\bar{B}$ is compact. As the product law is continuous (even polynomial), the set $\{xy : \ x, y \in \bar{B}\}$ is also compact. Therefore, there is a constant $C > 0$ such that

$$\forall x, y \in \bar{B} \qquad |xy| \le C.$$

Let $x, y \in G$. If both of them are 0, there is nothing to prove. If not, let $t > 0$ be given by $t^{-1} = |x| + |y| > 0$. Then $D_t(x)$ and $D_t(y)$ are in $\bar{B}$, so that

$$t|xy| = |D_t(xy)| = |D_t(x)D_t(y)| \le C,$$

and this concludes the proof. □

Note that the constant $C$ in Proposition 3.1.38 satisfies necessarily $C \ge 1$ since $|0| = 0$ implies $|x| \le C|x|$ for all $x \in G$. It is natural to ask whether a homogeneous Lie group $G$ may admit a homogeneous quasi-norm $|\cdot|$ which is actually a norm or, equivalently, which satisfies the triangle inequality with constant $C = 1$. For instance, on the Heisenberg group $\mathbb{H}_{n_o}$, the homogeneous quasi-norm given in Example 3.1.18 turns out to be a norm (cf. [Cyg81]). In the stratified case, the norm built from the control distance of the sub-Laplacian, often called the Carnot-Caratheodory distance, is also 1-homogeneous (see, e.g., [Pan89] or [BLU07, Section 5.2]). This can be generalised to all homogeneous Lie groups.

**Theorem 3.1.39.** *Let $G$ be a homogeneous Lie group. Then there exist a homogeneous quasi-norm on $G$ which is a norm, that is, a homogeneous quasi-norm $|\cdot|$ which satisfies the triangle inequality*

$$|xy| \le |x| + |y| \qquad \forall x, y \in G.$$

A proof of Theorem 3.1.39 by Hebisch and Sikora uses the correspondence between homogeneous norms and convex sets, see [HS90]. Here we sketch a different proof. Its idea may be viewed as an adaptation of a part of the proof that the control distance in the stratified case is a distance. Our proof may be simpler than the stratified case though, since we define a distance without using 'horizontal' curves.

*Sketch of the proof of Theorem 3.1.39.* If $\gamma : [0, T] \to G$ is a smooth curve, its tangent vector $\gamma'(t_o)$ at $\gamma(t_o)$ is usually defined as the element of the tangent space $T_{\gamma(t_o)}G$ at $\gamma(t_o)$ such that

$$\gamma'(t_o)(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=t_o}, \quad f \in C^\infty(G).$$

It is more convenient for us to identify the tangent vector of $\gamma$ at $\gamma(t_o)$ with an element of the Lie algebra $\mathfrak{g} = T_0 G$. We therefore define $\tilde{\gamma}'(t_o) \in \mathfrak{g}$ via

$$\tilde{\gamma}'(t_o)(f) := \frac{d}{dt} f(\gamma(t_o)^{-1}\gamma(t))\Big|_{t=t_o}, \quad f \in C^\infty(G).$$

We now fix a basis $\{X_j\}_{j=1}^n$ of $\mathfrak{g}$ such that $D_r X_j = r^{\upsilon_j} X_j$. We also define the map $|\cdot|_\infty : \mathfrak{g} \to [0,\infty)$ by

$$|X|_\infty := \max_{j=1,\ldots,n} |x_j|^{\frac{1}{\upsilon_j}}, \quad X = \sum_{j=1}^n x_j X_j \in \mathfrak{g}.$$

Given a piecewise smooth curve $\gamma : [0,T] \to G$, we define its length adapted to the group structure by

$$\tilde{\ell}(\gamma) := \int_0^T |\tilde{\gamma}'(t)|_\infty dt.$$

If $x$ and $y$ are in $G$, we denote by $d(x,y)$ the infimum of the lengths $\tilde{\ell}(\gamma)$ of the piecewise smooth curves $\gamma$ joining $x$ and $y$. Since two points $x$ and $y$ can always be joined by a smooth compact curve, e.g. $\gamma(t) = ((1-t)x)\, ty$, the quantity $d(x,y)$ is always finite. Hence we have obtained a map $d : G \times G \to [0,\infty)$. It is a routine exercise to check that $d$ is symmetric and satisfies the triangle inequality in the sense that we have for all $x,y,z \in G$, that

$$d(x,y) = d(y,x) \quad \text{and} \quad d(x,y) \leq d(x,z) + d(z,y).$$

Moreover, one can check easily that $\tilde{\ell}(D_r(\gamma)) = r\tilde{\ell}(\gamma)$ and $\tilde{\ell}(z\gamma) = \tilde{\ell}(\gamma)$, thus we also have for all $x,y,z \in G$ and $r > 0$, that

$$d(zx,zy) = d(x,y) \quad \text{and} \quad d(rx,ry) = rd(x,y). \tag{3.23}$$

Let us show that $d$ is non-degenerate, that is, $d(x,y) = 0 \Rightarrow x = y$. First let $|\cdot|_E$ be the Euclidean norm on $\mathfrak{g} \sim \mathbb{R}^n$ such that the basis $\{X_j\}_{j=1}^n$ is orthonormal. We endow each tangent space $T_x G$ with the Euclidean norm obtained by left translation of the Euclidean norm $|\cdot|_E$. Hence we have for any smooth curve $\gamma$ at any point $t_o$

$$|\gamma'(t_o)|_{T_{\gamma(t_o)}G} = |\tilde{\gamma}'(t_o)|_E.$$

Now we see that if $X = \sum_{j=1}^n x_j X_j \in \mathfrak{g}$ is such that

$$|X|_{E,\infty} := \max_{j=1,\ldots,n} |x_j| \leq 1,$$

then

$$|X|_E \asymp |X|_{E,\infty} \leq |X|_\infty.$$

This implies that if $\gamma : [0,T] \to G$ is a smooth curve satisfying

$$\forall t \in [0,T] \qquad |\gamma'(t)|_{T_{\gamma(t)}G} < 1, \tag{3.24}$$

then

$$\ell(\gamma) \le C\tilde{\ell}(\gamma), \tag{3.25}$$

where $\ell$ is the usual length

$$\ell(\gamma) := \int_0^T |\gamma'(t)|_{T_{\gamma(t)}G} dt,$$

and $C > 0$ a positive constant independent of $\gamma$.

Let $d_G$ be the Riemaniann distance induced by our choice of metric on the manifold $G$, that is, the infimum of the lengths $\ell(\gamma)$ of the piecewise smooth curves $\gamma$ joining $x$ and $y$. Very well known results in Riemaniann geometry imply that $d_G$ induces the same topology as the Euclidean topology. Moreover, there exists a small open set $\Omega$ containing 0 such that any point in $\Omega$ may be joined to 0 by a smooth curve satisfying (3.24) at any point. Then (3.25) yields that we have $d_G(0,x) \le Cd(0,x)$ for any $x \in \Omega$. This implies that $d$ is non-degenerate since $d$ is invariant under left-translation and is 1-homogeneous in the sense of (3.23),

Checking that the associated map $x \mapsto |x| = d(0,x)$ is a quasi-norm concludes the sketch of the proof of Theorem 3.1.39. □

Even if homogeneous norms do exist, it is often preferable to use homogeneous quasi-norms. Because the triangle inequality is up to a constant in this case, we do not necessarily have the inequality $||xy| - |x|| \le C|y|$. However, the following lemma may help:

**Proposition 3.1.40.** *We fix a homogeneous quasi-norm $|\cdot|$ on $G$. For any $f \in C^1(G\backslash\{0\})$ homogeneous of degree $\nu \in \mathbb{C}$, for any $b \in (0,1)$ there is a constant $C = C_b > 0$ such that*

$$|f(xy) - f(x)| \le C|y|\ |x|^{\operatorname{Re}\nu - 1} \quad \textit{whenever} \quad |y| \le b|x|.$$

Indeed, applying it to a $C^1(G\backslash\{0\})$ homogeneous quasi-norm, we obtain

$$\forall b \in (0,1) \quad \exists C = C_b > 0 \quad \forall x,y \in G \quad |y| \le b|x| \Longrightarrow \big||xy| - |x|\big| \le C|y|. \tag{3.26}$$

*Proof of Proposition 3.1.40.* Let $f \in C^1(G\backslash\{0\})$. Both sides of the desired inequality are homogeneous of degree $\operatorname{Re}\nu$ so it suffices to assume that $|x| = 1$ and $|y| \le b$. By Proposition 3.1.37 and the continuity of multiplication, the set $\{xy : |x| = 1 \,\text{and}\, |y| \le b\}$ is a compact which does not contain 0. So by the (Euclidean) mean value theorem on $\mathbb{R}^n$, we get

$$|f(xy) - f(x)| \le C|y|_E.$$

We conclude using the next lemma. □

The next lemma shows that locally a homogeneous quasi-norm and the Euclidean norm are comparable:

**Lemma 3.1.41.** *We fix a homogeneous quasi-norm* $|\cdot|$ *on* $G$. *Then there exist* $C_1, C_2 > 0$ *such that*

$$C_1|x|_E \leq |x| \leq C_2|x|_E^{\frac{1}{\upsilon_n}} \quad \textit{whenever} \quad |x| \leq 1.$$

*Proof of Lemma 3.1.41.* By Proposition 3.1.37, the unit sphere $\{y : |y| = 1\}$ is compact and does not contain 0. Hence the Euclidean norm assumes a positive maximum $C_1^{-1}$ and a positive minimum $C_2^{-\upsilon_n}$ on it, for some $C_1, C_2 > 0$.

Let $x \in G$. We may assume $x \neq 0$. Then we can write it as $x = ry$ with $|y| = 1$ and $r = |x|$. We observe that since

$$|ry|_E^2 = \sum_{j=1}^{n} y_j^2 r^{2\upsilon_j},$$

we have if $r \leq 1$

$$r^{\upsilon_n}|y|_E \leq |ry|_E \leq r|y|_E.$$

Hence for $r = |x| \leq 1$, we get

$$|x|_E = |ry|_E \leq r|y|_E \leq |x|C_1^{-1} \quad \text{and} \quad |x|_E = |ry|_E \geq r^{\upsilon_n}|y|_E \geq |x|^{\upsilon_n}C_2^{-\upsilon_n},$$

implying the statement. □

### 3.1.7 Polar coordinates

There is an analogue of polar coordinates on homogeneous Lie groups.

**Proposition 3.1.42.** *Let* $G$ *be a homogeneous Lie group equipped with a homogeneous quasi-norm* $|\cdot|$. *Then there is a (unique) positive Borel measure* $\sigma$ *on the unit sphere*

$$\mathfrak{S} := \{x \in G : |x| = 1\},$$

*such that for all* $f \in L^1(G)$, *we have*

$$\int_G f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{3.27}$$

In order to prove this claim, we start with the following averaging property:

**Lemma 3.1.43.** *Let* $G$ *be a homogeneous Lie group equipped with a homogeneous quasi-norm* $|\cdot|$. *If* $f$ *is a locally integrable function on* $G\backslash\{0\}$, *homogeneous of degree* $-Q$, *then there exists a constant* $m_f \in \mathbb{C}$ (the average value of $f$) *such that for all* $u \in L^1((0,\infty), r^{-1}dr)$, *we have*

$$\int_G f(x)u(|x|)dx = m_f \int_0^\infty u(r)r^{-1}dr. \tag{3.28}$$

The proof of Lemma 3.1.43 yields the formula for $m_f$ in terms of the homogeneous quasi-norm $|\cdot|$,

$$m_f = \int_{1\le|x|\le e} f(x)dx. \tag{3.29}$$

However, in Lemma 3.1.45 we will give an invariant meaning to this value.

*Proof of Lemma 3.1.43.* Let $f$ be locally integrable function on $G\backslash\{0\}$, homogeneous of degree $-Q$. We set for any $r>0$,

$$\varphi(r) := \begin{cases} \int_{1\le|x|\le r} f(x)dx & \text{if } r\ge 1, \\ -\int_{r\le|x|\le 1} f(x)dx & \text{if } r<1. \end{cases}$$

The mapping $\varphi:(0,\infty)\to\mathbb{C}$ is continuous and one easily checks that

$$\varphi(rs) = \varphi(r)+\varphi(s) \qquad \text{for all } r,s>0,$$

by making the change of variable $x\mapsto sx$ and using the homogeneity of $f$. It follows that $\varphi(r)=\varphi(e)\ln r$ and we set

$$m_f := \varphi(e).$$

Then the equation (3.28) is easily satisfied when $u$ is the characteristic function of an interval. By taking the linear combinations and limits of such functions, the equation (3.28) is also satisfied when $u\in L^1((0,\infty),r^{-1}dr)$. $\square$

*Proof of Proposition 3.1.42.* For any continuous function $f$ on the unit sphere $\mathfrak{S}$, we define the homogeneous function $\tilde f$ on $G\backslash\{0\}$ by

$$\tilde f(x) := |x|^{-Q} f(|x|^{-1}x).$$

Then $\tilde f$ satisfies the hypotheses of Lemma 3.1.43. The map $f\mapsto m_{\tilde f}$ is clearly a positive functional on the space of continuous functions on $\mathfrak{S}$. Hence it is given by integration against a regular positive measure $\sigma$ (see, e.g. [Rud87, ch.VI]).

For $u\in L^1((0,\infty),r^{-1}dr)$, we have

$$\begin{aligned} \int f(|x|^{-1}x)u(|x|)dx &= \int \tilde f(x)|x|^Q u(|x|)dx = m_{\tilde f}\int_{r=0}^{\infty} r^{Q-1}u(r)dr \\ &= \int_0^\infty\int_{\mathfrak{S}} f(y)u(r)r^{Q-1}d\sigma(y)dr. \end{aligned}$$

Since linear combinations of functions of the form $f(|x|^{-1}x)u(|x|)$ are dense in $L^1(G)$, the proposition follows. $\square$

We view the formula (3.27) as a change in polar coordinates.

*Example* 3.1.44. For $0 < a < b < \infty$ and $\alpha \in \mathbb{C}$, we have

$$\int_{a<|x|<b} |x|^{\alpha-Q} dx = C \begin{cases} \alpha^{-1}(b^\alpha - a^\alpha) & \text{if } \alpha \neq 0 \\ \ln\left(\frac{b}{a}\right) & \text{if } \alpha = 0 \end{cases} \qquad \text{with } C = \sigma(\mathfrak{S}).$$

And if $\alpha \in \mathbb{R}$ and $f$ is a measurable function on $G$ such that $f(x) = O(|x|^{\alpha-Q})$ then $f$ is integrable either near $\infty$ if $\alpha < 0$, or near 0 if $\alpha > 0$.

The measure $\sigma$ in the polar coordinates decomposition actually has a smooth density. We will not need this fact and will not prove it here, but refer to [FR66] and [Goo80].

Now, the polar change of coordinates depends on the choice of a homogeneous quasi-norm to fix the unit sphere. But it turns out that the average value of the $(-Q)$-homogeneous function considered in Lemma 3.1.43 does not. Let us prove this fact for the sake of completeness.

**Lemma 3.1.45.** *Let $G$ be a homogeneous Lie group and let $f$ be a locally integrable function on $G\backslash\{0\}$, homogeneous of degree $-Q$.*

*Given a homogeneous quasi-norm, let $\sigma$ be the Radon measure on the unit sphere $\mathfrak{S}$ giving the polar change of coordinate (3.27). Then the average value of $f$ defined in (3.28) is given by*

$$m_f = \int_{\mathfrak{S}} f d\sigma. \tag{3.30}$$

*This average value $m_f$ is independent of the choice of the homogeneous quasi-norm.*

*Proof of Lemma 3.1.45.* For any homogeneous quasi-norm, using the polar change of coordinates (3.27), we obtain

$$\begin{aligned}\int_{a<|x|<b} f(x)dx &= \int_a^b \int_{\mathfrak{S}} f(rx)d\sigma(x)r^{Q-1}dr \\ &= \int_a^b \int_{\mathfrak{S}} f(x)d\sigma(x)r^{-1}dr = \int_a^b r^{-1}dr \int_{\mathfrak{S}} f(x)d\sigma(x) = \left(\ln\frac{b}{a}\right) m_f.\end{aligned}$$

This shows (3.30), taking $a = 1$ and $b = e$, see (3.29) and the proof of Lemma 3.1.43.

Let $|\cdot|$ and $|\cdot|'$ be two homogeneous quasi-norms on $G$. We denote by

$$\bar{B}_r := \{x \in G : |x| \leq r\} \quad \text{and} \quad \bar{B}'_r := \{x \in G : |x|' \leq r\},$$

the closed balls around 0 of radius $r$ for $|\cdot|$ and $|\cdot|'$, respectively. By Proposition 3.1.35, Part (ii), there exists a constant $a > 0$ such that $\bar{B}'_a \subset \bar{B}_1$. We also have $\bar{B}'_a \subset \bar{B}'_{2a} \subset \bar{B}_2$ and, with the usual sign convention for integration, we have

$$\int_{\bar{B}_2\backslash\bar{B}_1} = \int_{\bar{B}_2\backslash\bar{B}'_a} - \int_{\bar{B}_1\backslash\bar{B}'_a} = \int_{\bar{B}_2\backslash\bar{B}'_{2a}} + \int_{\bar{B}'_{2a}\backslash\bar{B}'_a} - \int_{\bar{B}_1\backslash\bar{B}'_a}.$$

Using the homogeneities of $f$ and of the Haar measure, we see, after the changes of variables $x = 2y$ and $x = az$, that

$$\int_{\bar{B}_2 \setminus \bar{B}'_{2a}} f(x)dx = \int_{\bar{B}_1 \setminus \bar{B}'_a} f(y)dy \quad \text{and} \quad \int_{\bar{B}'_{2a} \setminus \bar{B}'_a} f(x)dx = \int_{\bar{B}'_2 \setminus \bar{B}'_1} f(z)dz.$$

Hence

$$\int_{\bar{B}_2 \setminus \bar{B}_1} f = \int_{\bar{B}'_2 \setminus \bar{B}'_1} f.$$

Using the first computations of this proof, the left and right hand sides are equal to $(\ln b/a)m_f$ and $(\ln b/a)m'_f$, respectively, where $m_f$ and $m'_f$ are the average values for $|\cdot|$ and $|\cdot|'$. Thus $m_f = m'_f$. □

### 3.1.8 Mean value theorem and Taylor expansion

Here we prove the mean value theorem and describe the Taylor series on homogeneous Lie groups. Naturally, the space $C^1(G)$ here is the space of functions $f$ such that $X_j f$ are continuous on $G$ for all $j$, etc. The following mean value theorem can be partly viewed as a refinement of Proposition 3.1.40.

**Proposition 3.1.46.** *We fix a homogeneous quasi-norm $|\cdot|$ on $G$. There exist group constants $C_0 > 0$ and $\eta > 1$ such that for all $f \in C^1(G)$ and all $x, y \in G$, we have*

$$|f(xy) - f(x)| \le C_0 \sum_{j=1}^{n} |y|^{\upsilon_j} \sup_{|z| \le \eta |y|} |(X_j f)(xz)|.$$

In order to prove this proposition, we first prove the following property.

**Lemma 3.1.47.** *The map $\phi : \mathbb{R}^n \to G$ defined by*

$$\phi(t_1, \ldots, t_n) = \exp_G(t_1 X_1)\, \exp_G(t_2 X_2) \ldots \exp_G(t_n X_n),$$

*is a global diffeomorphism.*

*Moreover, fixing a homogeneous quasi-norm $|\cdot|$ on $G$, there is a constant $C_1 > 0$ such that*

$$\forall (t_1, \ldots, t_n) \in \mathbb{R}^n,\ j = 1, \ldots, n, \qquad |t_j|^{\frac{1}{\upsilon_j}} \le C_1 |\phi(t_1, \ldots, t_n)|.$$

The first part of the lemma is true for any nilpotent Lie group (see Remark 1.6.7 Part (ii)). But we will not use this fact here.

*Proof.* Clearly the map $\phi$ is smooth. By the Baker-Campbell-Hausdorff formula (see Theorem 1.3.2), the differential $d\phi(0) : \mathbb{R}^n \to T_0 G$ is the isomorphism

$$d\phi(0)(t_1, \ldots, t_n) = \sum_{j=1}^{n} t_j X_j|_0,$$

so that $\phi$ is a local diffeomorphism near 0 (this is true for any Lie group). More precisely, there exist $\delta, C' > 0$ such that $\phi$ is a diffeomorphism from $U$ to the ball $B_\delta := \{x \in G : |x| < \delta\}$ with

$$\phi^{-1}(B_\delta) = U \subset \{(t_1, \ldots, t_n) : \max_{j=1,\ldots,n} |t_j|^{\frac{1}{\upsilon_j}} < C'\}.$$

We now use the dilations and for any $r > 0$, we see that

$$\begin{aligned}
\phi(r^{\upsilon_1}t_1, \ldots, r^{\upsilon_n}t_n) &= \exp_G(r^{\upsilon_1}t_1X_1)\ldots\exp_G(r^{\upsilon_n}t_nX_n) \\
&= (r\exp_G(t_1X_1))\ldots(r\exp_G(t_nX_n)) \\
&= r\left(\exp_G(t_1X_1)\ldots\exp_G(t_nX_n)\right),
\end{aligned}$$

hence

$$\phi(r^{\upsilon_1}t_1, \ldots, r^{\upsilon_n}t_n) = r\phi(t_1, \ldots, t_n). \tag{3.31}$$

If $\phi(t_1, \ldots, t_n) = \phi(s_1, \ldots, s_n)$, formula (3.31) implies that for all $r > 0$, we have

$$\phi(r^{\upsilon_1}t_1, \ldots, r^{\upsilon_n}t_n) = \phi(r^{\upsilon_1}s_1, \ldots, r^{\upsilon_n}s_n).$$

For $r$ sufficiently small, this forces $t_j = s_j$ for all $j$ since $\phi$ is a diffeomorphism on $U$. So the map $\phi : \mathbb{R}^n \to G$ is injective.

Moreover, any $x \in G\backslash\{0\}$ can be written as

$$x = ry \quad \text{with} \quad r := \frac{2}{\delta}|x| \quad \text{and} \quad y := r^{-1}x \in \overline{B_{\frac{\delta}{2}}} \subset \phi(U).$$

We may write $y = \phi(s_1, \ldots s_n)$ with $|s_j|^{\frac{1}{\upsilon_j}} \leq C'$ and formula (3.31) then implies that $x = \phi(t_1, \ldots, t_n)$ is in $\phi(\mathbb{R}^n)$ with $t_j := r^{\upsilon_j}s_j$ satisfying $|t_j|^{\frac{1}{\upsilon_j}} \leq C'r$. Setting $C_1 = 2C'/\delta$, the assertion follows. □

*Proof of Proposition 3.1.46.* First let us assume that $y = \exp_G(tX_j)$. Then

$$\begin{aligned}
f(xy) - f(x) &= \int_0^t \partial_{s'=s}\left\{f(x\exp_G(s'X_j))\right\} ds \\
&= \int_0^t \partial_{s'=0}\left\{f(x\exp_G(sX_j)\exp_G(s'X_j))\right\} ds \\
&= \int_0^t X_jf(x\exp_G(sX_j))ds,
\end{aligned}$$

and hence

$$\begin{aligned}
|f(xy) - f(x)| &\leq |t| \sup_{0\leq s\leq t} |X_jf(x\exp_G(sX_j))| \\
&\leq |t| \sup_{|z|\leq|y|} |X_jf(xz)|.
\end{aligned}$$

Since $|\exp_G(sX_j)| = |s|^{\frac{1}{\upsilon_j}}|\exp_G X_j|$ and hence $|y| = |t|^{\frac{1}{\upsilon_j}}|\exp_G X_j|$, setting

$$C_2 := \max_{k=1,\dots,n} |\exp_G X_k|^{-\upsilon_k},$$

we obtain

$$|f(xy) - f(x)| \le C_2 |y|^{\upsilon_j} \sup_{|z|\le|y|} |X_j f(xz)|. \tag{3.32}$$

We now prove the general case, so let $y$ be any point of $G$. By Lemma 3.1.47, it can be written uniquely as $y = y_1 y_2 \dots y_n$ with $y_j = \exp_G(t_j X_j)$, and hence

$$|y_j| = |t|^{\frac{1}{\upsilon_j}}|\exp_G X_j| \le C_1 C_3 |y| \quad \text{where} \quad C_3 := \max_{k=1,\dots,n} |\exp_G X_k|, \tag{3.33}$$

and $C_1$ is as in Lemma 3.1.47. We write

$$\begin{aligned} |f(xy) - f(x)| \le\ & |f(xy_1 \dots y_n) - f(xy_1 \dots y_{n-1})| \\ & + |f(xy_1 \dots y_{n-1}) - f(xy_1 \dots y_{n-2})| + \dots + |f(xy_1) - f(x)|, \end{aligned}$$

and applying (3.32) to each term, we obtain

$$|f(xy) - f(x)| \le \sum_{j=1}^{n} C_2 |y_j|^{\upsilon_j} \sup_{|z|\le|y_j|} |X_j f(xy_1 \dots y_{j-1} z)|.$$

Let $C_4 \ge 1$ be the constant of the triangle inequality for $|\cdot|$ (see Proposition 3.1.38). If $|z| \le |y_j|$, then $z' = y_1 \dots y_{j-1} z$ satisfies

$$\begin{aligned} |z'| &\le C_4(|y_1 \dots y_{j-1}| + |y_j|) \le C_4\big(C_4(|y_1 \dots y_{j-2}| + |y_{j-1}|) + |y_j|\big) \\ &\le C_4^2 (|y_1 \dots y_{j-2}| + |y_{j-1}| + |y_j|) \le \dots \le C_4^{j-1}(|y_1| + |y_2| + \dots |y_j|) \\ &\le C_4^{j-1} j C_1 C_3 |y|, \end{aligned}$$

using (3.33). Therefore, setting $\eta := C_4^n n C_1 C_3$, using again (3.33), we have obtained

$$|f(xy) - f(x)| \le C_2 \sum_{j=1}^{n} (C_1 C_3 |y|)^{\upsilon_j} \sup_{|z'|\le\eta|y|} |X_j f(xz')|,$$

completing the proof. □

*Remark* 3.1.48. Let us make the following remarks.

1. In the same way, we can prove the following version of Proposition 3.1.46 for right-invariant vector fields: a homogeneous quasi-norm $|\cdot|$ being fixed on $G$, there exists group constants $C > 0$ and $b > 0$ such that for all $f \in C^1(G)$ and all $x, y \in G$, we have

$$|f(yx) - f(x)| \le C \sum_{j=1}^{n} |y|^{\upsilon_j} \sup_{|z|\le b|y|} |(\tilde{X}_j f)(zx)|.$$

2. If the homogeneous Lie group $G$ is stratified, a more precise version of the mean value theorem exists involving only the vector fields of the first stratum, see Folland and Stein [FS82, (1.41)], but we will not use this fact here.
3. The statement and the proof of the mean value theorem can easily be adapted to hold for functions which are valued in a Banach space, the modulus being replaced by the Banach norm.

**Taylor expansion**

In view of Corollary 3.1.31, we can define Taylor polynomials:

**Definition 3.1.49.** The *Taylor polynomial* of a suitable function $f$ at a point $x \in G$ of homogeneous degree $\leq M \in \mathbb{N}_0$ is the unique $P \in \mathcal{P}_{\leq M}$ such that

$$\forall \alpha \in \mathbb{N}_0^n,\ [\alpha] \leq M \qquad X^\alpha P(0) = X^\alpha f(x).$$

More precisely, we have defined the *left* Taylor polynomial, and a similar definition using the right-invariant differential operators $\tilde{X}^\alpha$ yields the right Taylor polynomial. However, in this monograph we will use only left Taylor polynomials.

We may use the following notation for the Taylor polynomial $P$ of a function $f$ at $x$ and for its *remainder of order $M$*:

$$P_{x,M}^{(f)} := P \quad \text{and} \quad R_{x,M}^{(f)}(y) := f(xy) - P(y). \tag{3.34}$$

For instance, $P_{x,M}^{(f)}(0) = f(x)$. We will also extend the notation for negative $M$ with

$$P_{x,M}^{(f)} := 0 \quad \text{and} \quad R_{x,M}^{(f)}(y) := f(xy) \quad \text{when } M < 0.$$

With this notation, we easily see (whenever it makes sense), the following properties.

**Lemma 3.1.50.** *For any $M \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and suitable function $f$, we have*

$$X^\alpha P_{x,M}^{(f)} = P_{x,M-[\alpha]}^{(X^\alpha f)} \quad \textit{and} \quad X^\alpha R_{x,M}^{(f)} = R_{x,M-[\alpha]}^{(X^\alpha f)}.$$

*Proof.* It is easy to check that the polynomial $P_o := X^\alpha P_{x,M}^{(f)}$ is homogeneous of degree $M - [\alpha]$. Furthermore, using (3.19), it satisfies for every $\beta \in \mathbb{N}_0^n$, such that $[\alpha] + [\beta] \leq M$, the equality

$$\begin{aligned} X^\beta P_o(0) &= X^\beta X^\alpha P_{x,M}^{(f)}(0) \\ &= \sum_{\substack{|\gamma| \leq |\alpha| + |\beta| \\ [\gamma] = [\alpha] + [\beta]}} c'_{\alpha,\beta,\gamma} X^\gamma P_{x,M}^{(f)}(0) = \sum_{\substack{|\gamma| \leq |\alpha| + |\beta| \\ [\gamma] = [\alpha] + [\beta]}} c'_{\alpha,\beta,\gamma} X^\gamma f(x) \\ &= X^\beta X^\alpha f(x). \end{aligned}$$

This shows the claim. $\square$

In Definition 3.1.49 the suitable functions $f$ are distributions on a neighbourhood of $x$ in $G$ whose (distributional) derivatives $X^\alpha f$ are continuous in a neighbourhood of $x$ for $[\alpha] \leq M$. We will see in the sequel that in order to control (uniformly) a remainder of a function $f$ of order $M$ we would like $f$ to be at least $(k+1)$ times continuously differentiable, i.e. $f \in C^{k+1}(G)$, where $k \in \mathbb{N}_0$ is equal to

$$\lceil M \rfloor := \max\{|\alpha| \, : \, \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \leq M\}; \tag{3.35}$$

this is indeed a maximum over a finite set because of (3.16).

We can now state and prove Taylor's inequality.

**Theorem 3.1.51.** *We fix a homogeneous quasi-norm $|\cdot|$ on $G$ and obtain a corresponding constant $\eta$ from the mean value theorem (see Proposition 3.1.46). For any $M \in \mathbb{N}_0$, there is a constant $C_M > 0$ such that for all functions $f \in C^{\lceil M \rfloor + 1}(G)$ and all $x, y \in G$, we have*

$$|R^{(f)}_{x,M}(y)| \leq C_M \sum_{\substack{|\alpha| \leq \lceil M \rfloor + 1 \\ [\alpha] > M}} |y|^{[\alpha]} \sup_{|z| \leq \eta^{\lceil M \rfloor + 1}|y|} |(X^\alpha f)(xz)| \, ,$$

*where $R^{(f)}_{x,M}$ and $\lceil M \rfloor$ are defined by (3.34) and (3.35).*

Theorem 3.1.51 for $M = 0$ boils down exactly to the mean value theorem as stated in Proposition 3.1.46. Similar comments as in Remark 3.1.48 for the mean value theorem are also valid for Taylor's inequality.

*Proof.* Under the hypothesis of the theorem, a remainder $R^{(f)}_{x,M}$ is always $C^1$ and vanishes at 0. Let us apply the mean value theorem (see Proposition 3.1.46) at the point 0 to the remainders $R^{(f)}_{x,M}$, $R^{(X_{j_0} f)}_{x,M-\upsilon_{j_0}}$, $R^{(X_{\upsilon_{j_1}} X_{\upsilon_{j_0}} f)}_{x,M-(\upsilon_{j_0}+\upsilon_{j_1})}$, and so on as long as $M - (\upsilon_{j_0} + \ldots + \upsilon_{j_k}) \geq 0$; using this together with Lemma 3.1.50, we obtain

$$\begin{aligned}
\left| R^{(f)}_{x,M}(y_0) \right| &\leq C_0 \sum_{j_0=1}^{n} |y_0|^{\upsilon_{j_0}} \sup_{|y_1| \leq \eta |y_0|} \left| R^{(X_{\upsilon_{j_0}} f)}_{x,M-\upsilon_{j_0}}(y_1) \right| , \\
\left| R^{(X_{\upsilon_{j_0}} f)}_{x,M-\upsilon_{j_0}}(y_1) \right| &\leq C_0 \sum_{j_1=1}^{n} |y_1|^{\upsilon_{j_1}} \sup_{|y_2| \leq \eta |y_1|} \left| R^{(X_{\upsilon_{j_1}} X_{\upsilon_{j_0}} f)}_{x,M-(\upsilon_{j_0}+\upsilon_{j_1})}(y_2) \right| , \\
&\vdots \\
\left| R^{(X_{\upsilon_{j_k}} \ldots X_{\upsilon_{j_0}} f)}_{x,M-(\upsilon_{j_0}+\ldots+\upsilon_{j_k})}(y_k) \right| &\leq C_0 \sum_{j_k=1}^{n} |y_k|^{\upsilon_{j_k}} \sup_{|y_{k+1}| \leq \eta |y_k|} \left| R^{(X_{\upsilon_{j_{k+1}}} \ldots X_{\upsilon_{j_0}} f)}_{x,M-(\upsilon_{j_0}+\ldots+\upsilon_{j_{k+1}})}(y_k) \right| .
\end{aligned}$$

We combine these inequalities together, to obtain

$$\left| R^{(f)}_{x,M}(y_0) \right| \leq C_0^{k+1} \eta^k \sum_{\substack{j_i = 1, \ldots, n \\ i = 0, \ldots, k+1}} |y_0|^{\upsilon_{j_0} + \ldots + \upsilon_{j_k}} \sup_{|y_{k+1}| \leq \eta^{k+1} |y_0|} \left| R^{(X^{\upsilon_{j_{k+1}}} \ldots X^{\upsilon_{j_0}} f)}_{x,M-(\upsilon_{j_0}+\ldots+\upsilon_{j_{k+1}})}(y_k) \right| .$$

The process stops exactly for $k = \lceil M \rfloor$ by the very definition of $\lceil M \rfloor$. For this value of $k$, Corollary 3.1.32 and the change of discrete variable $\alpha := \upsilon_{j_0} e_{j_0} + \dots \upsilon_{j_{k+1}} e_{j_{k+1}}$ (where $e_j$ denotes the multi-index with 1 in the $j$-th place and zeros elsewhere) yield the result. $\square$

*Remark* 3.1.52.

1. We can consider Taylor polynomials for right-invariant vector fields. The corresponding Taylor estimates would then approximate $f(yx)$ with a polynomial in $y$. See Part 1 of Remark 3.1.48, about the mean value theorem for the case of order 0. Note that in Theorem 3.1.51 we consider $f(xy)$ and its approximation by a polynomial in $y$.
2. If the homogeneous Lie group $G$ is stratified, a more precise versions of Taylor's inequality exists involving only the vector fields of the first stratum, see Folland and Stein [FS82, (1.41)], but we will not use this fact here.
3. The statement and the proof of Theorem 3.1.51 can easily be adapted to hold for functions which are valued in a Banach space, the modulus being replaced by the Banach norm.
4. One can derive explicit formulae for Taylor's polynomials and the remainders on homogeneous Lie groups, see [Bon09] (see also [ACC05] for the case of Carnot groups), but we do not require these here.

As a corollary of Theorem 3.1.51 that will be useful to us later, the right-derivatives of Taylor polynomials and of the remainder will have the following properties, slightly different from those for the left derivatives in Lemma 3.1.50.

**Corollary 3.1.53.** *Let $f \in C^\infty(G)$. For any $M \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, we have*

$$\tilde{X}^\alpha P^{(f)}_{x,M} = P^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]} \quad \textit{and} \quad \tilde{X}^\alpha R^{(f)}_{x,M} = R^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}.$$

*Proof.* Recall from (1.12) that for any $X \in \mathfrak{g}$ identified with a left-invariant vector field, we have

$$\tilde{X}_y\{f(xy)\} = \frac{d}{dt} f(xe^{tX}y)_{t=0} = X_x\{f(xy)\},$$

and recursively, we obtain

$$\tilde{X}_y^\alpha\{f(xy)\} = X_x^\alpha\{f(xy)\}. \tag{3.36}$$

Therefore, we have

$$\begin{aligned}
&\tilde{X}^\alpha P^{(f)}_{x,M}(y) - P^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y) \\
&\quad = \tilde{X}_y^\alpha \left\{ f(xy) - R^{(f)}_{x,M}(y) \right\} - \left\{ X_x^\alpha f(xy) - R^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y) \right\} \\
&\quad = -\tilde{X}^\alpha R^{(f)}_{x,M}(y) + R^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y).
\end{aligned} \tag{3.37}$$

By Corollary 3.1.30, we can write

$$\begin{aligned}\tilde{X}^\alpha R^{(f)}_{x,M}(y) &= \sum_{|\beta|\leq|\alpha|,\,[\beta]\geq[\alpha]} Q_{\alpha,\beta}(y) X^\beta R^{(f)}_{x,M}(y)\\ &= \sum_{|\beta|\leq|\alpha|,\,[\beta]\geq[\alpha]} Q_{\alpha,\beta}(y) R^{(X^\beta f)}_{x,M-[\beta]}(y),\end{aligned}$$

where each $Q_{\alpha,\beta}$ is a homogeneous polynomial of degree $[\beta]-[\alpha]$.

Fixing a homogeneous quasi-norm $|\cdot|$ on $G$, the Taylor inequality (Theorem 3.1.51) applied to $R^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}$ and $R^{(X^\beta f)}_{x,M-[\beta]}$ implies that, for $|y|\leq 1$,

$$|R^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y)| \leq C|y|^{M-[\alpha]+1} \quad \text{and} \quad |R^{(X^\beta f)}_{x,M-[\beta]}(y)| \leq C|y|^{M-[\beta]+1}.$$

Hence

$$|\tilde{X}^\alpha R^{(f)}_{x,M}(y)| \leq C|y|^{M-[\alpha]+1}.$$

Going back to (3.37), we have obtained that its left hand side can be estimated as

$$|\tilde{X}^\alpha P^{(f)}_{x,M}(y) - P^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y)| \leq C|y|^{M-[\alpha]+1}.$$

But $\tilde{X}^\alpha P^{(f)}_{x,M}(y) - P^{(X_x^\alpha f(x\,\cdot))}_{0,M-[\alpha]}(y)$ is a polynomial of homogeneous degree at most $M-[\alpha]$. Therefore, this polynomial is identically 0. This concludes the proof of Corollary 3.1.53. □

### 3.1.9 Schwartz space and tempered distributions

The Schwartz space on a homogeneous Lie group $G$ is defined as the Schwartz space on any connected simply connected nilpotent Lie group, namely, by identifying $G$ with the underlying vector space of its Lie algebra (see Definition 1.6.8). The vector space $\mathcal{S}(G)$ is naturally endowed with a Fréchet topology defined by any of a number of families of seminorms.

In the 'traditional' Schwartz seminorm on $\mathbb{R}^n$ (see (1.13)) we can replace (without changing anything for the Fréchet topology):

- $\left(\frac{\partial}{\partial x}\right)^\alpha$ and the isotropic degree $|\alpha|$ by $X^\alpha$ and the homogeneous degree $[\alpha]$, respectively, in view of Section 3.1.5,
- the Euclidean norm by the norm $|\cdot|_p$ given in (3.21), and then by any homogeneous norm since homogeneous quasi-norms are equivalent (cf. Proposition 3.1.35).

Hence we choose the following family of seminorms for $\mathcal{S}(G)$, where $G$ is a homogeneous Lie group:

$$\|f\|_{\mathcal{S}(G),N} := \sup_{[\alpha]\leq N,\, x\in G} (1+|x|)^N |X^\alpha f(x)| \qquad (N\in\mathbb{N}_0),$$

after having fixed a homogeneous quasi-norm $|\cdot|$ on $G$.

Another equivalent family is given by a similar definition with the right-invariant vector fields $\tilde{X}^\alpha$ replacing $X^\alpha$.

The following lemma proves, in particular, that translations, taking the inverse, and convolutions, are continuous operations on Schwartz functions.

**Lemma 3.1.54.** *Let $f \in \mathcal{S}(G)$ and $N \in \mathbb{N}$. Then we have*

$$\begin{aligned} \left\|f(y\,\cdot\,)\right\|_{\mathcal{S}(G),N} &\leq C_N(1+|y|)^N \|f\|_{\mathcal{S}(G),N} \qquad (y \in G), & (3.38)\\ \left\|\tilde{f}\right\|_{\mathcal{S}(G),N} &\leq C_N \|f\|_{\mathcal{S}(G),(\upsilon_n+1)N} \quad \textit{where} \quad \tilde{f}(x) = f(x^{-1}), & (3.39)\\ \left\|f(\,\cdot\,y)\right\|_{\mathcal{S}(G),N} &\leq C_N(1+|y|)^{(\upsilon_n+1)N} \|f\|_{\mathcal{S}(G),(\upsilon_n+1)^2N} \quad (y \in G). & (3.40) \end{aligned}$$

*Moreover,*

$$\left\|f(y\,\cdot\,) - f\right\|_{\mathcal{S}(G),N} \longrightarrow_{y\to 0} 0 \quad \textit{and} \quad \left\|f(\,\cdot\,y) - f\right\|_{\mathcal{S}(G),N} \longrightarrow_{y\to 0} 0. \qquad (3.41)$$

*The group convolution of two Schwartz functions $f_1, f_2 \in \mathcal{S}(G)$ satisfies*

$$\|f_1 * f_2\|_{\mathcal{S}(G),N} \leq C_N \|f_1\|_{\mathcal{S}(G),N+Q+1} \|f_2\|_{\mathcal{S}(G),N}. \qquad (3.42)$$

*Proof.* Let $C_o \geq 1$ be the constant of the triangle inequality, cf. Proposition 3.1.38. We have easily that

$$\forall x, y \in G \qquad (1+|x|) \leq C_o(1+|y|)(1+|yx|). \qquad (3.43)$$

Thus,

$$\begin{aligned} \left\|f(y\,\cdot\,)\right\|_{\mathcal{S}(G),N} &\leq \sup_{[\alpha]\leq N,\, x\in G} \left(C_o(1+|y|)(1+|yx|)\right)^N |X^\alpha f(yx)| \\ &\leq C_o^N(1+|y|)^N \|f\|_{\mathcal{S}(G),N}. \end{aligned}$$

This shows (3.38).

For (3.39), using (1.11) and Corollary 3.1.30, we have

$$\begin{aligned} \left\|\tilde{f}\right\|_{\mathcal{S}(G),N} &\leq \sup_{[\alpha]\leq N,\, x\in G} (1+|x|)^N |(\tilde{X}^\alpha f)(x^{-1})| \\ &\leq \sup_{[\alpha]\leq N,\, x\in G} \sum_{\substack{\beta\in\mathbb{N}_0^n,\, |\beta|\leq|\alpha| \\ [\beta]\geq[\alpha]}} (1+|x|)^N |\left(Q_{\alpha,\beta} X^\beta f\right)(x^{-1})| \\ &\leq C_N \sup_{[\beta]\leq \upsilon_n N,\, x\in G} (1+|x'|)^{N+[\beta]} |X^\beta f(x')| \end{aligned}$$

by homogeneity of the polynomials $Q_{\alpha,\beta}$ and (3.16).

Since $f(\,\cdot\,y) = (\tilde{f}(y^{-1}\,\cdot\,))\tilde{}$, we deduce (3.40) from (3.38) and (3.39).

By the mean value theorem (cf. Proposition 3.1.46),

$$\begin{aligned}
\|f(y\,\cdot\,)-f\|_{\mathcal{S}(G),N} &= \sup_{[\alpha]\le N,\, x\in G}(1+|x|)^N|X^\alpha f(yx)-X^\alpha f(x)| \\
&\le C\sum_{j=1}^{n}|y|^{\upsilon_j}\sup_{\substack{[\alpha]\le N\\ x\in G,\,|z|\le\eta|y|}}(1+|x|)^N|(X_jX^\alpha f)(xz)| \\
&\le C\sum_{j=1}^{n}|y|^{\upsilon_j}\,\|f\|_{\mathcal{S}(G),N+\upsilon_n}\,, \qquad (3.44)
\end{aligned}$$

and this proves (3.41) for the left invariance. The proof is similar for the right invariance and is left to the reader.

Since using (3.43) we have

$$\begin{aligned}
(1+|x|)^N\,|X^\alpha(f_1*f_2)(x)| &\le \int_G(1+|x|)^N|f_1(y)|\,|X^\alpha f_2(y^{-1}x)|dy \\
&\le C_o^N\int_G(1+|y|)^N|f_1(y)|(1+|y^{-1}x|)^N|X^\alpha f_2(y^{-1}x)|dy \\
&\le C_o^N\sup_{z\in G}(1+|z|)^N|X^\alpha f_2(z)|\int_G(1+|y|)^N|f_1(y)|dy,
\end{aligned}$$

we obtain (3.42) by the convergence in Example 3.1.44. □

The space of tempered distributions $\mathcal{S}'(G)$ is the (continuous) dual of $\mathcal{S}(G)$. Hence a linear form $f$ on $\mathcal{S}(G)$ is in $\mathcal{S}'(G)$ if and only if

$$\exists N\in\mathbb{N}_0,\ C>0 \qquad \forall\phi\in\mathcal{S}(G)\quad |\langle f,\phi\rangle|\le C\|\phi\|_{\mathcal{S}(G),N}. \qquad (3.45)$$

The topology of $\mathcal{S}'(G)$ is given by the family of seminorms given by

$$\|f\|_{\mathcal{S}'(G),N}:=\sup\{|\langle f,\phi\rangle|,\|\phi\|_{\mathcal{S}(G),N}\le 1\},\quad f\in\mathcal{S}'(G),\ N\in\mathbb{N}_0.$$

Now, with these definitions, we can repeat the construction in Section 1.5 and define convolution of a distribution in $\mathcal{S}'(G)$ with the test function in $\mathcal{S}(G)$. Then we have

**Lemma 3.1.55.** *For any $f\in\mathcal{S}'(G)$ there exist $N\in\mathbb{N}$ and $C>0$ such that*

$$\forall\phi\in\mathcal{S}(G)\qquad \forall x\in G\qquad |(\phi*f)(x)|\le C(1+|x|)^N\|\phi\|_{\mathcal{S}(G),N}. \qquad (3.46)$$

*The constant $C$ may be chosen of the form $C=C'\|f\|_{\mathcal{S}'(G),N'}$ for some $C'$ and $N'$ independent of $f$.*

*For any $f\in\mathcal{S}'(G)$ and $\phi\in\mathcal{S}(G)$, $\phi*f\in C^\infty(G)$. Moreover, if $f_\ell\longrightarrow_{\ell\to\infty}f$ in $\mathcal{S}'(G)$ then for any $\phi\in\mathcal{S}(G)$,*

$$\phi*f_\ell\longrightarrow_{\ell\to\infty}\phi*f$$

*in* $C^\infty(G)$.

*Furthermore, if* $f \in \mathcal{S}'(G)$ *is compactly supported then* $\phi * f \in \mathcal{S}(G)$ *for any* $\phi \in \mathcal{S}(G)$.

*Proof.* Let $f \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$. By definition of the convolution in Definition 1.5.3 and continuity of $f$ (see (3.45)) we have

$$\begin{aligned} |(\phi * f)(x)| &= |\langle f, \tilde{\phi}(\cdot\, x^{-1})\rangle| \leq C\|\tilde{\phi}(\cdot\, x^{-1})\|_{\mathcal{S}(G),N} \\ &\leq C(1+|x^{-1}|)^{(\upsilon_n+1)N}\|\tilde{\phi}\|_{\mathcal{S}(G),(\upsilon_n+1)^2N} \qquad \text{(by (3.40))} \\ &\leq C(1+|x|)^{(\upsilon_n+1)N}\|\phi\|_{\mathcal{S}(G),(\upsilon_n+1)^3N} \qquad \text{(by (3.39))}. \end{aligned}$$

This shows (3.46). Consequently

$$\tilde{X}^\alpha(\phi * f) = (\tilde{X}^\alpha \phi) * f$$

is also bounded for every $\alpha \in \mathbb{N}_0^n$ and hence $\phi * f$ is smooth. The convergence statement then follows from the definition of the convolution for distributions.

Let us now assume that the distribution $f$ is compactly supported. Its support is included in the ball of radius $R$ for $R$ large enough. There exists $N \in \mathbb{N}_0$ such that

$$\begin{aligned} |(\phi * f)(x)| &= |\langle f, \tilde{\phi}(\cdot\, x^{-1})\rangle| \leq C \sup_{|y|\leq R,\, |\alpha|\leq N} \left| \left(\frac{\partial}{\partial y}\right)^\alpha (\phi(xy^{-1}))\right| \\ &\leq C_R \sup_{|y|\leq R,\, [\alpha]\leq \upsilon_n N} \left| \tilde{X}_y^\alpha\{\phi(xy^{-1})\}\right|, \end{aligned}$$

using (3.16) and (3.17). By (1.11), we have

$$\tilde{X}_y^\alpha\{\phi(xy^{-1})\} = (-1)^{|\alpha|}(X^\alpha\phi)(xy^{-1}),$$

and so for every $M \in \mathbb{N}_0$ with $M \geq [\alpha]$, we obtain

$$\left|\tilde{X}_y^\alpha\{\phi(xy^{-1})\}\right| = \left|X^\alpha\phi(xy^{-1})\right| \leq \|\phi\|_{\mathcal{S}(G),M}(1+|xy^{-1}|)^{-M}.$$

By (3.43), we have also

$$(1+|xy^{-1}|)^{-1} \leq C_o(1+|y|)(1+|x|)^{-1}.$$

Therefore, for every $M \in \mathbb{N}$ with $M \geq \upsilon_n N$ we get

$$\begin{aligned} |(\phi * f)(x)| &\leq C_R \sup_{|y|\leq R} C_o^M(1+|y|)^M(1+|x|)^{-M}\|\phi\|_{\mathcal{S}(G),M} \\ &\leq C_R'(1+|x|)^{-M}\|\phi\|_{\mathcal{S}(G),M}. \end{aligned}$$

This shows $\phi * f \in \mathcal{S}(G)$. $\square$

We note that there are certainly different ways of introducing the topology of the Schwartz spaces by different choices of families of seminorms.

**Lemma 3.1.56.** *Other families of Schwartz seminorms defining the same Fréchet topology on $\mathcal{S}(G)$ are*

- $\phi \mapsto \max_{[\alpha],[\beta]\leq N} \|x^\alpha X^\beta \phi\|_p$
- $\phi \mapsto \max_{[\alpha],[\beta]\leq N} \|X^\beta x^\alpha \phi\|_p$
- $\phi \mapsto \max_{[\beta]\leq N} \|(1+|\cdot|)^N X^\beta \phi\|_p$

*(for the first two we don't need a homogeneous quasi-norm) where $p \in [1,\infty]$.*

*Proof.* The first two families with the usual Euclidean derivatives instead of left-invariant vector fields are known to give the Fréchet topologies. Therefore, by e.g. using Proposition 3.1.28, this is also the case for the first two families.

The last family would certainly be equivalent to the first one for the homogeneous quasi-norm $|\cdot|_p$ in (3.21), for $p$ being a multiple of $\upsilon_1, \ldots, \upsilon_n$, since $|x|_p^p$ is a polynomial. Therefore, the last family also yields the Fréchet topology for any choice of homogeneous quasi-norm since any two homogeneous quasi-norms are equivalent by Proposition 3.1.35. $\square$

### 3.1.10 Approximation of the identity

The family of dilations gives an easy way to define approximations to the identity.

If $\phi$ is a function on $G$ and $t > 0$, we define $\phi_t$ by

$$\phi_t := t^{-Q} \phi \circ D_{t^{-1}} \quad \text{i.e.} \quad \phi_t(x) = t^{-Q} \phi(t^{-1}x).$$

If $\phi$ is integrable then $\int \phi_t$ is independent of $t$.

We denote by $C_o(G)$ the space of continuous functions on $G$ which vanish at infinity:

**Definition 3.1.57.** We denote by $C_o(G)$ the space of continuous function $f : G \to \mathbb{C}$ such that for every $\epsilon > 0$ there exists a compact set $K$ outside which we have $|f| < \epsilon$.

Endowed with the supremum norm $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(G)}$, $C_o(G)$ is a Banach space.

We also denote by $C_c(G)$ the space of continuous and compactly supported functions on $G$. It is easy to see that $C_c(G)$ is dense in $L^p(G)$ for $p \in [1,\infty)$ and in $C_o(G)$ (in which case we set $p = \infty$).

**Lemma 3.1.58.** *Let $\phi \in L^1(G)$ and $\int_G \phi = c$.*

(i) *For every $f \in L^p(G)$ with $1 \le p < \infty$ or every $f \in C_o(G)$ with $p = \infty$, we have*

$$\phi_t * f \underset{t\to 0}{\longrightarrow} cf \quad \text{in } L^p(G) \text{ or } C_o(G), \text{ i.e. } \quad \|\phi_t * f - cf\|_{L^p(G)} \underset{t\to 0}{\longrightarrow} 0.$$

*The same holds for $f * \phi_t$.*

(ii) *If $\phi \in \mathcal{S}(G)$, then for any $\psi \in \mathcal{S}(G)$ and $f \in \mathcal{S}'(G)$, we have*

$$\phi_t * \psi \underset{t\to 0}{\longrightarrow} c\psi \quad \text{in } \mathcal{S}(G) \quad \text{and} \quad \phi_t * f \underset{t\to 0}{\longrightarrow} cf \quad \text{in } \mathcal{S}'(G).$$

*The same holds for $\psi * \phi_t$ and $f * \psi_t$.*

The proof is very similar to its Euclidean counterpart.

*Proof.* Let $\phi \in L^1(G)$ and $c = \int_G \phi$. If $f \in C_c(G)$ then

$$\begin{aligned}
(\phi_t * f)(x) - cf(x) &= \int_G t^{-Q}\phi(t^{-1}y)f(y^{-1}x)dy - cf(x) \\
&= \int_G \phi(z)f((tz)^{-1}x)dz - \int_G \phi(z)dz f(x) \\
&= \int_G \phi(z)\left(f((tz)^{-1}x) - f(x)\right)dz.
\end{aligned}$$

Hence by the Minkowski inequality we have

$$\|\phi_t * f - cf\|_p \le \int_G |\phi(z)| \left\|f((tz)^{-1}\cdot\,) - f\right\|_p dz.$$

Since $\left\|f((tz)^{-1}\cdot\,) - f\right\|_p \le 2\|f\|_p$, this shows (i) for any $f \in C_c(G)$ by the Lebesgue dominated convergence theorem. Let $f$ be in $L^p(G)$ or $C_o(G)$ (in this case $p = \infty$). By density of $C_c(G)$, for any $\epsilon > 0$, we can find $f_\epsilon \in C_c(G)$ such that $\|f - f_\epsilon\|_p \le \epsilon$. We have

$$\|\phi_t * (f - f_\epsilon)\|_p \le \|\phi_t\|_1 \|f - f_\epsilon\|_p \le \|\phi\|_1 \epsilon,$$

thus

$$\begin{aligned}
\|\phi_t * f - cf\|_p &\le \|\phi_t * (f - f_\epsilon)\|_p + |c|\|f_\epsilon - f\|_p + \|\phi_t * f_\epsilon - cf_\epsilon\|_p \\
&\le (\|\phi\|_1 + |c|)\epsilon + \|\phi_t * f_\epsilon - cf_\epsilon\|_p.
\end{aligned}$$

Since $\|\phi_t * f_\epsilon - cf_\epsilon\|_p \to 0$ as $t \to 0$, there exists $\eta > 0$ such that

$$\forall t \in (0, \eta) \qquad \|\phi_t * f_\epsilon - cf_\epsilon\|_p < \epsilon.$$

Hence if $0 < t < \eta$, we have

$$\|\phi_t * f - cf\|_p \le (\|\phi\|_1 + |c| + 1)\epsilon.$$

This shows the convergence of $\phi_t * f - cf$ for any $f \in L^p(G)$ or $C_o(G)$.

With the notation $\tilde{\cdot}$ for the operation given by $\tilde{g}(x) = g(x^{-1})$, we also have

$$(f * g)\tilde{} = \tilde{g} * \tilde{f}.$$

Hence applying the previous result to $\tilde{f}$ and $\tilde{\phi}$, we obtain the convergence of $f * \phi_t - cf$.

Let us prove (ii) for $\phi, \psi \in \mathcal{S}(G)$. We have as above

$$(\phi_t * \psi)(x) - c\psi(x) = \int_G \phi(z) \left(\psi((tz)^{-1}x) - \psi(x)\right) dz,$$

thus

$$\begin{aligned} \|\phi_t * \psi - c\psi\|_{\mathcal{S}(G),N} &\leq \int_G |\phi(z)| \, \left\|\psi((tz)^{-1}\cdot) - \psi\right\|_{\mathcal{S}(G),N} dz \\ &\leq \int_G |\phi(z)| \, C \sum_{j=1}^n |(tz)^{-1}|^{\upsilon_j} \, \|\psi\|_{\mathcal{S}(G),N+\upsilon_n} dz \end{aligned}$$

by (3.44). And this shows

$$\|\phi_t * \psi - c\psi\|_{\mathcal{S}(G),N} \leq C \sum_{j=1}^n \|\phi\|_{\mathcal{S}(G),Q+1+\upsilon_j} \, \|\psi\|_{\mathcal{S}(G),N+\upsilon_n} \, t^{\upsilon_j} \underset{t\to 0}{\longrightarrow} 0.$$

Hence we have obtained the convergence of $\phi_t * \psi - c\psi$. As above, applying the previous result to $\tilde{\psi}$ and $\tilde{\phi}$, we obtain the convergence of $\psi * \phi_t$.

Let $f \in \mathcal{S}'(G)$. By (1.14) for distributions, we see for any $\psi \in \mathcal{S}(G)$, that

$$\langle f * \phi_t, \psi \rangle = \langle f, \psi * \tilde{\phi}_t \rangle \underset{t\to 0}{\longrightarrow} c\langle f, \psi \rangle$$

by the convergence just shown above. This shows that $f * \phi_t$ converges to $f$ in $\mathcal{S}'(G)$. As above, applying the previous result to $\tilde{f}$ and $\tilde{\phi}$, we obtain the convergence of $f * \phi_t$. $\square$

In the sequel we will need (only in the proof of Theorem 4.4.9) the following collection of technical results. Recall that a simple function is a measurable function which takes only a finite number of values.

**Lemma 3.1.59.** *Let $\mathscr{B}$ denote the space of simple and compactly supported functions on $G$. Then we have the following properties.*

(i) *The space $\mathscr{B}$ is dense in $L^p(G)$ for any $p \in [1, \infty)$.*

(ii) *If $\phi \in \mathcal{S}(G)$ and $f \in \mathscr{B}$, then $\phi * f$ and $f * \phi$ are in $\mathcal{S}(G)$.*

(iii) *For every $f \in \mathscr{B}$ and $p \in [1,\infty]$,*

$$\phi_t * f \underset{t\to 0}{\longrightarrow} (\int_G \phi) f$$

*in $L^p(G)$. The same holds for $f * \phi_t$.*

*Proof.* Part (i) is well-known (see, e.g., Rudin [Rud87, ch. 1]).

As a convolution of a Schwartz function $\phi$ with a compactly supported tempered distribution $f \in \mathscr{B}$, $f * \phi$ and $\phi * f$ are Schwartz by Lemma 3.1.55. This proves (ii).

Part (iii) follows from Lemma 3.1.58 (i) for $1 \leq p < \infty$. For the case $p = \infty$, we proceed as in the first part of the proof of Lemma 3.1.58 (i) taking $f$ not in $C_c(G)$ but a simple function with compact support. □

*Remark* 3.1.60. In Section 4.2.2 we will see that the heat semi-group associated to a positive Rockland operator gives an approximation of the identity $h_t$, $t > 0$, which is commutative:

$$h_t * h_s = h_s * h_t = h_{s+t}.$$

## 3.2 Operators on homogeneous Lie groups

In this section we analyse operators on a (fixed) homogeneous Lie group $G$. We first study sufficient conditions for a linear operator to extend boundedly from some $L^p$-space to an $L^q$-space. We will be particularly interested in the case of left-invariant homogeneous linear operators. In the last section, we will focus our attention on such operators which are furthermore differential and on the possible existence of their fundamental solutions. As an application, we will give a version of Liouville's Theorem which holds on homogeneous Lie groups. All these results have well-known Euclidean counterparts.

All the operators we consider here will be linear so we will not emphasise their linearity in every statement.

### 3.2.1 Left-invariant operators on homogeneous Lie groups

The Schwartz kernel theorem (see Theorem 1.4.1) says that, under very mild hypothesis, an operator on a smooth manifold has an integral representation. An easy consequence is that a left-invariant operator on a Lie group has a convolution kernel.

**Corollary 3.2.1** (Kernel theorem on Lie groups)**.** *We have the following statements.*

- *Let $G$ be a connected Lie group and let $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ be a continuous linear operator which is invariant under left-translations, i.e.*

$$\forall x_o \in G,\ f \in \mathcal{D}(G) \qquad T(f(x_o\,\cdot)) = (Tf)(x_o\,\cdot).$$

*Then there exists a unique distribution $\kappa \in \mathcal{D}'(G)$ such that*

$$Tf_1 : x \longmapsto f_1 * \kappa(x) = \int_G \kappa(y^{-1}x) f_1(y) dy.$$

*In other words, $T$ is a convolution operator with (right convolution) kernel $\kappa$. The converse is also true.*

- *Let $G$ be a connected simply connected nilpotent Lie group identified with $\mathbb{R}^n$ endowed with a polynomial law (see Proposition 1.6.6). Let $T : \mathcal{S}(G) \to \mathcal{S}'(G)$ be a continuous linear operator which is invariant under left translations, i.e.*

$$\forall x_o \in G,\, f \in \mathcal{S}(G) \qquad T(f(x_o\,\cdot)) = (Tf)(x_o\,\cdot).$$

*Then there exists a unique distribution $\kappa \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$Tf_1 : x \longmapsto f_1 * \kappa(x) = \int_G \kappa(y^{-1}x) f_1(y) dy.$$

*In other words, $T$ is a convolution operator with (right convolution) kernel $\kappa$. The converse is also true.*

*In both cases, for any test function $f_1$, the function $Tf_1$ is smooth. Furthermore, the map $\kappa \mapsto T$ is an isomorphism of topological vector spaces.*

A similar statement holds for right-invariant operators.

We omit the proof: it relies on approaching the kernels $\kappa(x, y)$ by continuous functions for which the invariance forces them to be of the form $\kappa(y^{-1}x)$. The converses are much easier and have been shown in Section 1.5.

In this monograph, we will often use the following notation:

**Definition 3.2.2.** Let $T$ be an operator on a connected Lie group $G$ which is continuous as an operator $\mathcal{D}(G) \to \mathcal{D}'(G)$ or as $\mathcal{S}(G) \to \mathcal{S}'(G)$. Its right convolution kernel $\kappa$, as given in Corollary 3.2.1, is denoted by

$$T\delta_0 = \kappa.$$

In the case of left-invariant differential operators, we obtain easily the following properties.

**Proposition 3.2.3.** *If $T$ is a left-invariant differential operator on a connected Lie group $G$, then its kernel is by definition the distribution $T\delta_0 \in \mathcal{D}'(G)$ such that*

$$\forall \phi \in \mathcal{D}(G) \qquad T\phi = \phi * T\delta_0.$$

*The distribution $T\delta_0 \in \mathcal{S}'(G)$ is supported at the origin. The equality*

$$f * T\delta_0 = Tf$$

*holds for any $f \in \mathcal{E}'(G)$, the left-hand side being the group convolution of a distribution with a compactly supported distribution. The equality*

$$T\delta_0 * f = \tilde{T}f$$

*for the right-invariant differential operator corresponding to $T$ also holds for any $f \in \mathcal{E}'(G)$.*

*The kernel of $T^t\delta_0$ is given formally by*

$$T^t\delta_0(x) = T\delta_0(x^{-1}).$$

*If $T = X^\ell$, for a left-invariant vector field $X$ on $G$ and $\ell \in \mathbb{N}$, then the distribution $(-1)^\ell X^\ell \delta_0(x^{-1})$ is the left convolution kernel of the right-invariant differential operator $\tilde{T}$.*

We can also see from (1.14) and Definition 1.5.4 that the adjoint of the bounded on $L^2(G)$ operator $Tf = f * \kappa$ is the convolution operator $T^*f = f * \tilde{\kappa}$, well defined on $\mathcal{D}(G)$, with the right convolution kernel given by

$$\tilde{\kappa}(x) = \bar{\kappa}(x^{-1}). \tag{3.47}$$

The transpose operation is defined in Definition A.1.5, and for left-invariant differential operators it takes the form given by (1.10). Clearly the transpose of a left-invariant differential operator on $G$ is a left-invariant differential operator on $G$.

*Proof.* A left-invariant differential operator is necessarily continuous as $\mathcal{D}(G) \to \mathcal{D}(G)$. Hence it admits the kernel $T\delta_0$. We have for $\phi \in \mathcal{D}(G)$ with $\tilde{\phi}(x) = \phi(x^{-1})$ that

$$\langle T\delta_0, \tilde{\phi}\rangle = (\phi * T\delta_0)(0) = T\phi(0).$$

So if $0 \notin \operatorname{supp}\phi$ then $\langle T\delta_0, \phi\rangle = 0$. This shows that $T\delta_0$ is supported at 0.

If $\phi, \psi \in \mathcal{D}(G)$, then

$$\langle \phi * T\delta_0, \psi\rangle = \langle T\phi, \psi\rangle = \langle \phi, T^t\psi\rangle = \langle \phi, \psi * T^t\delta_0\rangle.$$

By (1.14) this shows that $T^t\delta_0 = (T\delta_0)\tilde{}$. Furthermore, if $f \in \mathcal{D}'(G)$, then

$$\langle Tf, \phi\rangle = \langle f, T^t\phi\rangle = \langle f, \phi * T^t\delta_0\rangle = \langle f, \phi * (T\delta_0)\tilde{}\rangle = \langle f * T\delta_0, \phi\rangle.$$

This shows $Tf = f * T\delta_0$.

Now we can check easily (see (1.11)) that

$$\tilde{X}f = -(X\tilde{f})\tilde{}$$

and, more generally,

$$\tilde{X}^\ell f = (-1)^\ell (X^\ell \tilde{f})\tilde{}$$

for $\ell \in \mathbb{N}$. Since the equality $(f * g)\check{} = \tilde{g} * \tilde{f}$ holds as long as it makes sense, this shows that

$$(-1)^\ell (X^\ell \delta_0)\check{} * f = \tilde{T} f.$$

□

In fact, our primary concern will be to study operators of a different nature, and their possible extensions to some $L^p$-spaces. This (i.e. the $L^p$-boundedness) is certainly not the case for general differential operators.

Assuming that an operator is continuous as $\mathcal{S}(G) \to \mathcal{S}'(G)$ or as $\mathcal{D}(G) \to \mathcal{D}'(G)$ is in practice a very mild hypothesis. It ensures that a potential extension into a bounded operator $L^p(G) \to L^q(G)$ is necessarily unique, by density of $\mathcal{D}(G)$ in $L^p(G)$. Hence we may abuse the notation, and keep the same notation for an operator which is continuous as $\mathcal{S}(G) \to \mathcal{S}'(G)$ or as $\mathcal{D}(G) \to \mathcal{D}'(G)$ and its possible extension, once we have proved that it gives a bounded operator from $L^p(G)$ to $L^q(G)$.

We want to study in the context of homogeneous Lie groups the condition which implies that an operator as above extends to a bounded operator from $L^p(G)$ to $L^q(G)$.

As the next proposition shows, only the case $p \leq q$ is interesting.

**Proposition 3.2.4.** *Let $G$ be a homogeneous Lie group and let $T$ be a linear left-invariant operator bounded from $L^p(G)$ to $L^q(G)$, for some (given) finite $p, q \in [1, \infty)$. If $p > q$ then $T = 0$.*

The proof is based on the following lemma:

**Lemma 3.2.5.** *Let $f \in L^p(G)$ with $1 \leq p < \infty$. Then*

$$\lim_{x \to \infty} \|f - f(x\,\cdot)\|_{L^p(G)} = 2^{\frac{1}{p}} \|f\|_{L^p(G)}.$$

*Proof of Lemma 3.2.5.* First let us assume that the function $f$ is continuous with compact support $E$. For $x_o \in G$, the function $f(x_o\,\cdot)$ is continuous and supported in $x_o^{-1}E$. Therefore, if $x_o$ is not in $EE^{-1} = \{yz \ : \ y \in E, \ z \in E^{-1}\}$, then $f$ and $f(x_o\,\cdot)$ have disjoint supports, and

$$\|f - f(x_o\,\cdot)\|_p^p = \int_E |f|^p + \int_{x_o^{-1}E} |f(x_o\,\cdot)|^p = 2\|f\|_p^p.$$

Now we assume that $f \in L^p(G)$. For each sufficiently small $\epsilon > 0$, let $f_\epsilon$ be a continuous function with compact support $E_\epsilon \subset \{|x| \leq \epsilon^{-1}\}$ satisfying $\|f - f_\epsilon\|_p < \epsilon$. We claim that for any sufficiently small $\epsilon > 0$, we have

$$|x_o| > 2\epsilon^{-1} \Longrightarrow \Big| \|f - f(x_o\,\cdot)\|_p - 2^{\frac{1}{p}} \|f\|_p \Big| \leq (2 + 2^{\frac{1}{p}})\epsilon. \tag{3.48}$$

Indeed, using the triangle inequality, we obtain

$$\left| \|f - f(x_o \,\cdot)\|_p - 2^{\frac{1}{p}} \|f\|_p \right| \le \left| \|f - f(x_o \,\cdot)\|_p - 2^{\frac{1}{p}} \|f_\epsilon\|_p \right| + 2^{\frac{1}{p}} \left| \|f_\epsilon\|_p - \|f\|_p \right|.$$

For the last term of the right-hand side we have

$$\left| \|f_\epsilon\|_p - \|f\|_p \right| \le \|f_\epsilon - f\|_p < \epsilon,$$

whereas for the first term, if $x_o \notin E_\epsilon E_\epsilon^{-1}$, using the first part of the proof and then the triangle inequality, we get

$$\begin{aligned} \left| \|f - f(x_o \,\cdot)\|_p - 2^{\frac{1}{p}} \|f_\epsilon\|_p \right| &= \left| \|f - f(x_o \,\cdot)\|_p - \|f_\epsilon - f_\epsilon(x_o \,\cdot)\|_p \right| \\ &\le \|(f - f(x_o \,\cdot)) - (f_\epsilon - f_\epsilon(x_o \,\cdot))\|_p \\ &\le \|f - f_\epsilon\|_p + \|f(x_o \,\cdot) - f_\epsilon(x_o \,\cdot)\|_p < 2\epsilon. \end{aligned}$$

This shows (3.48) and concludes the proof of Lemma 3.2.5. □

*Proof of Proposition 3.2.4.* Let $f \in \mathcal{D}(G)$. As $T$ is left-invariant, we have

$$\|(Tf)(x_o \,\cdot) - Tf\|_q = \left\|T\big(f(x_o \,\cdot) - f\big)\right\|_q \le \|T\|_{\mathscr{L}(L^p(G),L^q(G))} \|f(x_o \,\cdot) - f\|_p \,.$$

Taking the limits as $x_o$ tends to infinity, by Lemma 3.2.5, we get

$$2^{\frac{1}{q}} \|Tf\|_q \le \|T\|_{\mathscr{L}(L^p(G),L^q(G))} 2^{\frac{1}{p}} \|f\|_p \,.$$

But then

$$\|T\|_{\mathscr{L}(L^p(G),L^q(G))} \le 2^{\frac{1}{p}-\frac{1}{q}} \|T\|_{\mathscr{L}(L^p(G),L^q(G))}.$$

Hence $p > q$ implies $\|T\|_{\mathscr{L}(L^p(G),L^q(G))} = 0$ and $T = 0$. □

As in the Euclidean case, Proposition 3.2.4 is all that can be proved in the general framework of left-invariant bounded operators from $L^p(G)$ to $L^q(G)$. However, if we add the property of homogeneity more can be said and we now focus our attention on this case.

### 3.2.2 Left-invariant homogeneous operators

The next statement says that if the operator $T$ is left-invariant, homogeneous and bounded from $L^p(G)$ to $L^q(G)$, then the indices $p$ and $q$ must be related in the same way as in the Euclidean case but with the topological dimension being replaced by the homogeneous dimension $Q$.

**Proposition 3.2.6.** *Let $T$ be a left-invariant linear operator on $G$ which is bounded from $L^p(G)$ to $L^q(G)$ for some (given) finite $p, q \in [1, \infty)$. If $T$ is homogeneous of degree $\nu \in \mathbb{C}$ (and $T \neq 0$), then*

$$\frac{1}{q} - \frac{1}{p} = \frac{\operatorname{Re} \nu}{Q}.$$

*Proof.* We compute easily,

$$\|f \circ D_t\|_p = t^{-\frac{Q}{p}} \|f\|_p, \quad f \in L^p(G),\ t > 0.$$

Thus, since $T$ is homogeneous of degree $\nu$, we have

$$\begin{aligned} t^{\mathrm{Re}\,\nu - \frac{Q}{q}} \|Tf\|_q = \|t^\nu \big(Tf\big) \circ D_t\|_q = \|T\big(f \circ D_t\big)\|_q \leq \|T\|_{\mathscr{L}(L^p(G),L^q(G))} \|f \circ D_t\|_p \\ = \|T\|_{\mathscr{L}(L^p(G),L^q(G))} t^{-\frac{Q}{p}} \|f\|_p, \end{aligned}$$

so

$$\forall t > 0 \qquad \|T\|_{\mathscr{L}(L^p(G),L^q(G))} \leq t^{-\mathrm{Re}\,\nu + \frac{Q}{q} - \frac{Q}{p}} \|T\|_{\mathscr{L}(L^p(G),L^q(G))}.$$

Hence we must have

$$-\mathrm{Re}\,\nu + \frac{Q}{q} - \frac{Q}{p} = 0$$

as claimed. □

Combining together Propositions 3.2.4 and 3.2.6, we see that it makes sense to restrict one's attention to

$$\frac{\mathrm{Re}\,\nu}{Q} \in (-1, 0].$$

The case $\mathrm{Re}\,\nu = 0$ is the most delicate and we leave it aside for the moment (see Section 3.2.5). We shall discuss instead the case

$$-Q < \mathrm{Re}\,\nu < 0.$$

Let us observe that the homogeneity of the operator is equivalent to the homogeneity of its kernel:

**Lemma 3.2.7.** *Let $T$ be a continuous left-invariant linear operator as $\mathcal{S}(G) \to \mathcal{S}'(G)$ or as $\mathcal{D}(G) \to \mathcal{D}'(G)$, where $G$ is a homogeneous Lie group. Then $T$ is $\nu$-homogeneous if and only if its (right) convolution kernel is $-(Q+\nu)$-homogeneous.*

*Proof.* On one hand we have

$$T(f(r\,\cdot))(x) = \int_G f(ry)\kappa(y^{-1}x)dy,$$

and on the other hand,

$$\begin{aligned} Tf\,(rx) &= \int_G f(z)\kappa(z^{-1}rx)dz = \int_G f(ry)\kappa((ry)^{-1}rx)r^Q dy \\ &= r^Q \int_G f(ry)(\kappa \circ D_r)(y^{-1}x)dy. \end{aligned}$$

Now the statement follows from these and the uniqueness of the kernel. □

The following proposition gives a sufficient condition on the homogeneous kernel so that the corresponding left-invariant homogeneous operator extends to a bounded operator from $L^p(G)$ to $L^q(G)$.

**Proposition 3.2.8.** *Let $T$ be a linear continuous operator as $\mathcal{S}(G) \to \mathcal{S}'(G)$ or as $\mathcal{D}(G) \to \mathcal{D}'(G)$ on a homogeneous Lie group $G$. We assume that the operator $T$ is left-invariant and homogeneous of degree $\nu$, that*

$$\operatorname{Re}\nu \in (-Q, 0),$$

*and that the (right convolution) kernel $\kappa$ of $T$ is continuous away from the origin.*

*Then $T$ extends to a bounded operator from $L^p(G)$ to $L^q(G)$ whenever $p, q \in (1, \infty)$ satisfy*

$$\frac{1}{q} - \frac{1}{p} = \frac{\operatorname{Re}\nu}{Q}.$$

*The integral kernel $\kappa$ then can also be identified with a locally integrable function at the origin.*

We observe that, by Corollary 3.2.1, $\kappa$ is a distribution (in $\mathcal{S}'(G)$ or $\mathcal{D}'(G)$) on $G$. The hypothesis on $\kappa$ says that its restriction to $G\backslash\{0\}$ coincides with a continuous function $\kappa_o$ on $G\backslash\{0\}$.

*Proof of Proposition 3.2.8.* We fix a homogeneous norm $|\cdot|$ on $G$. We denote by $\bar{B}_R := \{x : |x| \leq R\}$ and $\mathfrak{S} := \{x : |x| = 1\}$ the ball of radius $R$ and the unit sphere around 0. By Lemma 3.2.7, $\kappa_o$ is a continuous homogeneous function of degree $-(Q+\nu)$ on $G\backslash\{0\}$. Denoting by $C$ its maximum on the unit sphere, we have

$$\forall x \in G\backslash\{0\} \qquad |\kappa_o(x)| \leq \frac{C}{|x|^{Q+\operatorname{Re}\nu}}.$$

Hence $\kappa_o$ defines a locally integrable function on $G$, even around 0, and we keep the same notation for this function. Therefore, the distribution $\kappa' = \kappa - \kappa_o$ on $G$ is, in fact, supported at the origin. It is also homogeneous of degree $-Q-\nu$. Due to the compact support of $\kappa'$, $|\langle \kappa', f\rangle|$ is controlled by some $C^k$ norm of $f$ on a fixed small neighbourhood of the origin. But, because of its homogeneity, and using (3.9), we get

$$\forall t > 0 \qquad \langle \kappa', f\rangle = t^{-Q-\nu}\langle \kappa' \circ D_{\frac{1}{t}}, f\rangle = t^{-\nu}\langle \kappa', f \circ D_t\rangle.$$

Letting $t$ tend to 0, the $C^k$ norms of $f \circ D_t$ remain bounded, so that $\langle \kappa', f\rangle = 0$ since $\operatorname{Re}\nu < 0$. This shows that $\kappa' = 0$ and so $\kappa = \kappa_o$.

Note that the weak $L^r(G)$-norm of $\kappa$ is finite for $r = Q/(Q + \operatorname{Re}\nu)$. Indeed, if $s > 0$,

$$|\kappa_o(x)| > s \Longrightarrow |x|^{Q+\operatorname{Re}\nu} \leq \frac{C}{s},$$

so that

$$|\{x \, : \, |\kappa_o(x)| > s\}| \leq \left| B_{(C/s)^{\frac{1}{Q+\mathrm{Re}\,\nu}}} \right| \leq c \left( \frac{C}{s} \right)^{\frac{Q}{Q+\mathrm{Re}\,\nu}},$$

with $c = |B_1|$, and hence

$$\|\kappa_o\|_{w-L^r(G)} \leq c\, C^{\frac{Q}{Q+\mathrm{Re}\,\nu}} \quad \text{with } r = \frac{Q}{Q+\mathrm{Re}\,\nu}.$$

The proposition is now easy using the generalisation of Young's inequalities (see Proposition 1.5.2), so that we get that $T$ is bounded from $L^p(G)$ to $L^q(G)$ for

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{r} - 1 = \frac{\mathrm{Re}\,\nu}{Q},$$

as claimed. □

We may use the usual vocabulary for homogeneous kernels as in [Fol75] and [FS82]:

**Definition 3.2.9.** Let $G$ be a homogeneous Lie group and let $\nu \in \mathbb{C}$.

A distribution $\kappa \in \mathcal{D}'(G)$ which is smooth away from the origin and homogeneous of degree $\nu - Q$ is called a *kernel of type* $\nu$ on $G$.

A (right) convolution operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ whose convolution kernel is of type $\nu$ is called an *operator of type* $\nu$. That is, $T$ is given via

$$T(\phi) = \phi * \kappa,$$

where $\kappa$ kernel of type $\nu$.

*Remark* 3.2.10. We will mainly be interested in the $L^p \to L^q$-boundedness of operators of type $\nu$. Thus, by Propositions 3.2.4 and 3.2.6, we will restrict ourselves to $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu \in [0, Q)$.

If $\mathrm{Re}\,\nu \in (0, Q)$, then a $(\nu - Q)$-homogeneous function in $C^\infty(G\backslash\{0\})$ is integrable on a neighbourhood of 0 and hence extends to a distribution in $\mathcal{D}'(G)$, see the proof of Proposition 3.2.8. Hence, in the case $\mathrm{Re}\,\nu \in (0, Q)$, the restriction to $G\backslash\{0\}$ yields a one-to-one correspondence between the $(\nu - Q)$-homogeneous functions in $C^\infty(G\backslash\{0\})$ and the kernels of type $\nu$.

We will see in Remark 3.2.29 that the case $\mathrm{Re}\,\nu = 0$ is more subtle.

In view of Lemma 3.2.7 and Proposition 3.2.8, we have the following statement for operators of type $\nu$ with $\mathrm{Re}\,\nu \in (0, Q)$.

**Corollary 3.2.11.** *Let $G$ be a homogeneous Lie group and let $\nu \in \mathbb{C}$ with*

$$\mathrm{Re}\,\nu \in (0, Q).$$

*Any operator of type $\nu$ is $(-\nu)$-homogeneous and extends to a bounded operator from $L^p(G)$ to $L^q(G)$ whenever $p, q \in (1, \infty)$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{\mathrm{Re}\,\nu}{Q}.$$

As we said earlier the case of a left-invariant operator which is homogeneous of degree 0 is more complicated and is postponed until the end of Section 3.2.4. In the meantime, we make a useful parenthesis about the Calderón-Zygmund theory in our context.

### 3.2.3 Singular integral operators on homogeneous Lie groups

In the case of $\mathbb{R}$, a famous example of a left-invariant 0-homogeneous operator is the Hilbert transform. This particular example has motivated the development of the theory of singular integrals in the Euclidean case as well as in other more general settings. In Section A.4, the interested reader will find a brief presentation of this theory in the setting of spaces of homogeneous type (due to Coifman and Weiss). In this section here, we check that homogeneous Lie groups are spaces of homogeneous type and we obtain the corresponding theorem of singular integrals together with some useful consequences for left-invariant operators. We also propose a definition of Calderón-Zygmund kernels on homogeneous Lie groups, thereby extending the one on Euclidean spaces (cf. Section A.4).

First let us check that homogeneous Lie groups equipped with a quasi-norm are spaces of homogeneous type in the sense of Definition A.4.2 and that the Haar measure is doubling (see Section A.4):

**Lemma 3.2.12.** *Let $G$ be a homogeneous Lie groups and let $|\cdot|$ be a quasi-norm. Then the set $G$ endowed with the usual Euclidean topology together with the quasi-distance*

$$d : (x, y) \mapsto |y^{-1}x|$$

*is a space of homogeneous type and the Haar measure has the doubling property given in (A.5).*

*Proof of Lemma 3.2.12.* We keep the notation of the statement. The defining properties of a quasi-norm and the fact that it satisfies the triangular inequality up to a constant (see Proposition 3.1.38) imply easily that $d$ is indeed a quasi-distance on $G$ in the sense of Definition A.4.1. By Proposition 3.1.37, the corresponding quasi-balls $B(x,r) := \{y \in G : d(x,y) < r\}$, $x \in G$, $r > 0$, generate the usual topology of the underlying Euclidean space. Hence the first property listed in Definition A.4.2 is satisfied.

By Remark 3.1.34, the quasi-balls satisfy $B(x,r) = xB(0,r)$ and $B(0,r) = D_r(B(0,1))$. By (3.6), the volume of $B(0,r)$ is $|B(0,r)| = r^Q|B(0,1)|$. Hence we have obtained that the volume of any open quasi-ball is $|B(x,r)| = r^Q|B(0,1)|$. This implies that the Haar measure satisfies the doubling condition given in (A.5). We can now conclude the proof of the statement with Lemma A.4.3. □

Lemma 3.2.12 implies that we can apply the theorem of singular integrals on spaces of homogeneous type recalled in Theorem A.4.4 and we obtain:

**Theorem 3.2.13** (Singular integrals)**.** *Let $G$ be a homogeneous Lie group and let $T$ be a bounded linear operator on $L^2(G)$, i.e.*

$$\exists C_o \qquad \forall f \in L^2 \quad \|Tf\|_2 \leq C_o \|f\|_2. \tag{3.49}$$

*We assume that the integral kernel $\kappa$ of $T$ coincides with a locally integrable function away from the diagonal, that is, on $(G \times G)\backslash\{(x,y) \in G \times G : x = y\}$. We also assume that there exist $C_1, C_2 > 0$ satisfying*

$$\forall y, y_o \in G \qquad \int_{|y_o^{-1}x| > C_1 |y_o^{-1}y|} |\kappa(x,y) - \kappa(x,y_o)| dx \leq C_2, \tag{3.50}$$

*for a quasi-norm $|\cdot|$.*

*Then for all $p$, $1 < p \leq 2$, $T$ extends to a bounded operator on $L^p$ because*

$$\exists A_p > 0 \qquad \forall f \in L^2 \cap L^p \quad \|Tf\|_p \leq A_p \|f\|_p;$$

*for $p = 1$, the operator $T$ extends to a weak-type (1,1) operator since*

$$\exists A_1 > 0 \qquad \forall f \in L^2 \cap L^1 \quad \mu\{x \ : \ |Tf(x)| > \alpha\} \leq A_1 \frac{\|f\|_1}{\alpha};$$

*the constants $A_p$, $1 \leq p \leq 2$, depend only on $C_o$, $C_1$ and $C_2$.*

*Remark* 3.2.14.

- The $L^2$-boundedness, that is, Condition (3.49), implies that the operator satisfies the Schwartz kernel theorem (see Theorem 1.4.1) and thus yields the existence of a distributional integral kernel. We still need to assume that this distribution is locally integrable away from the diagonal.
- Since any two quasi-norms on $G$ are equivalent (see Proposition 3.1.35), if the kernel condition in (3.50) holds for one quasi-norm, it then holds for any quasi-norm (maybe with different constants $C_1, C_2$).

As recalled in Section A.4, the notion of Calderón-Zygmund kernels in the Euclidean setting appear naturally as sufficient conditions (often satisfied 'in practice') for (A.7) to be satisfied by the kernel of the operator and the kernel of its formal adjoint. This leads us to define the Calderón-Zygmund kernels in our setting as follows:

**Definition 3.2.15.** A *Calderón-Zygmund kernel on a homogeneous Lie group $G$* is a measurable function $\kappa_o$ defined on $(G \times G)\backslash\{(x,y) \in G \times G : x = y\}$ satisfying for some $\gamma$, $0 < \gamma \leq 1$, $C_1 > 0$, $A > 0$, and a homogeneous quasi-norm $|\cdot|$ the inequalities

$$\begin{aligned}
|\kappa_o(x,y)| &\leq A|y^{-1}x|^{-Q}, \\
|\kappa_o(x,y) - \kappa_o(x',y)| &\leq A\frac{|x^{-1}x'|^\gamma}{|y^{-1}x|^{Q+\gamma}} \qquad \text{if } C_1|x^{-1}x'| \leq |y^{-1}x|, \\
|\kappa_o(x,y) - \kappa_o(x,y')| &\leq A\frac{|y^{-1}y'|^\gamma}{|y^{-1}x|^{Q+\gamma}} \qquad \text{if } C_1|y^{-1}y'| \leq |y^{-1}x|.
\end{aligned}$$

A linear continuous operator $T$ as $\mathcal{D}(G) \to \mathcal{D}'(G)$ or as $\mathcal{S}(G) \to \mathcal{S}'(G)$ is called a *Calderón-Zygmund operator* if its integral kernel coincides with a Calderón-Zygmund kernel on $(G \times G)\backslash\{(x,y) \in G \times G : x = y\}$.

*Remark* 3.2.16. 1. In other words, we have modified the definition of a classical Calderón-Zygmund kernel (as in Section A.4)

- by replacing the Euclidean norm by a homogeneous quasi-norm
- and, more importantly, the topological (Euclidean) dimension of the underlying space $n$ by the homogeneous dimension $Q$.

2. By equivalence of homogeneous quasi-norms, see Proposition 3.1.35, the definition does not depend on a particular choice of a homogeneous quasi-norm as we can change the constants $C_1, A$.

As in the Euclidean case, we have

**Proposition 3.2.17.** *Let $G$ be a homogeneous Lie group and let $T$ be a bounded linear operator on $L^2(G)$.*

*If $T$ is a Calderón-Zygmund operator on $G$ (in the sense of Definition 3.2.15), then $T$ is bounded on $L^p(G)$, $p \in (1,\infty)$, and weak-type (1,1).*

*Proof of Proposition 3.2.17.* Let $T$ be a bounded operator on $L^2(G)$ and $\kappa : (x,y) \mapsto \kappa(x,y)$ its distributional kernel. Then its formal adjoint $T^*$ is also bounded on $L^2(G)$ with the same operator norm. Furthermore its distributional kernel is $\kappa^{(*)} : (x,y) \mapsto \bar{\kappa}(y,x)$. We assume that $\kappa$ coincides with a Calderón-Zygmund kernel $\kappa_o$ away from the diagonal. We fix a quasi-norm $|\cdot|$. The first inequality in Definition 3.2.15 shows that $\kappa_o$ and $\kappa_o^{(*)}$ coincide with locally integrable functions away from the diagonal. Using the last inequality, we have for any $y, y_o \in G$,

$$\int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|} |\kappa_o(x,y) - \kappa_o(x,y_o)|dx \leq A \int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|} \frac{|y^{-1}y_o|^\gamma}{|y_o^{-1}x|^{Q+\gamma}} dx$$

and, using the change of variable $x' = y_o^{-1}x$, we have

$$\begin{aligned}
\int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|} \frac{1}{|y_o^{-1}x|^{Q+\gamma}} dx &= \int_{|x'|\geq C_1|y_o^{-1}y|} |x'|^{-(Q+\gamma)} dx' \\
&\leq \int_{|x'|\geq C_1|y_o^{-1}y|} |x'|^{-(Q+\gamma)} dx' \\
&= c \int_{r=C_1|y_o^{-1}y|}^{+\infty} r^{-(Q+\gamma)} r^{Q-1} dr = c_1 |y_o^{-1}y|^{-\gamma},
\end{aligned}$$

having also used the polar coordinates (Proposition 3.1.42) with $c$ denoting the mass of the Borel measure on the unit sphere, and $c_1$ a new constant (of $C_1$, $\gamma$ and $Q$). Hence we have obtained

$$\int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|} |\kappa_o(x,y) - \kappa_o(x,y_o)|dx \leq c_1 A.$$

Similarly for $\kappa_o^{(*)}$, we have

$$\begin{aligned}\int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|}|\kappa_o^{(*)}(x,y)-\kappa_o^{(*)}(x,y_o)|dx &= \int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|}|\kappa_o(y,x)-\kappa_o(y_o,x)|dx\\ &\leq A\int_{|y_o^{-1}x|\geq C_1|y_o^{-1}y|}\frac{|y_o^{-1}y|^\gamma}{|y_o^{-1}x|^{Q+\gamma}}dx,\end{aligned}$$

having used the second inequality in Definition 3.2.15. The same computation as above shows that the last left-hand side is bounded by $c_1 A$. Hence $\kappa_o$ and $\kappa_o^{(*)}$ satisfy (3.50). Proposition 3.2.17 now follows from Theorem 3.2.13. □

*Remark* 3.2.18. As in the Euclidean case, Calderón-Zygmund kernels do not necessarily satisfy the other condition of the $L^2$-boundedness (see (3.49)) and a condition of 'cancellation' is needed in addition to the Calderón-Zygmund condition to ensure the $L^2$-boundedness. Indeed, one can prove adapting the Euclidean case (see the proof of Proposition 1 in [Ste93, ch.VII §3]) that if $\kappa_o$ is a Calderón-Zygmund kernel satisfying the inequality

$$\exists c>0 \qquad \forall x\neq y \qquad \kappa_o(x,y)\geq c|y^{-1}x|^{-Q},$$

then there does not exist an $L^2$-bounded operator $T$ having $\kappa_o$ as its kernel.

The following statement gives sufficient conditions for a kernel to be Calderón-Zygmund in terms of derivatives:

**Lemma 3.2.19.** *Let $G$ be a homogeneous Lie group. If $\kappa_o$ is a continuously differentiable function on $(G\times G)\backslash\{(x,y)\in G\times G : x=y\}$ satisfying the inequalities for any $x,y\in G$, $x\neq y$, $j=1,\ldots,n$,*

$$\begin{aligned}|\kappa_o(x,y)| &\leq A|y^{-1}x|^{-Q},\\ |(X_j)_x\kappa_o(x,y)| &\leq A|y^{-1}x|^{-(Q+\upsilon_j)},\\ |(X_j)_y\kappa_o(x,y)| &\leq A|y^{-1}x|^{-(Q+\upsilon_j)},\end{aligned}$$

*for some constant $A>0$ and homogeneous quasi-norm $|\cdot|$, then $\kappa_o$ is a Calderón-Zygmund kernel in the sense of Definition 3.2.15 with $\gamma=1$.*

Again, if these inequalities are satisfied for one quasi-norm, then they are satisfied for all quasi-norms, maybe with different constants $A>0$.

*Proof of Lemma 3.2.19.* We fix a quasi-norm $|\cdot|$. We assume that it is a norm without loss of generality because of the remark just above and the existence of a homogeneous norm (Theorem 3.1.39); although we could give a proof without this hypothesis, it simplifies the constants below. Let $\kappa_o$ be as in the statement. Using the Taylor expansion (Theorem 3.1.51) or the Mean Value Theorem (Proposition 3.1.46), we have

$$|\kappa_o(x',y)-\kappa_o(x,y)|\leq C_o\sum_{j=1}^n|x^{-1}x'|^{\upsilon_j}\sup_{|z|\leq\eta|x^{-1}x'|}|(X_j)_{x_1=xz}\kappa_o(x_1,y)|.$$

Using the second inequality in the statement, we have

$$\sup_{|z|\leq\eta|x^{-1}x'|}|(X_j)_{x_1=xz}\kappa_o(x_1,y)|\leq A\sup_{|z|\leq\eta|x^{-1}x'|}|y^{-1}xz|^{-(Q+\upsilon_j)}.$$

The reverse triangle inequality yields

$$|y^{-1}xz|\geq|y^{-1}x|-|z|\geq\frac{1}{2}|y^{-1}x|\qquad\text{if } |z|\leq\frac{1}{2}|y^{-1}x|.$$

Hence, if $2\eta|x^{-1}x'|\leq|y^{-1}x|$, then we have

$$\sup_{|z|\leq\eta|x^{-1}x'|}|y^{-1}xz|^{-(Q+\upsilon_j)}\leq 2^{Q+\upsilon_j}|y^{-1}x|^{-(Q+\upsilon_j)},$$

and we have obtained

$$\begin{aligned}|\kappa_o(x,y)-\kappa_o(x',y)| &\leq C_o\sum_{j=1}^n|x^{-1}x'|^{\upsilon_j}2^{Q+\upsilon_j}|y^{-1}x|^{-(Q+\upsilon_j)}\\ &\leq C_o\left(\sum_{j=1}^n(2\eta)^{-(\upsilon_j-1)}2^{Q+\upsilon_j}\right)|x^{-1}x'||y^{-1}x|^{-(Q-1)}.\end{aligned}$$

This shows the second inequality in Definition 3.2.15.

We proceed in a similar way to prove the third inequality in Definition 3.2.15: the Taylor expansion yields

$$|\kappa_o(x,y)-\kappa_o(x,y')|\leq C_o\sum_{j=1}^n|y^{-1}y'|^{\upsilon_j}\sup_{|z|\leq\eta|y^{-1}y'|}|(X_j)_{y_1=yz}\kappa_o(x,y_1)|$$

while one checks easily

$$\begin{aligned}\sup_{|z|\leq\eta|y^{-1}y'|}|(X_j)_{y_1=yz}\kappa_o(x,y_1)| &\leq A\sup_{|z|\leq\eta|y^{-1}y'|}|(yz)^{-1}x|^{-(Q+\upsilon_j)}\\ &\leq A2^{Q+\upsilon_j}|y^{-1}x|^{-(Q+\upsilon_j)},\end{aligned}$$

when $2\eta|y^{-1}y'|\leq|y^{-1}x|$. We conclude in the same way as above and this shows that $\kappa_o$ is a Calderón-Zygmund kernel. □

**Corollary 3.2.20.** *Let $G$ be a homogeneous Lie group and let $\kappa$ be a continuously differentiable function on $G\backslash\{0\}$. If $\kappa$ satisfies for any $x\in G\backslash\{0\}$, $j=1,\ldots,n$,*

$$\begin{aligned}|\kappa(x)| &\leq A|x|^{-Q},\\ |X_j\kappa(x)| &\leq A|x|^{-(Q+\upsilon_j)},\\ |\tilde{X}_j\kappa(x)| &\leq A|x|^{-(Q+\upsilon_j)},\end{aligned}$$

*for some constant $A>0$ and homogeneous quasi-norm $|\cdot|$, then*

$$\kappa_o:(x,y)\mapsto\kappa(y^{-1}x)$$

*is a Calderón-Zygmund kernel in the sense of Definition 3.2.15 with $\gamma=1$.*

Corollary 3.2.20 will be useful when dealing with convolution kernels which are smooth away from the origin, in particular when they are also $(-Q)$-homogeneous, see Theorem 3.2.30.

*Proof of Corollary 3.2.20.* Keeping the notation of the statement, using properties (1.11) of left and right invariant vector fields, we have

$$\begin{aligned}(X_j)_x\kappa_o(x,y) &= (X_j\kappa)(y^{-1}x),\\ (X_j)_y\kappa_o(x,y) &= -(\tilde{X}_j\kappa)(y^{-1}x).\end{aligned}$$

The statement now follows easily from Lemma 3.2.19. □

Often, the convolution kernel decays quickly enough at infinity and the main singularity to deal with is about the origin. The next statement is an illustration of this idea:

**Corollary 3.2.21.** *Let $G$ be a homogeneous Lie group and let $T$ be a linear operator which is bounded on $L^2(G)$ and invariant under left translations.*

*We assume that its distributional convolution kernel coincides on $G\backslash\{0\}$ with a continuously differentiable function $\kappa$ which satisfies*

$$\begin{aligned}\int_{|x|\geq 1/2}|\kappa(x)|dx &\leq A,\\ \sup_{0<|x|\leq 1}|x|^Q|\kappa(x)| &\leq A,\\ \sup_{0<|x|\leq 1}|x|^{Q+\upsilon_j}|X_j\kappa(x)| &\leq A, \quad j=1,\ldots,n,\end{aligned}$$

*for some constant $A>0$ and a homogeneous quasi-norm $|\cdot|$. Then $T$ is bounded on $L^p(G)$, $p\in(1,\infty)$, and is weak-type (1,1).*

*Proof.* Let $\chi\in\mathcal{D}(G)$ be $[0,1]$-valued function such that $\chi\equiv 0$ on $\{|x|\geq 1\}$ and $\chi\equiv 1$ on $\{|x|\leq 1/2\}$. As $\int_{|x|\geq 1/2}|\kappa(x)|dx$ is finite, $(1-\chi)\kappa$ is integrable and the convolution operator with convolution kernel $(1-\chi)\kappa$ is bounded on $L^p(G)$ for $p\in[1,\infty]$. Hence it suffices to prove that the kernel $\kappa_o$ given via $\kappa_o(x,y)=(\chi\kappa)(y^{-1}x)$ is Calderón-Zygmund.

From the estimates satisfied by $\kappa$, it is clear that the quantities

$$\sup_{x\in G\backslash\{0\}}|x|^Q|(\chi\kappa)(x)| \quad \text{and} \quad \sup_{x\in G\backslash\{0\}}|x|^{-(Q+\upsilon_j)}|X_j(\chi\kappa)(x)|$$

are finite. As each $\tilde{X}_j$ may be expressed as a combination of $X_k$ with homogeneous polynomial coefficients, see Section 3.1.5, we have for any (regular enough) function $f$ with compact support

$$\sup_{x\in G\backslash\{0\}}|x|^{-(Q+\upsilon_j)}|\tilde{X}_jf(x)| \leq C\sup_{\substack{x\in G\backslash\{0\}\\ k=1,\ldots,n}}|x|^{-(Q+\upsilon_k)}|X_kf(x)|.$$

Consequently, the quantities $\sup_{x\in G\setminus\{0\}} |x|^{-(Q+\upsilon_j)}|\tilde{X}_j(\chi\kappa)(x)|$ are also bounded. Applying Lemma 3.2.19 to $\kappa_o$ defined above, one checks easily that it is a Calderón-Zygmund kernel. Applying Proposition 3.2.17 concludes the proof of Corollary 3.2.21. □

This closes our parenthesis about the Calderón-Zygmund theory in our context, and we can go back to the study of left-invariant homogeneous operators, this time of homogeneous degree 0.

### 3.2.4 Principal value distribution

As we will see in the sequel, many interesting operators for our analysis on a homogeneous Lie group $G$ will be given by convolution operators with (right convolution distributional) kernels homogeneous of degree $\nu$ with $\operatorname{Re}\nu = -Q$. In most of the 'interesting' cases, the distribution $\kappa$ will be given by a locally integrable function away from the origin; denoting by $\kappa_o$ the restriction of $\kappa$ to $G\setminus\{0\}$, one may wonder if there is a one-to-one correspondence between $\kappa$ and $\kappa_o$. As in the Euclidean case, this leads to the notion of the principal value distribution and we adapt the ideas here to fit the homogeneous context; in particular, the topological (Euclidean) dimension is replaced by the homogeneous dimension $Q$.

*So the question is:* Considering a locally integrable function $\kappa_o$ on $G\setminus\{0\}$ which is homogeneous of degree $\nu$ with $\operatorname{Re}\nu = -Q$, does there exist a distribution $\kappa \in \mathcal{D}'(G)$ on $G$, homogeneous of the same degree $\nu$ on $G$, whose restriction to $G\setminus\{0\}$ coincides with $\kappa_o$? that is,

$$\langle \kappa, f\rangle = \int_{G\setminus\{0\}} \kappa_o(x) f(x) dx,$$

whenever $f \in \mathcal{D}(G)$ and $0 \notin \operatorname{supp} f$. In other words, can the functional

$$\mathcal{D}(\mathbb{R}^n\setminus\{0\}) \ni f \longmapsto \int_{G\setminus\{0\}} \kappa_o(x) f(x) dx$$

be extended to a continuous functional on $\mathcal{D}(\mathbb{R}^n)$?

*Remark* 3.2.22. 1. We observe that if such an extension exists, it is not unique in general. For $\nu = -Q$, the reason is that the Dirac $\delta_0$ at the origin is homogeneous of degree $-Q$ (see Example 3.1.20), so that if $\kappa$ is a solution, then $\kappa + c\delta_0$ for any constant $c$ is another solution. (However, see Proposition 3.2.27.)

2. The second observation is that the answer is negative in general:

*Example* 3.2.23. Let $|\cdot|$ be some fixed homogeneous quasi-norm on $G$ smooth away from the origin. The function defined by $\kappa_o(x) = |x|^\nu$ with $\nu = -Q + i\tau$, $\tau \in \mathbb{R}$, is homogeneous of degree $\nu$ on $G\setminus\{0\}$ but can not be extended into a homogeneous distribution $\kappa \in \mathcal{D}'(G)$ of homogeneous degree $\nu$.

*Proof of Example 3.2.23.* Indeed, let us assume that such a distribution $\kappa$ exists for this $\kappa_o$. Homogeneity of degree $-Q+i\tau$ means that

$$\langle \kappa, \psi \circ D_t \rangle = t^{-i\tau} \langle \kappa, \psi \rangle, \quad t>0,\ \psi \in \mathcal{D}(G).$$

Let $B_\delta := \{x \in G : |x| < \delta\}$ be the ball around 0 of radius $\delta$. Let $\phi \in \mathcal{D}(G)$ be a real-valued function supported on $D_2(B_\delta)\backslash B_\delta$, such that

$$\int_G (\phi(x) - \phi(2x))\, |x|^{-Q} dx \neq 0.$$

We now define

$$\psi(x) := |x|^{-i\tau}\phi(x) \quad \text{and} \quad f := \psi - 2^{i\tau}(\psi \circ D_2),\ x \in G\backslash\{0\}.$$

Immediately we notice that

$$f(x) = |x|^{-i\tau}(\phi(x) - \phi(2x))$$

and, therefore, both $\psi$ and $f$ are supported inside $D_4(B_\delta)\backslash B_\delta$ and are smooth. We compute

$$\langle \kappa_o, f \rangle = \int_G (\phi(x) - \phi(2x))\, |x|^{-Q} dx \neq 0$$

by the choice of $\phi$. On the other hand,

$$\langle \kappa, f \rangle = \langle \kappa, \psi \rangle - 2^{i\tau} \langle \kappa, \psi \circ D_2 \rangle = 0.$$

We have obtained a contradiction. $\square$

The next statement answers the question above under the assumption that $\kappa_o$ is also continuous on $G\backslash\{0\}$.

**Proposition 3.2.24.** *Let $G$ be a homogeneous Lie group and let $\kappa_o$ be a continuous homogeneous function on $G\backslash\{0\}$ of degree $\nu$ with $\operatorname{Re}\nu = -Q$.*

*Then $\kappa_o$ extends to a homogeneous distribution in $\mathcal{D}'(G)$ if and only if its average value, defined in Lemmata 3.1.43 and 3.1.45, is $m_{\kappa_o} = 0$.*

*Proof.* Let us fix a homogeneous quasi-norm $|\cdot|$. We denote by $\sigma$ the measure on the unit sphere $\mathfrak{S} = \{x : |x| = 1\}$ which gives the polar change of coordinates (see Proposition 3.1.42) and $|\sigma|$ its total mass.

By Lemma 3.1.41, there exists $c > 0$ such that

$$|x| \leq 1 \Longrightarrow |x|_E \leq c|x|. \tag{3.51}$$

First let us assume $m_{\kappa_o} = 0$. Therefore, for any $a, b \in [0, \infty)$,

$$\int_{a<|x|<b} \kappa_o(x) dx = \int_{r=a}^{b} \int_{\mathfrak{S}} \kappa_o(rx) d\sigma(x) r^{Q-1} dr = m_{\kappa_o} \int_{r=a}^{b} r^\nu r^{Q-1} dr = 0,$$

see Section 3.1.7. We claim that, for each $f \in \mathcal{D}(G)$,

$$\exists \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \kappa_o(x) f(x) dx \ < \infty. \tag{3.52}$$

Indeed, let us check the Cauchy condition for $0 < \epsilon < \epsilon'$. We see that

$$\begin{aligned} \left| \int_{|x|>\epsilon} \kappa_o(x) f(x) dx - \int_{|x|>\epsilon'} \kappa_o(x) f(x) dx \right| &= \left| \int_{\epsilon<|x|<\epsilon'} \kappa_o(x) f(x) dx \right| \\ &= \left| \int_{\epsilon<|x|<\epsilon'} \kappa_o(x) \left( f(x) - f(0) \right) dx \right| \\ &\leq \int_{\epsilon<|x|<\epsilon'} |\kappa_o(x)| \, |f(x) - f(0)| \, dx. \end{aligned}$$

The (Euclidean) mean value theorem and the estimate (3.51) imply

$$|f(x) - f(0)| \leq \|\nabla f\|_\infty |x|_E \leq \|\nabla f\|_\infty c|x| \quad \text{if } |x| < 1.$$

Since $\kappa_o$ is $\nu$-homogeneous with $\operatorname{Re} \nu = -Q$, denoting by $C_o$ the maximum of $|\kappa_o|$ on the unit sphere $\{x : |x| = 1\}$, we have

$$\forall x \in G \backslash \{0\} \qquad |\kappa_o(x)| \leq C_o |x|^{-Q}.$$

Hence if $\epsilon' < 1$,

$$\begin{aligned} \left| \int_{|x|>\epsilon} \kappa_o(x) f(x) dx - \int_{|x|>\epsilon'} \kappa_o(x) f(x) dx \right| &\leq \int_{\epsilon<|x|<\epsilon'} \|\nabla f\|_\infty c C_o |x|^{1-Q} dx \\ &= \|\nabla f\|_\infty c C_o (\epsilon' - \epsilon). \end{aligned}$$

This implies the Cauchy condition. Therefore, Claim (3.52) is proved and we denote the limit by

$$\langle \kappa, f \rangle := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \kappa_o(x) f(x) dx, \quad f \in \mathcal{D}(G). \tag{3.53}$$

This clearly defines a linear functional. Moreover, this functional is continuous since if $f \in \mathcal{D}(G)$ is supported in a ball $\bar{B}_R = \{x : |x| \leq R\}$ for $R$ large enough, then, for $\epsilon < 1$,

$$\begin{aligned} \left| \int_{|x|>\epsilon} \kappa_o(x) f(x) dx \right| &\leq \left| \int_{\epsilon<|x|<1} \kappa_o(x) f(x) dx \right| + \left| \int_{1<|x|} \kappa_o(x) f(x) dx \right| \\ &\leq \|\nabla f\|_\infty c C_o (1 - \epsilon) + C_o \int_{1<|x|\leq R} |f(x)| dx \\ &\leq C_R (\|\nabla f\|_\infty + \|f\|_\infty). \end{aligned}$$

For the converse, we proceed by contradiction: let us assume that $\kappa$ exists and that $m_{\kappa_o} \neq 0$. Then

$$\kappa_o - \frac{m_{\kappa_o}}{|\sigma|}|x|^\nu$$

is a continuous homogeneous distribution of $G\backslash\{0\}$ of degree $\nu$ with mean average

$$\begin{aligned} \int_{\mathfrak{S}} \left(\kappa_o(x) - \frac{m_{\kappa_o}}{|\sigma|}|x|^\nu\right) d\sigma(x) &= \int_{\mathfrak{S}} \kappa_o(x) d\sigma(x) - \frac{m_{\kappa_o}}{|\sigma|} \int_{\mathfrak{S}} d\sigma(x) \\ &= m_{\kappa_o} - m_{\kappa_o} = 0. \end{aligned}$$

Hence it admits an extension into a homogeneous distribution by the first part of the proof. But this would imply that $|x|^\nu$ has such an extension and this is impossible by Example 3.2.23. □

*Remark* 3.2.25. (i) In view of the proof above, the hypothesis of continuity in Proposition 3.2.24 (and also in Proposition 3.2.27) can be relaxed into the following condition: $\kappa_o$ is locally integrable and locally bounded on $G\backslash\{0\}$.

This ensures that all the computations make sense and, since the unit sphere of a given homogeneous quasi-norm is compact, $|\kappa_o|$ is bounded there.

We will not use this fact.

(ii) By Lemma 3.1.45 the condition $m_{\kappa_o} = 0$ is independent of the homogeneous quasi-norm. However, the distribution defined in (3.53) depends on the choice of a particular homogeneous quasi-norm. For instance, one can show that the function on $\mathbb{R}^2$ given in polar coordinates by

$$\kappa_o(re^{i\theta}) = \frac{\cos 4\theta}{r^2},$$

admits two different extensions $\kappa$ via the procedure (3.53) when considering the Euclidean norm $(x, y) \mapsto (x^2 + y^2)^{1/2}$ and the $\ell^1$-norm $(x, y) \mapsto |x| + |y|$.

**Definition 3.2.26.** The distribution given in (3.53) is called a *principal value distribution* denoted by

$$p.v.\ \kappa_o(x).$$

The notation is ambiguous unless a homogeneous norm is specified.

The next proposition states that, modulo a Dirac distribution at the origin, the only possible extension is the principal value distribution:

**Proposition 3.2.27.** *Let $\kappa$ be a homogeneous distribution of degree $\nu$ with* $\mathrm{Re}\,\nu = -Q$ *on a homogeneous Lie group $G$. We assume that the restriction of $\kappa$ to $G\backslash\{0\}$ coincides with a continuous function $\kappa_o$.*

*Then $\kappa_o$ is homogeneous of degree $\nu$ on $G\backslash\{0\}$ and $m_{\kappa_o} = 0$. Moreover, after the choice of a homogeneous norm,*

$$\kappa(x) = p.v.\ \kappa_o(x) + c\delta_o,$$

*for some constant $c \in \mathbb{C}$, with $c = 0$ if $\nu \neq -Q$.*

*Proof.* By Proposition 3.2.24, $m_{\kappa_o} = 0$. Then

$$\kappa' := \kappa - p.v.\ \kappa_o$$

is also homogeneous of degree $\nu$ and supported at the origin.

Let $f \in \mathcal{D}(G)$ with $f(0) = 0$. Due to the compact support of $\kappa'$, $|\langle \kappa', f\rangle|$ is controlled by some $C^k$ norm of $f$ on a fixed small neighbourhood of the origin. But, because of its homogeneity of degree $\nu$ with $\operatorname{Re}\nu = -Q$,

$$\forall t > 0 \qquad |\langle \kappa', f\rangle| = |\langle \kappa', f \circ D_t\rangle|.$$

Letting $t$ tend to 0, the note that the $C^k$ norms of $f \circ D_t$ remain bounded. Let us show that as $t \to 0$, we actually have $\langle \kappa', f \circ D_t\rangle \to 0$. We claim that $f \circ D_t \to 0$ in $C^k(U)$ for a neighbourhood $U$ of 0. Indeed,

$$X^\alpha(f \circ D_t) = t^{[\alpha]}(X^\alpha f) \circ D_t \to 0 \quad \text{ as } t \to 0,$$

provided that $\alpha \neq 0$. On the other hand, also $(f \circ D_t)(x) = f(tx) \to f(0) = 0$ as $t \to 0$, and same for the $L^\infty$ norm over the set $U$. Thus, we have proved that $\langle \kappa', f\rangle = 0$ for any $f \in \mathcal{D}(G)$ vanishing at 0.

We now fix a function $\chi \in \mathcal{D}(G)$ with $\chi(0) = 1$. For any $f \in \mathcal{D}(G)$,

$$\langle \kappa', f\rangle = \langle \kappa', f - f(0)\chi\rangle + f(0)\langle \kappa', \chi\rangle = f(0)\langle \kappa', \chi\rangle,$$

since $f - f(0)\chi \in \mathcal{D}(G)$ vanishes at 0. This shows $\kappa' = c\delta_0$ where $c = \langle \kappa', \chi\rangle$. But $\delta_0$ is homogeneous of degree $-Q$, see Example 3.1.20, whereas $\kappa'$ is homogeneous of degree $\nu$. So $c = 0$ if $\nu \neq -Q$.

Alternatively, we can also argue as follows. By Proposition 1.4.2 we must have

$$\kappa' = \kappa - p.v.\ \kappa_o = \sum_{|\alpha| \le j} a_\alpha \partial^\alpha \delta_0$$

for some $j$ and some constants $a_\alpha$. Now, we know by Example 3.1.20 that $\delta_0$ is homogeneous of degree $-Q$, and by Proposition 3.1.23 that $\partial^\alpha \delta_0$ is homogeneous of degree $-Q - [\alpha]$. Since $\kappa'$ is homogeneous of degree $-Q$, it follows that all $a_\alpha = 0$ for $-Q - [\alpha] \neq \nu$. The statement now follows since, if $\nu \neq -Q$, we must have all $a_\alpha = 0$, and if $\nu = -Q$, we take $c = a_0$. □

Using the vocabulary of kernels of type $\nu$, see Definition 3.2.9, Proposition 3.2.24 implies easily:

**Corollary 3.2.28.** *Let $G$ be a homogeneous Lie group and let $\kappa_o$ be a smooth homogeneous function on $G\backslash\{0\}$ of degree $\nu$ with $\operatorname{Re}\nu = -Q$. Then $\kappa_o$ extends to a homogeneous distribution in $\mathcal{D}'(G)$ if and only if its average value, defined in Lemmata 3.1.43 and 3.1.45, is $m_{\kappa_o} = 0$. In this case, the extension is a kernel of type $\nu$.*

*Remark* 3.2.29. Remark 3.2.10 explained the correspondence between the kernels of type $\nu$ and their restriction to $G\backslash\{0\}$ in the case $\operatorname{Re}\nu \in (0, Q)$.

With Corollary 3.2.28, we obtain the case $\operatorname{Re}\nu = 0$: the restriction to $G\backslash\{0\}$ yields a correspondence between

- the $(\nu - Q)$-homogeneous functions in $C^\infty(G\backslash\{0\})$ with vanishing mean value
- and the kernels of type $\nu$.

It is one-to-one if $\nu \neq 0$ but if $\nu = 0$, we have to consider the kernels of type $\nu$ modulo $\mathbb{C}\delta_0$.

### 3.2.5 Operators of type $\nu = 0$

We can now go back to our original motivation, that is, a condition on a left-invariant homogeneous operator of degree 0 to obtain continuity on every $L^p(G)$. Our condition here is that the operator is of type 0, or more generally of type $\nu$, $\operatorname{Re}\nu = 0$.

**Theorem 3.2.30.** *Let $G$ be a homogeneous Lie group and let $\nu \in \mathbb{C}$ with*

$$\operatorname{Re}\nu = 0.$$

*Any operator of type $\nu$ on $G$ is $(-\nu)$-homogeneous and extends to a bounded operator on $L^p(G)$, $p \in (1, \infty)$.*

The proof consists in showing that the operator is Calderón-Zygmund (in the sense of Definition 3.2.15) and bounded on $L^2(G)$. Note that the cancellation condition (see Remark 3.2.18), is provided by $m_{\kappa_o} = 0$, see Proposition 3.2.27.

*Proof.* Let $\kappa \in \mathcal{D}'(G)$ be a kernel of type $\nu$, $\operatorname{Re}\nu = 0$. We denote by $\kappa_o$ its smooth restriction to $G\backslash\{0\}$. One checks easily that $\kappa_o$ satisfies the hypotheses of Corollary 3.2.20. Consequently, $\kappa_o$ is a Calderón-Zygmund kernel in the sense of Definition 3.2.15. By the Singular Integral Theorem, more precisely its form given in Proposition 3.2.17, to prove the $L^p$-boundedness for every $p \in (1, \infty)$, it suffices to prove the case $p = 2$.

Fixing a homogeneous norm $|\cdot|$ smooth away from the origin, by Proposition 3.2.27, we may assume that $\kappa$ is the principal value distribution of $\kappa_o$ (see Definition 3.2.26). We want to show that

$$f \mapsto f * p.v.\, \kappa_o$$

is bounded on $L^2(G)$. For this, we will apply the Cotlar-Stein lemma (see Theorem A.5.2) to the operators

$$T_j : f \mapsto f * K_j, \quad j \in \mathbb{Z},$$

where

$$K_j(x) = \kappa_o(x) 1_{2^{-j} \leq |x| \leq 2^{-j+1}}(x).$$

We claim that

$$\max\left(\|T_j^*T_k\|_{\mathscr{L}(L^2(G))}, \|T_jT_k^*\|_{\mathscr{L}(L^2(G))}\right) \le C2^{-|j-k|}. \tag{3.54}$$

Assuming this claim, by the Cotlar-Stein lemma, $\sum_j T_j$ defines a bounded operator on $L^2(G)$ and its (right convolution) kernel is $\sum_j K_j$ which coincides, as a distribution, with $p.v.\,\kappa_o = \kappa$. This would conclude the proof.

Let us start to prove Claim (3.54). It is not difficult to see (see (3.47)) that the adjoint of the operator $T_j$ on $L^2(G)$ is the convolution operator with right convolution kernel given by

$$K_j^*(x) = \bar{K}_j(x^{-1}),$$

which is compactly supported. Therefore, the operators $T_j^*T_k$ and $T_jT_k^*$ are convolution operators with kernels $K_k * K_j^*$ and $K_k^* * K_j$, respectively. We observe that, by homogeneity of $\kappa_o$, for any $j \in \mathbb{N}_0$,

$$|K_j(x)| = 2^{jQ}|K_0(2^jx)| \quad \text{and so} \quad \|K_j\|_{L^1(G)} = \|K_0\|_{L^1(G)}.$$

By the Young convolution inequality (see Proposition 1.5.2), the operators $T_j$, $T_j^*T_k$ and $T_jT_k^*$ are bounded on $L^2(G)$ with operator norms

$$\begin{aligned}
\|T_j\|_{\mathscr{L}(L^2(G))} &\le \|K_j\|_1 = \|K_0\|_1, \\
\|T_j^*T_k\|_{\mathscr{L}(L^2(G))} &\le \|K_k * K_j^*\|_1 \le \|K_k\|_1\|K_j^*\|_1 = \|K_0\|_1^2, \\
\|T_jT_k^*\|_{\mathscr{L}(L^2(G))} &\le \|K_k^* * K_j\|_1 \le \|K_k^*\|_1\|K_j\|_1 = \|K_0\|_1^2.
\end{aligned}$$

In order to prove Claim (3.54) we need to obtain a better decay for $\|K_k * K_j^*\|_1$ and $\|K_k^* * K_j\|_1$ when $j$ and $k$ are 'far apart'. Since $\|K_k * K_j^*\|_1 = \|K_j * K_k^*\|_1$ and $\|K_k^* * K_j\|_1 = \|K_j^* * K_k\|_1$, we may assume $k > j$. Quantitatively we assume that $C_1 2^{j-k+1} < 1/2$ where $C_1 \ge 1$ is the constant appearing in (3.26) for $b = 1/2$.

We observe that the cancellation condition $m_{\kappa_o} = 0$ implies

$$\int_G K_k(x)dx = \int_{2^{-k}\le|x|\le 2^{-k+1}} \kappa_o(x)dx = m_{\kappa_o}\ln 2 = 0,$$

and so

$$\begin{aligned}
\left|K_k * K_j^*(x)\right| &= \left|\int_G K_k(y)K_j^*(y^{-1}x)dy\right| = \left|\int_G K_k(y)\left(K_j^*(y^{-1}x) - K_j^*(x)\right)dy\right| \\
&\le \int_G |K_k(y)|\left|K_j^*(y^{-1}x) - K_j^*(x)\right| dy \\
&\le \int_{2^{-k}\le|y|\le 2^{-k+1}} C_o|y|^{-Q}\left|K_j^*(y^{-1}x) - K_j^*(x)\right| dy,
\end{aligned}$$

where $C_o$ is the maximum of $|\kappa_o|$ on the unit sphere $\{|x| = 1\}$. Thus after the change of variable $z = 2^k y$,

$$\left|K_k * K_j^*(x)\right| \leq \int_{1\leq|z|\leq 2} C_o|z|^{-Q} \left|K_j^*((2^{-k}z)^{-1}x) - K_j^*(x)\right| dz.$$

We want to estimate the $L^1$-norm with respect to $x$ of the last expression. Hence we now look at

$$\int_G \left|K_j^*((2^{-k}z)^{-1}x) - K_j^*(x)\right| dx = \int_G \left|K_j\left(x_1\ 2^{-k}z\right) - K_j(x_1)\right| dx_1,$$

after the change of variable $x = x_1^{-1}$. Using $K_j = 2^{j\nu} K_0 \circ D_j$ and the change of variable $x_2 = 2^j x_1$, we obtain

$$\int_G \left|K_j\left(x_1\ 2^{-k}z\right) - K_j(x_1)\right| dx_1 = \int_G \left|K_0\left(x_2\ 2^{-k+j}z\right) - K_0(x_2)\right| dx_2.$$

Let $A_0 = \{1 \leq |x| \leq 2\}$ be the annulus with radii 1 and 2 around 0 and write momentarily $y^{-1} = 2^{-k+j}z$ with $z \in A_0$. We can write the last integral as

$$\int_G \left|K_0(xy^{-1}) - K_0(x)\right| dx = \int_{A_0\cap(A_0y)} + \int_{A_0\backslash(A_0y)} + \int_{(A_0y)\backslash A_0}.$$

For the last two integrals, we see with a change of variable $x = x'y^{-1}$ that

$$\int_{A_0\backslash(A_0y)} = \int_{A_0\backslash(A_0y)} |K_0(x)|\, dx = \int_{(A_0y)\backslash A_0} \left|K_0(x'y^{-1})\right| dx' = \int_{(A_0y)\backslash A_0},$$

and

$$\int_{A_0\backslash(A_0y)} |K_0| \leq \int_{\substack{|xy^{-1}|>2\\ 1\leq|x|\leq 2}} C_o|x|^{-Q} dx + \int_{\substack{|xy^{-1}|<1\\ 1\leq|x|\leq 2}} C_o|x|^{-Q} dx.$$

Thus

$$\int_G \left|K_0(xy^{-1}) - K_0(x)\right| dx = \int_{A_0\cap(A_0y)} |K_0(xy^{-1}) - K_0(x)| dx \tag{3.55}$$
$$+2C_o\left(\int_{\substack{|xy^{-1}|>2\\ 1\leq|x|\leq 2}} |x|^{-Q} dx + \int_{\substack{|xy^{-1}|<1\\ 1\leq|x|\leq 2}} |x|^{-Q} dx\right).$$

Since $y^{-1}$ is relatively small, by (3.26) we get for the two integrals above

$$\int_{\substack{|xy^{-1}|>2\\ 1\leq|x|\leq 2}} + \int_{\substack{|xy^{-1}|<1\\ 1\leq|x|\leq 2}} \leq \int_{2-C_1|y|<|x|\leq 2} + \int_{1\leq|x|<1+C_1|y|}$$
$$= \ln\frac{2}{2-C_1|y|} + \ln(1+C_1|y|) \leq C|y|,$$

(see Example 3.1.44), whereas by Proposition 3.1.40 we have for any $x \in A_0$,

$$\left|K_0(xy^{-1}) - K_0(x)\right| \leq C|y|\ |x|^{-Q-1},$$

and so

$$\int_{A_0 \cap (A_0 y)} \left|K_0(xy^{-1}) - K_0(x)\right| dx \leq C|y| \int_{1 \leq |x| \leq 2} |x|^{-Q-1} dx \leq C|y|.$$

We have obtained that the expression (3.55) is up to a constant less than $2^{-k+j}$ when $C_1 2^{j-k+1} < 1/2$ (and $y^{-1} = 2^{-k+j} z$, $z \in A_0$). This estimate gives

$$\begin{aligned} \|K_k * K_j^*\|_1 &\leq C_o \int_{z \in A_0} |z|^{-Q} \int_G \left|K_0(x\ 2^{-k+j} z) - K_0(x)\right| dx\ dz \\ &\leq C_o \int_{z \in A_0} |z|^{-Q+1} C 2^{-k+j}\ dz \leq C 2^{-k+j}. \end{aligned}$$

With a very minor modification, we can show in the same way that $\|K_k^* * K_j\|_1 \leq C 2^{-k+j}$.

This shows Claim (3.54) and concludes the proof of Theorem 3.2.30. □

*Remark* 3.2.31. In view of the proof, we can relax the smoothness condition in the hypotheses of Theorem 3.2.30: it suffices to assume that $\kappa_o \in C^1(G\backslash\{0\})$.

This ensures that we can apply Propositions 3.2.27 and 3.1.40 during the proof.

### 3.2.6 Properties of kernels of type $\nu$, $\operatorname{Re}\nu \in [0, Q)$

The kernels and operators of type $\nu$ have been defined in Definition 3.2.9. Summarising results of the previous section, namely Corollary 3.2.11 for $\operatorname{Re}\nu \in (0, Q)$, and Theorem 3.2.30 for $\operatorname{Re}\nu = 0$, we can unite them as

**Corollary 3.2.32.** *Let $G$ be a homogeneous Lie group and let $\nu \in \mathbb{C}$ with*

$$\operatorname{Re}\nu \in [0, Q).$$

*Any operator of type $\nu$ on $G$ is $(-\nu)$-homogeneous and extends to a bounded operator from $L^p(G)$ to $L^q(G)$ provided that*

$$\frac{1}{p} - \frac{1}{q} = \frac{\operatorname{Re}\nu}{Q}, \quad 1 < p \leq q < \infty.$$

When considering kernels of type $\nu$, we have regularly used the following property: if $\kappa$ is a kernel of type $\nu$ then, fixing a homogeneous quasi-norm $|\cdot|$ on $G$, $\kappa$ admits a maximum $C_\kappa$ on the unit sphere $\{|x| = 1\}$, and by homogeneity we have

$$\forall x \in G\backslash\{0\} \qquad |\kappa(x)| \leq C_\kappa |x|^{\operatorname{Re}\nu - Q}. \tag{3.56}$$

In particular, it is locally integrable if $\operatorname{Re}\nu > 0$ and defines a distribution on the whole group $G$ in this case. In the case when $\operatorname{Re}\nu = 0$, by Proposition 3.2.27, $\kappa$ also defines a distribution on $G$ of the form

$$\kappa = p.v.\,\kappa_1 + c\delta_0,$$

where $\kappa_1$ is of type $\nu$ with vanishing average value and $c \in \mathbb{C}$ is a constant.

We can also deduce the type of a kernel from the following lemma:

**Lemma 3.2.33.** *Let $\kappa$ be a kernel of type $\nu_\kappa$ with $\operatorname{Re}\nu_\kappa \in (0, Q)$. Let $T$ be a homogeneous differential operator of homogeneous degree $\nu_T$. If $\operatorname{Re}\nu_\kappa - \nu_T \in [0, Q)$ then $T\kappa$ defines a kernel of type $\nu_\kappa - \nu_T$.*

*Proof.* Clearly $T\kappa$ is a $(Q - \nu_\kappa + \nu_T)$-homogeneous distribution which coincides with a smooth function away from 0. □

*Remark* 3.2.34. We have obtained certain properties of convolution operators with kernels of type $\nu$ in Corollary 3.2.11 for $\operatorname{Re}\nu \in (0, Q)$, and in Theorem 3.2.30 for $\operatorname{Re}\nu = 0$. When composing two such types of operators, we have to deal with the convolution of two kernels and this is a problematic question in general. Indeed, the problems about convolving distributions on a non-compact Lie group are essentially the same as in the case of the abelian convolution on $\mathbb{R}^n$. The convolution $\tau_1 * \tau_2$ of two distributions $\tau_1, \tau_2 \in \mathcal{D}'(G)$ is well defined as a distribution provided that at most one of them has compact support, see Section 1.5. However, additional assumptions must be imposed in order to define convolutions of distributions with non-compact supports. Furthermore, the associative law

$$(\tau_1 * \tau_2) * \tau_3 = \tau_1 * (\tau_2 * \tau_3), \tag{3.57}$$

holds when at most one of the $\tau_j$'s has non-compact support, but not necessarily when only one of the $\tau_j$'s has compact support even if each convolution in (3.57) could have a meaning.

The following proposition establishes that there is no such pathology appearing when considering convolution with kernel of type $\nu$ with $\operatorname{Re}\nu \in [0, Q)$. This will be useful in the sequel.

**Proposition 3.2.35.** *Let $G$ be a homogeneous Lie group.*

*(i) Suppose $\nu \in \mathbb{C}$ with $0 \le \operatorname{Re}\nu < Q$, $p \ge 1$, $q > 1$, and $r \ge 1$ given by*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\operatorname{Re}\nu}{Q} - 1.$$

*If $\kappa$ is a kernel of type $\nu$, $f \in L^p(G)$, and $g \in L^q(G)$, then $f * (g * \kappa)$ and $(f * g) * \kappa$ are well defined as elements of $L^r(G)$, and they are equal.*

(ii) *Suppose $\kappa_1$ is a kernel of type $\nu_1 \in \mathbb{C}$ with $\operatorname{Re}\nu_1 > 0$ and $\kappa_2$ is a kernel of type $\nu_2 \in \mathbb{C}$ with $\operatorname{Re}\nu_2 \geq 0$. We assume $\operatorname{Re}(\nu_1+\nu_2) < Q$. Then $\kappa_1 * \kappa_2$ is well defined as a kernel of type $\nu_1+\nu_2$. Moreover, if $f \in L^p(G)$ where*

$$1 < p < Q/(\operatorname{Re}(\nu_1+\nu_2))$$

*then $(f*\kappa_1)*\kappa_2$ and $f*(\kappa_1*\kappa_2)$ belong to $L^q(G)$,*

$$\frac{1}{q} = \frac{1}{p} - \frac{\operatorname{Re}(\nu_1+\nu_2)}{Q},$$

*and they are equal.*

*Proof.* Let us prove Part (i). By Corollary 3.2.11, Theorem 3.2.30 and Young's inequality (see Proposition 1.5.2), the mappings $(f,g) \mapsto f*(g*\kappa)$ and $(f,g) \mapsto (f*g)*\kappa$ are continuous from $L^p(G)\times L^q(G)$ to $L^r(G)$. They coincide when they have compact support, and hence in general.

Let us prove Part (ii). We fix a homogeneous quasi-norm $|\cdot|$ smooth away from the origin. We will use the general properties of kernels of type $\nu$ explained at the beginning of this section, especially estimate (3.56).

Let $x \neq 0$ be given. We can find $\epsilon > 0$ such that the balls

$$B(0,\epsilon) := \{y : |y| < \epsilon\} \quad \text{and} \quad B(x,\epsilon) := \{y : |xy^{-1}| < \epsilon\},$$

do not intersect. We note that these balls are different from those in Definition 3.1.33 (that are used throughout this book) but in this proof only, it will be more convenient for us to work with the balls defined as above.

If $\operatorname{Re}\nu_1,\ \operatorname{Re}\nu_2 > 0$, then both $\kappa_1$ and $\kappa_2$ are locally integrable and

$$\left|\kappa_1(xy^{-1})\kappa_2(y)\right| \leq C_{x,\epsilon} \begin{cases} |y|^{\operatorname{Re}\nu_2 - Q} & \text{for } y \in B(0,\epsilon), \\ |xy^{-1}|^{\operatorname{Re}\nu_1 - Q} & \text{for } y \in B(x,\epsilon), \\ O(|y|^{\operatorname{Re}(\nu_1+\nu_2)-2Q}) & y \notin B(0,\epsilon)\cup B(x,\epsilon). \end{cases}$$

Thus we can integrate $\kappa_1(xy^{-1})\kappa_2(y)$ against $dy$ on $B(0,\epsilon)$, $B(x,\epsilon)$ and outside of $B(0,\epsilon)\cup B(x,\epsilon)$ to obtain the sum of three integrals absolutely convergent:

$$\left[\int_{y\in B(0,\epsilon)} + \int_{y\in B(x,\epsilon)} + \int_{\substack{|y|>\epsilon \\ |xy^{-1}|>\epsilon}}\right] \kappa_1(xy^{-1})\kappa_2(y)dy := \kappa(x).$$

This defines $\kappa(x)$ which is independent of $\epsilon$ small enough.

If $\operatorname{Re}\nu_2 = 0$, by Proposition 3.2.27, we may assume that $\kappa_2$ is the principal value of a homogeneous distribution with mean average 0 (see also Definition 3.2.26 and (3.53)). In this case, by smoothness of $\kappa_1$ away from 0 and Proposition 3.1.40,

$$\left|\left(\kappa_1(xy^{-1}) - \kappa_1(x)\right)\kappa_2(y)\right| \leq C_{x,\epsilon}|y|^{1-Q} \quad \text{for } y \in B(0,\epsilon),$$

and we obtain again the sum of three integrals absolutely convergent:

$$\int_{y\in B(0,\epsilon)} \left(\kappa_1(xy^{-1}) - \kappa_1(x)\right) \kappa_2(y)dy +$$
$$+ \left[\int_{y\in B(x,\epsilon)} + \int_{\substack{|y|>\epsilon \\ |xy^{-1}|>\epsilon}}\right] \kappa_1(xy^{-1})\kappa_2(y)dy =: \kappa(x).$$

This defines $\kappa(x)$ which is independent of $\epsilon$ small enough.

In both cases, we have defined a function $\kappa$ on $G\backslash\{0\}$. A simple change of variables shows that $\kappa$ is homogeneous of degree $\nu_1 + \nu_2 - Q$ (this is left to the reader interested in checking this fact).

Let us fix $\phi_1 \in \mathcal{D}(G)$ with $\phi_1 \equiv 1$ on $B(0, \epsilon/2)$ and $\phi_1 \equiv 0$ on the complement of $B(0, \epsilon)$. We fix again $x \neq 0$ and we set $\phi_2(y) = \phi_1(xy^{-1})$. Then $\phi_1$ and $\phi_2$ have disjoint supports and for $\operatorname{Re}\nu_2 > 0$ it is easy to check that for $z \in B(x, \epsilon/2)$ we have $\kappa(z) = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \int_G \phi_1(y)\kappa_1(zy^{-1})\kappa_2(y)dy, \\
I_2 &= \int_G \phi_2(y)\kappa_1(zy^{-1})\kappa_2(y)dy = \int_G \phi_2(y^{-1}z)\kappa_1(y)\kappa_2(y^{-1}z)dy, \\
I_3 &= \int_G (1 - \phi_1(y) - \phi_2(y))\kappa_1(zy^{-1})\kappa_2(y)dy,
\end{aligned}$$

with a similar formula for $\operatorname{Re}\nu_2 = 0$. The integrands of $I_1$, $I_2$, and $I_3$ depend smoothly on $z$. Furthermore, one checks easily that their derivatives in $z$ remains integrable. This shows that $\kappa$ is smooth near each point $x \neq 0$. Since $\operatorname{Re}(\nu_1+\nu_2) > 0$, $\kappa$ is locally integrable on the whole group $G$. Hence the distribution $\kappa \in \mathcal{D}'(G)$ is a kernel of type $\nu_1 + \nu_2$.

We can check easily for $\phi \in \mathcal{D}(G)$,

$$\langle \kappa, \phi\rangle = \langle \kappa_1, \phi * \tilde{\kappa}_2\rangle = \langle \kappa_2, \tilde{\kappa}_1 * \phi\rangle.$$

So having (1.14) and (1.15) we define $\kappa_1 * \kappa_2 := \kappa$.

Let $f \in L^p(G)$ where $p > 1$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{\operatorname{Re}(\nu_1 + \nu_2)}{Q} > 0.$$

We observe that $(f * \kappa_1) * \kappa_2$ and $f * \kappa$ are in $L^q(G)$ by Corollary 3.2.11, Theorem 3.2.30, and Young's inequality (see Proposition 1.5.2). To complete the proof, it suffices to show that the distributions $(f * \kappa_1) * \kappa_2$ and $f * (\kappa_1 * \kappa_2)$ are equal. For this purpose, we write $\kappa_1 = \kappa_1^0 + \kappa_1^\infty$ with

$$\kappa_1^0 := \kappa_1\ 1_{|x|\leq 1} \quad \text{and} \quad \kappa_1^\infty := \kappa_1\ 1_{|x|>1}.$$

If $r = Q/(Q - \operatorname{Re}\nu_1)$ then $\kappa_1^0 \in L^{r-\epsilon}(G)$ and $\kappa_1^\infty \in L^{r+\epsilon}(G)$ for any $\epsilon > 0$. We take $\epsilon$ so small that $r - \epsilon > 1$ and

$$p^{-1} + (r+\epsilon)^{-1} - \operatorname{Re}\nu_2/Q - 1 > 0.$$

By Part (i), $(f * \kappa_1^0) * \kappa_2$ and $f * (\kappa_1^0 * \kappa_2)$ coincide as elements of $L^s(G)$ where

$$s^{-1} = p^{-1} + (r-\epsilon)^{-1} - \operatorname{Re}\nu_2/Q - 1.$$

And $(f * \kappa_1^\infty) * \kappa_2$ and $f * (\kappa_1^\infty * \kappa_2)$ coincide as elements of $L^t(G)$ where

$$t^{-1} = p^{-1} + (r+\epsilon)^{-1} - \operatorname{Re}\nu_2/Q - 1.$$

Thus $(f*\kappa_1)*\kappa_2$ and $f*\kappa$ coincide as elements of $L^s(G)$ and $L^t(G)$. This concludes the proof of Part (ii) and of Proposition 3.2.35. $\square$

### 3.2.7 Fundamental solutions of homogeneous differential operators

On open sets or manifolds, general results about the existence of fundamental kernels of operators hold, see e.g. [Tre67, Theorems 52.1 and 52.2]. On a Lie group, we can study the case when the fundamental kernels are of the form $\kappa(x-y)$ in the abelian case and $\kappa(y^{-1}x)$ on a general Lie group, where $\kappa$ is a distribution, often called a fundamental solution. It is sometimes possible and desirable to obtain the existence of such fundamental solutions for left or right invariant differential operators.

In this section, we first give a definition and two general statements valid on any connected Lie group, and then analyse in more detail the situation on homogeneous Lie groups.

**Definition 3.2.36.** Let $L$ be a left-invariant differential operator on a connected Lie group $G$. A distribution $\kappa$ in $\mathcal{D}'(G)$ is called a *(global) fundamental solution* of $L$ if

$$L\kappa = \delta_0.$$

A distribution $\tilde{\kappa}$ on a neighbourhood $\Omega$ of 0 is called a *local fundamental solution* of $L$ (at 0) if $L\tilde{\kappa} = \delta_0$ on $\Omega$.

On $(\mathbb{R}^n, +)$, global fundamental solutions are often called *Green functions.*

*Example* 3.2.37. Fundamental solutions for the Laplacian $\Delta = \sum_j \partial_j^2$ on $\mathbb{R}^n$ are well-known

$$G(x) = \begin{cases} \frac{c_n}{|x|^{n-2}} + p(x) & \text{if} \quad n \geq 3 \\ c_2 \ln|x| + p(x) & \text{if} \quad n = 2 \\ x 1_{[0,\infty)}(x) + p(x) & \text{if} \quad n = 1 \end{cases}$$

where $c_n$ is a (known) constant of $n$, $p$ is any polynomial of degree $\leq 1$, and $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$.

Example 3.2.37 shows that fundamental solutions are not unique, unless some hypotheses, e.g. homogeneity (besides existence), are added.

Although, in practice, 'computing' fundamental solutions is usually difficult, they are useful and important objects.

**Lemma 3.2.38.** *Let $L$ be a left-invariant differential operator with smooth coefficients on a connected Lie group $G$.*

1. *If $L$ admits a fundamental solution $\kappa$, then for every distribution $u \in \mathcal{D}'(G)$ with compact support, the convolution $f = u * \kappa \in \mathcal{D}'(G)$ satisfies*

$$Lf = u$$

*on $G$.*

2. *An operator $L$ admits a local fundamental solution if and only if it is locally solvable at every point.*

For the definition of locally solvability, see Definition A.1.4.

*Proof.* For the first statement,

$$L\big(u * \kappa\big) = u * L\kappa = u * \delta_0 = u.$$

For the second statement, if $L$ is locally solvable, then at least at the origin, one can solve $L\tilde{\kappa} = \delta_0$ and this shows that $L$ admits a local fundamental solution.

Conversely, let us assume that $L$ admits a local fundamental solution $\tilde{\kappa}$ on the open neighbourhood $\Omega$ of 0. We can always find a function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ on an open neighbourhood $\Omega_1 \subsetneq \Omega$ of 0; we define $\kappa_1 \in \mathcal{D}'(\Omega)$ by $\kappa_1 := \chi\tilde{\kappa}$ and view $\kappa_1$ also as a distribution with compact support. Then it is easy to check that $L\kappa_1 = \delta_0$ on $\Omega_1$ but that

$$L\kappa_1 = \delta_0 + \Phi,$$

where $\Phi$ is a distribution whose support does not intersect $\Omega_1$.

Let $\Omega_0$ be an open neighbourhood of 0 such that

$$\Omega_0^{-1}\Omega_0 = \{x^{-1}y \,:\, x, y \in \Omega_0\} \subsetneq \Omega_1.$$

We can always find a function $\chi_1 \in \mathcal{D}(\Omega_0)$ which is equal to 1 on a neighbourhood $\Omega_0' \subsetneq \Omega_0$ of 0.

If now $u \in \mathcal{D}'(G)$, then the convolution $f = (\chi_1 u) * \kappa_1$ is well defined and

$$Lf = \chi_1 u + \chi_1 u * \Phi,$$

showing that $Lf = \chi_1 u$ on $\Omega_0$ and hence $Lf = u$ on $\Omega_0'$. Hence $L$ is locally solvable at 0. By left-invariance, it is locally solvable at any point. $\square$

Because of the duality between hypoellipticity and solvability, local fundamental solutions exist under the following condition:

**Proposition 3.2.39.** *Let $L$ be a left-invariant hypoelliptic operator on a connected Lie group $G$. Then $L^t$ is also left-invariant and it has a local fundamental solution.*

*Proof.* The first statement follows easily from the definition of $L^t$, and the second from the duality between solvability and hypoellipticity (cf. Theorem A.1.3) and Lemma 3.2.38. □

The next theorem describes some property of existence and uniqueness of global fundamental solutions in the context of homogeneous Lie groups.

**Theorem 3.2.40.** *Let $L$ be a $\nu$-homogeneous left-invariant differential operator on a homogeneous Lie group $G$. We assume that the operators $L$ and $L^t$ are hypoelliptic on a neighbourhood of 0. Then $L$ admits a fundamental solution $\kappa \in \mathcal{S}'(G)$ satisfying:*

*(a) if $\nu < Q$, the distribution $\kappa$ is homogeneous of degree $\nu - Q$ and unique,*

*(b) if $\nu \geq Q$, $\kappa = \kappa_o + p(x) \ln |x|$ where*

*(i) $\kappa_o \in \mathcal{S}'(G)$ is a homogeneous distribution of degree $\nu - Q$, which is smooth away from 0,*

*(ii) $p$ is a polynomial of degree $\nu - Q$ and,*

*(iii) $|\cdot|$ is any homogeneous quasi-norm, smooth away from the origin.*

*Necessarily $\kappa$ is smooth on $G\backslash\{0\}$.*

*Remark* 3.2.41. In case (a), the unique homogeneous fundamental solution is a kernel of type $\nu$, with the uniqueness understood in the class of homogeneous distributions of degree $\nu - Q$. For case (b), Example 3.2.37 shows that one can not hope to always have a homogeneous fundamental solution.

The rest of this section is devoted to the proof of Theorem 3.2.40.

The proofs of Parts (a) and (b) as presented here mainly follow the original proofs of these results due to Folland in [Fol75] and Geller in [Gel83], respectively.

*Proof of Theorem 3.2.40 Part (a).* Let $L$ be as in the statement and let $\nu < Q$. By Proposition 3.2.39, $L$ admits a local fundamental solution at 0: there exist a neighbourhood $\Omega$ of 0 and a distribution $\tilde{\kappa} \in \mathcal{D}'(\Omega)$ such that $L\tilde{\kappa} = \delta_0$ on $\Omega$. Note that by the hypoellipticity of $L$, $\tilde{\kappa}$ as well as any fundamental solution coincide with a smooth function away form 0. By shrinking $\Omega$ if necessary, we may assume that after having fixed a homogeneous quasi-norm, $\Omega$ is a ball around 0. So if $x \in \Omega$ and $r \in (0, 1]$ then $rx \in \Omega$.

Folland observed that if $\kappa$ exists then the distribution $h := \tilde{\kappa} - \kappa$ annihilates $L$ on $\Omega$, so it must be smooth on $\Omega$, while

$$\kappa(x) = r^{Q-\nu} \tilde{\kappa}(rx) - r^{Q-\nu} h(rx)$$

yields

$$\kappa(x) = \lim_{r\to 0} r^{Q-\nu}\tilde{\kappa}(rx)$$

and

$$h(x) = \tilde{\kappa}(x) - \lim_{r\to 0} r^{Q-\nu}\tilde{\kappa}(rx).$$

Going back to our proof, Folland's idea was to define $h_r \in \mathcal{D}'(\Omega)$ by

$$h_r := \tilde{\kappa} - r^{Q-\nu}\tilde{\kappa}\circ D_r \quad \text{ on } \Omega\backslash\{0\},\ r\in(0,1],$$

which makes sense in view of the smoothness of $\tilde{\kappa}$ on $\Omega\backslash\{0\}$. Since for any test function $\phi \in \mathcal{D}(\Omega)$,

$$\langle L(r^{Q-\nu}\tilde{\kappa}(r\,\cdot)),\phi\rangle = \langle r^{Q}(L\tilde{\kappa})(r\,\cdot)),\phi\rangle = \langle L\tilde{\kappa},\phi(r^{-1}\cdot)\rangle = \phi(r^{-1}0) = \phi(0),$$

we have $Lh_r = \delta_0 - \delta_0 = 0$. So $h_r$ is in $N_L(\Omega) \subset C^\infty(\Omega)$ where the $\mathcal{D}'(\Omega)$ and $C^\infty(\Omega)$ topologies agree, see Theorem A.1.6. Let us show that

$$\exists \lim_{r\to 0} h_r \in h \in C^\infty(\Omega); \tag{3.58}$$

for this it suffices to show that $\{h_r\}$ is a Cauchy family in $\mathcal{D}'(\Omega)$.

We observe that if $s \le r$, we have

$$\begin{aligned} h_s(x) - h_r(x) &= r^{Q-\nu}\tilde{\kappa}(rx) - s^{Q-\nu}\tilde{\kappa}(sx) \\ &= r^{Q-\nu}\left(\tilde{\kappa}(rx) - \left(\frac{s}{r}\right)^{Q-\nu}\tilde{\kappa}\big(\frac{s}{r}rx\big)\right) \\ &= r^{Q-\nu}h_{\frac{s}{r}}(rx). \end{aligned} \tag{3.59}$$

In particular, setting $s = r^2$ in (3.59) we obtain

$$h_{r^2} = r^{Q-\nu}h_r\circ D_r + h_r.$$

This formula yields, first by substituting $r$ by $r^2$,

$$\begin{aligned} h_{r^4} &= r^{2(Q-\nu)}h_{r^2}\circ D_{r^2} + h_{r^2} \\ &= r^{2(Q-\nu)}\left(r^{Q-\nu}h_r\circ D_r\circ D_{r^2} + h_r\circ D_{r^2}\right) + r^{Q-\nu}h_r\circ D_r + h_r \\ &= r^{3(Q-\nu)}h_r\circ D_{r^3} + r^{2(Q-\nu)}h_r\circ D_{r^2} + r^{Q-\nu}h_r\circ D_r + h_r. \end{aligned}$$

Continuing inductively, we obtain

$$h_{r^{2^\ell}} = \sum_{k=0}^{2^\ell-1} r^{k(Q-\nu)}h_r\circ D_{r^k}.$$

This implies

$$\forall n\in\mathbb{N}_0 \qquad \sup_{x\in(1-\epsilon)\Omega}|h_{r^{2^\ell}}(x)| \le (1-r^{Q-\nu})^{-1}\sup_{x\in(1-\epsilon)\Omega}|h_r(x)|,$$

and, since any $s \leq \frac{1}{2}$ can be expressed as $s = r^{2^\ell}$ for some $\ell \in \mathbb{N}_0$ and some $r \in [\frac{1}{4}, \frac{1}{2}]$,

$$\forall s \leq \frac{1}{2} \qquad \sup_{x \in (1-\epsilon)\Omega} |h_s(x)| \leq (1 - 2^{\nu - Q})^{-1} \sup_{\substack{x \in (1-\epsilon)\Omega \\ \frac{1}{4} \leq r \leq \frac{1}{2}}} |h_r(x)|.$$

Now the Schwartz-Treves lemma (see Theorem A.1.6) implies that the topologies of $\mathcal{D}'(\Omega)$ and $C^\infty(\Omega)$ on

$$N_L(\Omega) = \{f \in \mathcal{D}'(\Omega) \ : \ Tf = 0\} \subset C^\infty(\Omega)$$

coincide. Since $r \mapsto h_r$ is clearly continuous from $(0,1]$ to $\mathcal{D}'(\Omega) \cap N_L(\Omega)$, $\{h_r, \, r \in [\frac{1}{4}, \frac{1}{2}]\}$ and $\{h_r, \, r \in [\frac{1}{2}, 1]\}$ are compact in $\mathcal{D}(\Omega)$. Therefore, we have

$$\begin{aligned} \sup_{\substack{x \in (1-\epsilon)\Omega \\ 0 < s \leq 1}} |h_s(x)| &\leq \sup_{\substack{x \in (1-\epsilon)\Omega \\ 0 < s \leq \frac{1}{2}}} |h_s(x)| + \sup_{\substack{x \in (1-\epsilon)\Omega \\ \frac{1}{2} \leq s \leq 1}} |h_s(x)| \\ &\leq (1 - 2^{\nu - Q})^{-1} \sup_{\substack{x \in (1-\epsilon)\Omega \\ \frac{1}{4} \leq r \leq \frac{1}{2}}} |h_r(x)| + \sup_{\substack{x \in (1-\epsilon)\Omega \\ \frac{1}{2} \leq s \leq 1}} |h_s(x)| = C_\epsilon < \infty, \end{aligned}$$

that is, the $h_r$'s are uniformly bounded on $(1-\epsilon)\Omega$. But if $s < r$, (3.59) implies

$$\sup_{x \in (1-\epsilon)\Omega} |h_s(x) - h_r(x)| \leq r^{Q-\nu} \sup_{x \in (1-\epsilon)\Omega} \left| h_{\frac{s}{r}}(rx) \right| \leq C_\epsilon r^{Q-\nu} \underset{r \to 0}{\longrightarrow} 0.$$

This shows that $\{h_r\}_{r \in (0,1]}$ is a Cauchy family of $C(K)$ for any compact subset $K$ of $\Omega$. Therefore, $\{h_r\}_{r \in (0,1]}$ is a Cauchy family of $\mathcal{D}'(\Omega)$ and Claim (3.58) is proved. Let $h \in C^\infty(\Omega)$ be the limit of $\{h_r\}$. Necessarily $Lh = 0$. We set

$$\kappa := \tilde{\kappa} - h \in \mathcal{D}'(\Omega).$$

Now, on one hand

$$L\kappa = L\tilde{\kappa} - Lh = \delta_0$$

and $\kappa$ is smooth on $\Omega \backslash \{0\}$, and on the other,

$$\kappa(x) = \lim_{r \to 0} r^{Q-\nu} \tilde{\kappa}(rx),$$

so if $s \in (0,1]$, then

$$\kappa(sx) = \lim_{r \to 0} r^{Q-\nu} \tilde{\kappa}(srx) = \lim_{r' = rs \to 0} \left( \frac{r'}{s} \right)^{Q-\nu} \tilde{\kappa}(r'x) = s^{\nu - Q} \kappa(x).$$

By requiring that the formula $\kappa(sx) = s^{\nu-Q}\kappa(x)$ holds for all $s > 0$ and $x \neq 0$, we can extend $\kappa$ into a distribution defined on the whole space. The homogeneity of $L$ guarantees that the equation $L\kappa = \delta_0$ holds globally.

Finally, if $\kappa_1$ were another fundamental solution of $L$ satisfying (a), then $\kappa - \kappa_1$ would be $(\nu - Q)$-homogeneous with $\nu - Q < 0$; $\kappa - \kappa_1$ would also be smooth even at 0 since it annihilates $L$ on $G$. Thus $\kappa - \kappa_1 = 0$. $\square$

*Proof of Theorem 3.2.40 Part (b).* Let $L$ be as in the statement and let $\nu \geq Q$. Let also $\tilde{\kappa}$, $\Omega$ and $h_r$ be defined as in the proof of part (a).

Geller noticed that Folland's idea could be adapted by taking higher order derivatives. Indeed from (3.59), we have

$$X^\alpha h_s(x) - X^\alpha h_r(x) = r^{Q-\nu+[\alpha]} X^\alpha h_{\frac{s}{r}}(rx);$$

if $\alpha \in \mathbb{N}_0^n$ is large so that $Q - \nu + [\alpha] > 0$, we can proceed as for $h_r$ in the proof of Part (a) and obtain that $\{X^\alpha h_r\}_{r\in(0,1]}$ is a Cauchy family of $C^\infty(\Omega)$.

If $[\alpha] \leq \nu - Q$, the $C^\infty(\Omega)$-family $\{X^\alpha h_r\}_{r\in(0,1]}$ may not be Cauchy but by Taylor's theorem at the origin for homogeneous Lie groups, cf. Theorem 3.1.51,

$$\left| h_r(x) - P^{(h_r)}_{0,M}(x) \right| \leq C_M \sum_{\substack{|\alpha|\leq \lceil M \rfloor +1 \\ [\alpha]>M}} |x|^{[\alpha]} \sup_{|z|\leq \eta^{\lceil M\rfloor+1}|x|} \left| \left(X^\alpha h_r\right)(z) \right|,$$

for any $x$ such that $x$ and $\eta^{\lceil M\rfloor+1}x$ are in the ball $\Omega$. Choosing $M = \nu - Q$ and setting the polynomial $p_r(x) := P^{(h_r)}_{0,M}(x)$ and the ball $\Omega' := \eta^{-(\lceil M\rfloor+1)}\Omega$, this shows that the $C^\infty(\Omega')$-family $\{h_r - p_r\}_{r\in(0,1]}$ is Cauchy. We set

$$C^\infty(\Omega') \ni h := \lim_{r\to 0}(h_r - p_r), \quad \kappa_o := \tilde{\kappa} - h \in \mathcal{D}(\Omega').$$

Note that $Lp_r = 0$, since the polynomial $p_r$ is of degree $\nu - Q$ and the differential operator $L$ is $\nu$-homogeneous. Therefore, $L\kappa_o = \delta_0$ in $\Omega'$ and $\kappa_o \in C^\infty(\Omega'\backslash\{0\})$. Furthermore, if $[\alpha] > \nu - Q$ and $x \in \Omega'\backslash\{0\}$ then

$$\left(\frac{\partial}{\partial x}\right)^\alpha \kappa_o(x) = \lim_{r\to 0} r^{Q-\nu+[\alpha]} \left(\frac{\partial}{\partial x}\right)^\alpha \tilde{\kappa}(rx),$$

so if $s \in (0,1]$,

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\right)^\alpha \kappa_o(sx) &= \lim_{r\to 0} r^{Q-\nu+[\alpha]} \left(\frac{\partial}{\partial x}\right)^\alpha \tilde{\kappa}(rsx) \\
&= \lim_{r'=rs\to 0} \left(\frac{r'}{s}\right)^{Q-\nu+[\alpha]} \left(\frac{\partial}{\partial x}\right)^\alpha \tilde{\kappa}(r'x) = s^{\nu-Q-[\alpha]} \left(\frac{\partial}{\partial x}\right)^\alpha \kappa(x).
\end{aligned}$$

One could describe this property as $\left(\frac{\partial}{\partial x}\right)^\alpha \kappa_o$ being homogeneous on $\Omega'\backslash\{0\}$. We conclude the proof by applying Lemma 3.2.42 below. $\square$

In order to state Lemma 3.2.42, we first define the set $\mathcal{W}$ of all the possible homogeneous degrees $[\alpha]$, $\alpha \in \mathbb{N}_0^n$,

$$\mathcal{W} := \{\upsilon_1\alpha_1 + \ldots + \upsilon_n\alpha_n \ : \ \alpha_1, \ldots, \alpha_n \in \mathbb{N}_0\}. \tag{3.60}$$

In other words, $\mathcal{W}$ is the additive semi-group of $\mathbb{R}$ generated by 0 and $\mathcal{W}_A$.

For instance, in the abelian case $(\mathbb{R}^n, +)$ or on the Heisenberg group $\mathbb{H}_{n_o}$, with our conventions, $\mathcal{W} = \mathbb{N}_0$. This is also the case for a stratified Lie group or for a graded Lie group with $\mathfrak{g}_1$ non-trivial.

**Lemma 3.2.42.** *Let $B$ be an open ball around the origin of a homogeneous Lie group $G$ equipped with a smooth homogeneous quasi-norm $|\cdot|$. We consider the sets of functions $\mathcal{K}^\nu$ defined by*

$$\begin{aligned}
&\text{if } \nu\in\mathbb{R}\backslash\mathcal{W} \quad \mathcal{K}^\nu := \{f\in C^\infty(B\backslash\{0\}) \ : \ f \text{ is } \nu\text{-homogeneous}\}\,,\\
&\text{if } \nu\in\mathcal{W} \quad \mathcal{K}^\nu := \{f\in C^\infty(B\backslash\{0\}) \ : \ f = f_1 + p(x)\ln|x|\,,\\
&\qquad \text{where } f_1 \text{ is } \nu\text{-homogeneous and } p \text{ is a } \nu\text{-homogeneous polynomial}\}\,,
\end{aligned}$$

*where $\mathcal{W}$ was defined in (3.60), and we say that a function $f$ on $B$ or $B\backslash\{0\}$ is $\nu$-homogeneous when $f\circ D_s = s^\nu f$ on $B$ for all $s\in(0,1)$.*

*For any $\nu\in\mathbb{R}$ and $f\in C^\infty(B\backslash\{0\})$, if $\left(\frac{\partial}{\partial x}\right)^\alpha f\in\mathcal{K}^{\nu-[\alpha]}$ with $[\alpha]>\nu$, then there exists $p\in\mathcal{P}_{<\nu}$ such that $f-p\in\mathcal{K}^\nu$.*

Recall (see Definition 3.1.26) that $\mathcal{P}_{<M}$ denotes the set of polynomials $P$ on $G$ such that $D^\circ P<M$. It is empty if $M<0$.

*Proof of Lemma 3.2.42.* By induction it suffices to prove that for any $\nu\in\mathbb{R}$ and $f\in C^\infty(B\backslash\{0\})$,

$$\begin{aligned}
\frac{\partial(f-p_j)}{\partial x_j}\in\mathcal{K}^{\nu-\upsilon_j} \text{ with } p_j\in\mathcal{P}_{<\nu-\upsilon_j} \text{ for all } j=1,\ldots,n&\\
\Longrightarrow f-p\in\mathcal{K}^\nu \text{ for some } p\in\mathcal{P}_{<\nu}.&
\end{aligned}\tag{3.61}$$

To prove (3.61), we start by showing that for any $f\in C^\infty(B\backslash\{0\})$,

$$\frac{\partial f}{\partial x_j}\in\mathcal{K}^{\nu-\upsilon_j} \text{ for all } j=1,\ldots,n\Longrightarrow f-c\in\mathcal{K}^\nu \text{ for some } c\in\mathbb{C}.\tag{3.62}$$

By convention (see Definition 3.1.26), a homogeneous polynomial of homogeneous degree which is not in $\mathcal{W}$ is 0. With this in mind we continue the proof of (3.62) in a unified way. We consider $f\in C^\infty(B\backslash\{0\})$ satisfying the hypothesis of (3.62): for each $j=1,\ldots,n$, $\frac{\partial f}{\partial x_j}\in\mathcal{K}^{\nu-\upsilon_j}$ and there exists $p_j\in\mathcal{P}_{=\nu-\upsilon_j}$ such that $f-p_j\ln|\cdot|$ is a $\nu$-homogeneous function on $\backslash\{0\}$. We define

$$A(r,x):=f(rx)-r^\nu f(x),\quad x\in B,\ r\in(0,1].$$

We see that

$$\begin{aligned}
\frac{\partial A(r,x)}{\partial x_j} &= r^{\upsilon_j}\frac{\partial f}{\partial x_j}(rx)-r^\nu\frac{\partial f}{\partial x_j}(x)\\
&= r^{\upsilon_j}p_j(rx)\ln|rx|-r^\nu p_j(x)\ln|x| = r^\nu p_j(x)\ln r.
\end{aligned}$$

Note that for any $j,k$ we have

$$\frac{\partial p_j}{\partial x_k}=\frac{\partial p_k}{\partial x_j}\quad\text{since}\quad \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}A(r,x)=\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}A(r,x).$$

Because of this observation we can adapt the proof of the Poincaré Lemma to construct the polynomial

$$q(x) := c \sum_{k=1}^{n} \upsilon_k x_k p_k(x), \tag{3.63}$$

which is $\nu$-homogeneous and satisfies

$$\begin{aligned}
\frac{\partial q}{\partial x_j} &= c \sum_{k=1}^{n} \upsilon_k x_k \frac{\partial p_k(x)}{\partial x_j} + c\upsilon_j p_j(x) = c \sum_{k=1}^{n} \upsilon_k x_k \frac{\partial p_j(x)}{\partial x_k} + c\upsilon_j p_j(x) \\
&= c\partial_{t=1} \left(p_j(tx)\right) + c\upsilon_j p_j(x) = c(\nu - \upsilon_j) p_j(x) + c\upsilon_j p_j(x) \\
&= p_j(x),
\end{aligned}$$

by choosing $c = \nu^{-1}$ if $\nu \neq 0$; if $\nu = 0$, the polynomials $p_j$ and $q$ are zero. So we have

$$\frac{\partial}{\partial x_j} \left(A(r,x) - q(x) r^{\nu} \ln r\right) = 0 \quad \text{for all } j = 1, \ldots, n.$$

Therefore,

$$A(r,x) = q(x) r^{\nu} \ln r + a(r) \quad \text{for some } a \in C^{\infty}((0,1]).$$

Replacing $f$ by $f - (r^{\nu} \ln r)\, q$ we may assume that $q = 0$ in all the cases, so that

$$\forall r \in (0,1],\ x \in B \quad f(rx) - r^{\nu} f(x) = a(r). \tag{3.64}$$

Now if $0 < r, s < 1$, then using the formula just above twice, we get

$$\begin{aligned}
a(rs) &= f(rsx) - (rs)^{\nu} f(x) = a(r) + r^{\nu} f(sx) - (rs)^{\nu} f(x) \\
&= a(r) + r^{\nu} (a(s) + s^{\nu} f(x)) - (rs)^{\nu} f(x) \\
&= a(r) + r^{\nu} a(s).
\end{aligned}$$

Solving this functional equation and setting

$$f_o(x) := f(x) - a(|x|) \qquad (x \in G \backslash \{0\}),$$

for a particular solution $a$, we check easily that $f_o$ is $\nu$-homogeneous:

- If $\nu = 0$, then $a$ satisfies the functional equation

  $$a(rs) = a(r) + a(s)$$

  and must, therefore, be of the form $a(r) = C \ln(r)$ for some constant $C \in \mathbb{C}$. Using (3.64) we obtain

  $$f_o(rx) = f(rx) - a(|rx|) = f(x) + a(r) - a(|rx|) = f(x) - C \ln|x| = f_o(x).$$

- If $\nu \neq 0$, then $a$ satisfies the functional equation

$$a(r) + r^\nu a(s) = a(s) + s^\nu a(r)$$

and must therefore be of the form $a(r) = C(1-r^\nu)$ for some constant $C \in \mathbb{C}$. Using (3.64) we obtain

$$\begin{aligned} f_o(rx) &= f(rx) - C(1-|rx|^\nu) = r^\nu f(x) + C(1-r^\nu) - C(1-|rx|^\nu) \\ &= r^\nu \left(f(x) - C(1-|x|^\nu)\right) = r^\nu f_o(x). \end{aligned}$$

Hence (3.62) is proved and we can now go back to showing the main claim, that is, the one given in (3.61). Let $f$ and $p_j$ be as in the hypotheses of (3.61).

First we see that if $\nu < 0$, then all the polynomials $p_j$ are zero and, inspired by the construction of $q$ above, we check easily that

$$\frac{\partial}{\partial x_j}\left(\nu^{-1}\sum_{k=1}^n \upsilon_k x_k \frac{\partial f}{\partial x_k}\right) = \frac{\partial f}{\partial x_j},$$

thus $f$ and $\nu^{-1}\sum_{k=1}^n \upsilon_k x_k \frac{\partial f}{\partial x_k}$ must coincide so (3.61) is proved in this case.

Let us assume $\nu \geq 0$. We claim that

$$\forall j,k = 1,\ldots,n \qquad \frac{\partial p_k}{\partial x_j} = \frac{\partial p_j}{\partial x_k}. \tag{3.65}$$

This is certainly true if $\nu - \upsilon_j - \upsilon_k < 0$ since both are zero in this case. If instead $\nu - \upsilon_j - \upsilon_k \geq 0$ then the polynomial

$$\frac{\partial p_k}{\partial x_j} - \frac{\partial p_j}{\partial x_k} = \frac{\partial}{\partial x_j}\left(p_k - \frac{\partial f}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(p_j - \frac{\partial f}{\partial x_j}\right),$$

is in $\mathcal{K}^{\nu-\upsilon_j-\upsilon_k}$ and thus must be zero. Indeed if a polynomial $p$ is in some $\mathcal{K}^a$ then either $a \notin \mathcal{W}$ and then $p = 0$, or $a \in \mathcal{W}$ and $p(rx)$ is a polynomial in $r$ of degree $\leq a$ with $r^{-a}p(rx)$ unbounded unless $p = 0$; in both cases, $p = 0$.

Therefore, we can construct $q$ as above by (3.63) so that $\frac{\partial q}{\partial x_j} = p_j$. Then

$$\frac{\partial(f-q)}{\partial x_j} = \frac{\partial f}{\partial x_j} - p_j \in \mathcal{K}^{\nu-\upsilon_j} \text{ for all } j = 1,\ldots,n,$$

so $f - q \in \mathcal{K}^\nu$ by (3.62).

This concludes the proof of Claim (3.61) and of Lemma 3.2.42. □

*Remark* 3.2.43. The class of functions $\mathcal{K}^\nu$ defined in Lemma 3.2.42 is also used in the definition of the calculus by Christ et al. [CGGP92].

As an application of Theorem 3.2.40, let us extend Liouville's Theorem to homogeneous Lie groups.

### 3.2.8 Liouville's theorem on homogeneous Lie groups

Let us consider the following statement and proof of Liouville's Theorem in $\mathbb{R}^n$:

**Theorem 3.2.44** (Liouville)**.** *Every harmonic tempered distribution is a polynomial. This means that if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\Delta f = 0$ in the sense of distributions where $\Delta$ is the canonical Laplacian, then $f$ is a polynomial on $\mathbb{R}^n$.*

*Proof.* Let $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\Delta f = 0$. Then $|\xi|^2 \widehat{f} = 0$ where $\widehat{f}$ is the Euclidean Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ on $\mathbb{R}^n$. Hence the distribution $\widehat{f}$ is supported at the origin and must be a linear combination of derivatives of the Dirac distribution at 0, see Proposition 1.4.2. Consequently $f$ is a polynomial. $\square$

Liouville's Theorem and its proof given above are also valid for any homogeneous elliptic constant-coefficient differential operator on $\mathbb{R}^n$. We now show the following generalisation for homogeneous Lie groups:

**Theorem 3.2.45** (Liouville theorem on homogeneous Lie groups)**.** *Let $L$ be a homogeneous left-invariant differential operator on a homogeneous Lie group $G$. We assume that $L$ and $L^t$ are hypoelliptic on $G$. If the distribution $f \in \mathcal{S}'(G)$ satisfies $Lf = 0$ then $f$ is a polynomial.*

The rest of this section is devoted to the proof of Theorem 3.2.45. We follow the proof given by Geller in [Gel83].

Let $\widehat{\cdot}$ denote the Euclidean Fourier transform on $\mathbb{R}^n$ (cf. (2.25)). In view of the proof of Theorem 3.2.44, we want to show that the distribution $\widehat{f}$ is supported at 0. For this purpose, it suffices to show that any test function $\phi \in \mathcal{S}(G)$ whose Euclidean Fourier transform is supported away from 0, that is, $\operatorname{supp} \widehat{\phi} \not\ni 0$, can be written as $L^t\psi$ for some $\psi \in \mathcal{S}(G)$. Indeed, denoting momentarily $\iota(x) = -x$ for $x \in G$ identified with $\mathbb{R}^n$, and by $\check{\cdot}$ the inverse Fourier transform on $\mathbb{R}^n$, we have $\check{\phi} = \widehat{\phi} \circ \iota$, so that $\operatorname{supp} \check{\phi} = \operatorname{supp} \widehat{\phi}$, and

$$\langle \widehat{f}, \check{\phi} \rangle = \langle f, \phi \rangle = \langle f, L^t\psi \rangle = \langle Lf, \psi \rangle = 0.$$

The set of functions $\phi$ with $0 \notin \operatorname{supp} \widehat{\phi}$ is contained in

$$\mathcal{S}_o(\mathbb{R}^n) := \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) \ : \ \left(\frac{\partial}{\partial \xi}\right)^\alpha \widehat{\phi}(0) = 0, \ \forall \alpha \in \mathbb{N}_0^n \right\}.$$

We observe that the space $\mathcal{S}_o(\mathbb{R}^n)$ can be also described in terms of the group structure using the identification of $G$ with $\mathbb{R}^n$, as

$$\mathcal{S}_o(\mathbb{R}^n) = \mathcal{S}_o(G) = \{\phi \in \mathcal{S}(G) \ : \ \int_G x^\alpha \phi(x) dx = 0, \ \forall \alpha \in \mathbb{N}_0^n\}.$$

Indeed $\int_{\mathbb{R}^n} x^\alpha \phi(x) dx = c_\alpha (\frac{\partial}{\partial \xi})^\alpha \widehat{\phi}(0)$ with $c_\alpha$ a known non-zero constant. Here $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$ and the Haar measure on $G$ since these two measures coincide via the identification of $G$ with $\mathbb{R}^n$.

By Theorem 3.2.40, the operator $L^t$ has a fundamental solution $\kappa \in \mathcal{S}'(G)$ satisfying Part (a) or (b) of the statement. Thus we need only showing that for any $\phi \in \mathcal{S}_o(G)$, the function $\psi := \phi * \kappa$ is not only smooth (cf. Lemma 3.1.55) but also Schwartz. This is done in the following lemma:

**Lemma 3.2.46.** *If $\phi \in \mathcal{S}_o(G)$ is a Schwartz function and $\kappa \in \mathcal{S}'(G)$ is a homogeneous distribution smooth away from the origin or a distribution of the form $\kappa = p(x)\ln|x|$ where $p$ is a polynomial and $|\cdot|$ a homogeneous quasi-norm smooth away from the origin, then $\phi * \kappa \in \mathcal{S}(G)$.*

The end of this section is devoted to the proof of Lemma 3.2.46; this relies on consequences of the following versions of Hadamard's Lemma for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_o(\mathbb{R}^n)$:

**Lemma 3.2.47** (Hadamard)**.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $\int f = 0$. Then $f$ can be written as*

$$f = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \qquad \textit{with} \quad f_j \in \mathcal{S}(\mathbb{R}^n)$$

*In addition, if $f \in \mathcal{S}_o(\mathbb{R}^n)$, each function $f_j$ can be also taken in $\mathcal{S}_o(\mathbb{R}^n)$.*

*Proof of Lemma 3.2.47.* We fix $\chi_o \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi_o(\xi) = 1$ if $|\xi| \leq 1$ and $\chi_o(\xi) = 0$ if $|\xi| > 2$. Since $\int f = 0$ we have $\widehat{f}(0) = 0$ and

$$\widehat{f}(\xi) = \chi_o \widehat{f} + (1-\chi_o)\widehat{f} = (\chi_o \widehat{f}) - (\chi_o \widehat{f})(0) + (1-\chi_o)\widehat{f}.$$

We can write

$$(\chi_o \widehat{f})(\xi) - (\chi_o \widehat{f})(0) = \int_0^1 \partial_t \Big( \left(\chi_o \widehat{f}\right)(t\xi) \Big) dt = \sum_{j=1}^{n} \xi_j \int_0^1 \frac{\partial(\chi_o \widehat{f})}{\partial \xi_j}(t\xi) dt,$$

and

$$(1-\chi_o)\widehat{f}(\xi) = \sum_{j=1}^{n} \xi_j^2 \frac{1-\chi_o(\xi)}{|\xi|^2} \widehat{f}(\xi) \quad (\text{here } |\xi|^2 = \sum_{j=1}^{n} \xi_j^2).$$

We set

$$h_j(\xi) := \int_0^1 \frac{\partial(\chi_o \widehat{f})}{\partial \xi_j}(t\xi) dt + \xi_j \frac{1-\chi_o(\xi)}{|\xi|^2} \widehat{f}(\xi).$$

The first term is compactly supported (in the ball of radius 2), whereas the second one is well defined and is identically 0 on the unit ball. Since both terms are smooth, $h_j \in \mathcal{S}(\mathbb{R}^n)$. We have obtained $\widehat{f} = \sum_j \xi_j h_j$. We define $f_j \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{f_j} = c_j h_j$ where the constant $c_j$ is such that $\widehat{\partial_j} = c_j \xi_j$. Hence $f = \sum_j \frac{\partial f_j}{\partial x_j}$.

Moreover, since

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} h_j(0) = \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{\partial}{\partial \xi_j} \widehat{f}(0),$$

we see that if $f \in \mathcal{S}_o(\mathbb{R}^n)$ then $f_j \in \mathcal{S}_o(\mathbb{R}^n)$. $\square$

We will use the following consequence of Lemma 3.2.47 (in fact only the second point):

**Corollary 3.2.48.** • *If $f \in \mathcal{S}_o(\mathbb{R}^n)$, then for any $M \in \mathbb{N}_0$,*

$$f = \sum_{|\alpha|=M} \left(\frac{\partial}{\partial x}\right)^{\alpha} f_\alpha \qquad \textit{with} \quad f_\alpha \in \mathcal{S}_o(\mathbb{R}^n).$$

- *If $f \in \mathcal{S}_o(G)$ where $G$ is a homogeneous Lie groups, then for any $M \geq 1$, we can write $f$ as a finite sum*

$$f = \sum_{[\alpha]>M} X^\alpha f_\alpha$$

*with $f_\alpha \in \mathcal{S}_o(G)$.*

*Proof of Corollary 3.2.48.* Both points are obtained recursively, the first one from Lemma 3.2.47 and the second from the following observation: if $f \in \mathcal{S}_o(G)$, there exists $g_j \in \mathcal{S}_o(G)$ such that $f = \sum_{j=1}^n X_j g_j$. Indeed writing $f$ as in Lemma 3.2.47 and using (3.17) with Remark 3.1.29 (1), we set

$$g_j := f_j + \sum_{\substack{1\leq k\leq n\\ \upsilon_j<\upsilon_k}} (p_{j,k} f_j)$$

and we see that $g_j \in \mathcal{S}_o(G)$. □

We can now prove Lemma 3.2.46.

*Proof of Lemma 3.2.46.* Let $\kappa$ be a distribution as in the statement. We can always decompose $\kappa$ as the sum of $\kappa_0 + \kappa_\infty$, where $\kappa_0$ has compact support and $\kappa_\infty$ is smooth. Indeed, let $\chi \in \mathcal{D}(G)$ be identically 1 on a neighbourhood of the origin and define $\kappa_0$ by

$$\langle \kappa_0, \phi\rangle := \langle \kappa, \chi\phi\rangle.$$

Then

$$\kappa_\infty := \kappa - \kappa_0$$

coincides with $(1-\chi)\kappa_o$, where $\kappa_o$ is a smooth function on $G\backslash\{0\}$ either homogeneous or of the form $p(x)\ln|x|$; we denote by $\nu$ the homogeneous degree of the function $\kappa_o$ or of the polynomial $p$.

Let $\phi \in \mathcal{S}_o(G)$. Since the distribution $\kappa_0$ is compactly supported, we get, by Lemma 3.1.55, that $\phi * \kappa_0 \in \mathcal{S}(G)$. Since, by Corollary 3.2.48, we can write $\phi$ as a (finite) linear combination of $X^\alpha f$ with $f \in \mathcal{S}_o(G)$ and $[\alpha]$ as large as we want. We observe that

$$(X^\alpha f) * \kappa_\infty = f * \tilde{X}^\alpha \kappa_\infty$$

and that for $[\alpha]$ larger that $|\nu| + N + 1$ for $N \in \mathbb{N}_0$ fixed, we have

$$|\tilde{X}^\alpha \kappa_\infty(x)| \leq C_N (1 + |x|)^{-N}.$$

Thus

$$\begin{aligned} |(X^\alpha f) * \kappa_\infty(x)| &= |f * \tilde{X}^\alpha \kappa_\infty(x)| = \left| \int_G f(y) \tilde{X}^\alpha \kappa_\infty(y^{-1}x) dy \right| \\ &\leq \int_G |f(y)| C_N (1 + |y^{-1}x|)^{-N} dy \\ &\leq C_N C_o^N (1 + |x|)^{-N} \int_G |f(y)| (1 + |y|)^N dy, \end{aligned}$$

by (3.43). This shows that $\phi * \kappa_\infty \in \mathcal{S}(G)$. □

Hence Lemma 3.2.46 and Theorem 3.2.45 are proved.

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