Chapter 8
Mathematics in the Hands of Deaf Learners and Blind Learners: Visual–Gestural–Somatic Means of Doing and Expressing Mathematics

Lulu Healy, Elizabeth Becerra Ramos, Solange Hassan Ahmad Ali Fernandes, and Jurema Lindote Botelho Peixoto

8.1 Introduction

In this chapter, we focus on the linguistic resources used in the mathematics practices of students who lack, or who have very limited, access to one or other sensory field. Attention to students with disabilities is still relatively rare within the field of mathematics education as a whole. In the area of language diversity and mathematics learning, research has tended to consider diversity in relation to spoken languages rather than languages and linguistic resources expressed through other modalities, such as signed languages, or Sign,[1] and gestures. This scenario is beginning to change, at least with respect to those visual–gestural–somatic expressions described as gestures, with recent years bringing an increased attention to their communicative and cognitive functions in mathematical activities (see, for example, Edwards, Ferrera, & Russo-Moore, 2014; Nemirovsky, Kelton, & Rhodehamel, 2013; Radford, Edwards, & Arzarello, 2009) and an accompanying recognition of the multimodal nature of mathematical understandings (Radford, 2009; Roth, 2010; Roth & Thom, 2009). From these perspectives, the ways that

[1]Following Sacks (2000), we use the term Sign to denote all indigenous signed languages, though in this paper we limit our attention to MSL (Mexican Sign Language) and Libras (Brazilian Sign Language).

L. Healy (✉) • S.H.A.A. Fernandes
Anhanguera University of São Paulo, São Paulo, SP, Brazil
e-mail: lulu@baquara.com; solangehf@gmail.com

E.B. Ramos
Cinvestav-IPN, Mexico City, Mexico
e-mail: ebecerra@cinvestav.mx

J.L.B. Peixoto
State University of Santa Cruz, Illheus, Bahia, Brazil
e-mail: peixotojurema@gmail.com

R. Barwell et al. (eds.), *Mathematics Education and Language Diversity*, New ICMI Study Series, https://doi.org/10.1007/978-3-319-14511-2_8

linguistic resources are appropriated and used have their basis in the human body, its sensory and motor capacities and its location in space and time. For us, this suggests that a focus on the linguistic practices of those who do not hear with their ears or see with their eyes may open particular windows on mathematical cognition.

In case it may seem that we are treating mathematical cognition as an individual affair, bounded only by the body, we should stress that we see learners as essentially cultural beings and their participation in linguistic practices makes this particularly evident. Although bodily grounded, these practices are a central aspect in defining cultures, as every language, be it verbal, gestural, pictorial or of any another nature, is representative of a particular social group (Gee, 2014). The complexities associated with attending to different linguistic practices can result in tensions within the school context, especially if those practices that are emphasized do not take into account the specificities of the learners in question.

This has certainly been the case for deaf learners, whose history of participation in educational activities has been marked by a veritable battlefield related to linguistic modality. In the case of blind mathematics learners, on the other hand, considering linguistic resources of a visual-spatial nature, such as gestures, might seem misguided, since such resources will not be seen. However, there is evidence to suggest that even those who are born blind make use of gestures when speaking (Iverson & Goldin-Meadow, 1998) and when engaging in mathematics (Healy & Fernandes, 2011, 2014).

Whether we are speaking of Sign or gesture, Rotman (2009) has pointed to a general tendency to devalue communication systems which make use of the visual modality as compared to orally based ones and to assume that language should be identified with speaking, while communications using body movements are judged more primitive and nonintellectual. Perhaps the clearest example of this devaluation can be found in the history of Sign in the education of the deaf. In the following section, to consider how the attention given to visual rather than verbal communication forms has changed over time, we begin by recapitulating moments from this history.

8.2 The Rise, Fall and Rise of Sign and Gesture

For those who are born deaf into a hearing world, it is not so much the absence of sound but the consequences which derive from this which have dictated the ways in which they have been positioned and defined (Sacks, 2000). Before the mid-1700s, the deaf were treated as uneducable, their "inability" to speak indicative of an intellectual disability. It was only when attention began to be paid to signed languages that this view was challenged, initiating attempts to include deaf learners in the education system and also giving rise to a still ongoing debate about the type of language to be used in educational practice. The debate is frequently polarized in terms of use of signed languages verses oral methods.

On one side was the Frenchman Abbé de L'Epée, who, based on the signs used by the deaf people of Paris, elaborated the sign system, Methodical Sign. In 1760, he founded the first public school for the deaf in France, the "Institution Nationale des sourds-muets" in which this system was used. His interest in manual communication forms occurred as French philosophers were debating the origins of human languages and the sensualist Condillac was arguing that the language of action constituted the original, natural language. Condillac's view was that the transformation of embodied experiences into shared material signs (the "transformed sensations" that represent this language of action) holds the key to human knowledge.[2] Such a position implies no intellectual hierarchy between manual or verbal forms of communication.

At the same time, in Germany, the educator Samuel Heinicke was developing an oral/aural method to teach deaf people to speak. He was strongly opposed to the methods used by L'Epée and the ideological split was born (Moura, 2000). Almost a century later, in 1880, at the Congress of Milan participants "voted to proclaim that the German oral method should be the official method used in the schools of many nations" (Lang, 2003, p. 15). Deaf people were excluded from this vote. In announcing the congress recommendations, one of the organizers put it thus:

> Gesture is not the true language of man which suits the dignity of his nature. Gesture, instead of addressing the mind, addresses the imagination and the senses. Thus for us, it is an absolute necessity to prohibit that language and replace it with living speech…The fantastic language of signs exalts the senses and foments the passions, whereas speech elevates the mind much more naturally, with calm and truth. (Guilio Tarra quoted in Lane, 1984, pp. 391–394)

Following the Congress of Milan, Oralism, with its supposed intellectual and even moral superiority, dominated and, for many years, deaf learners were discouraged, and frequently physically forbidden, from using sign language during schooling.

It was only in the 1960s and 1970s that this dominance began to be challenged, particularly after the scientific recognition of American Sign Language (and consequently the sign languages of other countries throughout the world) as a true and natural language (Stokoe, 1960/2005). When the studies into the structure of sign languages began to emerge, formalist models of language drawn from structural linguistics were at their height. In this context, signs had to be shown to be equivalent to lexical items in spoken languages and phonological structures in sign languages corresponding to those of spoken languages needed to be identified (Armstrong & Wilcox, 2003). Clearly, in relation to deaf education and culture, recognition that the visual–gestural systems of communication used by the deaf are proper languages was (and is) fundamental. Yet, there were two perhaps unintended consequences of these attempts to demonstrate that the structures underlying signed languages were the same as spoken ones.

First, the issue of how the modality of Sign might impact on its nature has tended to be de-emphasized and, in particular, the iconicity of many signs downplayed, as

---

[2] According to Kendon (2008), at this point in time, distinctions were not drawn between gestures and Sign.

signifiers have had to be shown to be arbitrary in relation to what they signify. Second, though gestures might be produced in the same medium as signs, in the formalist view, they differ in that they are spontaneous and idiosyncratic and not constructed according to any standard forms, which meant, it could be argued, that they are not part of language. In order for Sign to be attributed the same status as spoken language, Sign and gesture were hence organized as distinct categories: Sign regarded as something linguistic and gesture seen as external to language, paralinguistic (Kendon, 2008; Wilcox & Morford, 2007).

It was only in the mid-1980s that gestures began to be seen as an integral part of language, with McNeill (1985) showing how they are used during speaking to constitute the conceptual content of the utterance. For Kendon (2008), though, while McNeill's work led to an increased attention to gestures, in general they were still seen as a kind of appendage or add-on to speech. A more radical view is offered by Rotman (2009), who argues rather that speech is a species of gesture, perceived by auditory rather than visual means, but a gesture nevertheless. This implies that gestures are as much a part of the set of language resources used to share experiences of the world as are the components of spoken and signed languages. It also brings us back to the beginning of this section, and to Condillac's conjecture that languages emerged from a process of transforming sensations (Hewes, 1996; LeBaron & Streeck, 2000). Hence, we might argue that the valuing of visual–gestural–somatic modalities of communication goes hand in hand with recognizing the embodied nature of our sense-making experiences.

In mathematics education, as well as in the area of linguistics, it is only recently that such recognition has gained space. As a consequence, we still know very little about what it means to learn and to do mathematics using the visual–gestural modality. To a certain extent, in relation to this modality, the learners on whom we focus in this chapter represent two extremes. The visual modality is ever present for the deaf, with evidence to suggest a preference for visual reasoning amongst those who do not hear (Bull, 2008; Kelly, 2008; Monteiro & Andrade, 2005; Nunes & Moreno, 2002), while, for the blind, spatial and visual information is not seen but felt or heard. By concentrating on these learners, then, what might be learnt about the role of visual–gestural–somatic language resources in mathematics learning?

8.3 Sensory Modalities and Knowledge Mediation

It was a related question, though concerned with learning more generally, which appears to have motivated the construct of knowledge mediation in the sociocultural perspective of Vygotsky. This construct has its roots in his work with differently-abled individuals (Vygotsky, 1997). In Vygotsky's view, all higher mental functions are mediated. A mediated mental function involves an indirect action on the world, which incorporates and transforms the natural, basic mental processes, extending their range and mode of functioning. The inclusion of the tool in activity alters the course both of the activity and of all the mental processes that enter into the instrumental act (Vygotsky, 1981). Material and semiotic tools do not just enable cognitive activity; they are part of the act of thinking *and* that which is being thought about (knowledge).

For Vygotsky, the use of language as an instrument in thinking is central to the ways that learners appropriate—that is, make their own—the forms of acting and communicating which characterize the social groups to which they belong. Generally speaking the instruments and languages of culture tend to be designed for those considered "normal," for those who have all the organs of the senses and the sensorial functions intact, meaning that they may not be accessible to some. In the perspective offered by Vygotsky, the solution for the inclusion of individuals with disabilities in social (cultural) activities lies in seeking ways to substitute the traditional mediational means with others, more suited to the specific ways in which they interact with others and with culturally defining objects.

Consistent with his ideas about mediation, he believed that while enabling intellectual development, the substitution of one tool by another (for example, signed rather than spoken language for the deaf or hands as seeing tools for the blind) could lead to the emergence of different developmental paths, since, just as the inclusion of any other tool in the process of activity alters its entire structure and flow, so too the substitution of the ear or the eye by another instrument would be expected to be associated with a profound restructuration of the intellect.

Here, we should make very clear that we are not referring to a state of deficiency, but to one of difference. To better understand the deaf mathematics learner, we need to better understand what it means to practice mathematics in the medium of sign language and how those whose cognitive processes are mediated by a visual–gestural–somatic language as opposed to a sequential-auditory language come to think mathematically. Similarly, to better understand the blind mathematics learner, we need to investigate how those who process visual data through touch or sound express the mathematical properties that they feel or hear.

We now turn to our attempts to contribute to the development of such understandings. We offer two examples involving deaf learners in order to consider in more detail the challenges associated with learning mathematics when visual–gestural–somatic language resources are the dominant forms of communication. We then extend the discussion of the relationships between sensory experiences, language and mathematics learning by including an example examining gesture use by blind learners.

In the light of the previous discussion, our aim in presenting these examples is threefold: to explore the characteristics of mathematical activity privileged by the visual–gestural–somatic forms of expression; to consider the relationships between lexical terms (be they signed or spoken) and gestures; and, more generally, to seek evidence of how mathematical activity and understandings are shaped by (and shape) different ways of sensing and acting and different ways of attempting to share these experiences with others.

8.4 Language Resources of Deaf Mathematics Students

The rise, fall, and recent rise again of interest in Sign and gesture has had, and continues to have, profound implications for the education of the deaf. In the countries in which we work, Brazil and Mexico, it is only during this century that Sign has begun to be (re-)considered as a medium for teaching and learning. Currently, at least some schools (both specialized and mainstream) are beginning to adopt approaches in which Sign is considered as the deaf learner's first language (L1) and the written version of Portuguese or Spanish as a second language (L2).

While linguistic research has demonstrated that signed languages are the natural languages of deaf communities (Cruz, 2008), Sign is not a universal language. There are a large variety of different signed languages across the world, many of which have developed independently of each other. However, the largest differences between signed languages appear to be lexical in nature, with grammatical features shared in most of the signed languages studied to date. This is the case for the two signed languages that figure in the examples to follow Libras and MSL.[3] Signs in both these languages are divided into five categories:

(a) The *form or configuration* which the hand or hands adopt when performing the sign.
(b) The *place or location* in which the sign is performed.
(c) The *movement* performed by the hand, which can have a variety of internal elements, including trajectory, direction, speed, rotation, muscular tension, and vibration among others.
(d) The *orientation* of the hand's palm.
(e) The *body or facial expression*, which accompanies the sign.

Differences in any one of these five parameters can give rise to assigning distinct meanings.

Their grammars, like the grammars of many other signed languages, make similar use of locations and orientations in space, the direction, quality and speed of movements, facial expressions, and sign orders. Thus, Sign is "written" in space, the signer manipulates the space to refer to spatial, temporal, and grammatical matters and different spatial planes are used to manage the timeline, present, past, and future.

Another shared construct of signed languages, and one that has been associated with a degree of linguistic controversy, is its iconicity. Because Sign is visual–gestural–somatic and the visual properties of entities and actions are so readily accessible, they are utilized in abundance in Sign. How iconicity is used in the emergence of language in creating novel practices and in historical change are questions that are beginning to attract research (Brentari, 2010) and the first two examples bring this discussion to the area of mathematics education.

[3] See Brito (1995) for a discussion of Libras and Cruz (2008) for a description of MSL.

Fig. 8.1 Signing 3 and beating his hand three times on the tables

8.4.1 *Example 1: The Visual Modality in Arithmetic Calculations*

The first study we consider was carried out in the State of Bahia in the Northeast of Brazil. The data reported here were part of a larger study whose aim was to survey school students' knowledge of additive and multiplicative structures.[4] Here, we focus on the strategies of a profoundly deaf student, Rodrigo[5] (aged 24) as he carried out two calculations: 32×3 and 65×6. Rodrigo only started to learn Libras as an adolescent, having begun his education in a specialized school before the policies on bilingual education had taken hold in Brazil. We focus here on the signs, gestures, and written expressions that emerged during his calculations.

Faced with the task of multiplying 32 by 3, Rodrigo began by signing with his left hand the number 3, then beating this configuration three times on the table (Fig. 8.1). This seems to have been a way of signalling to himself the calculation he should perform. To obtain the result, he signed 3 once again, this time to referring to the digit 3 in 32. He used his right hand, which he moved in space twice (Fig. 8.2). As his right hand was moved to the second location, he simultaneously signed 6 with his left. Maintaining the three on his right hand, he then used his left to sign 7, 8, 9, arriving at the first part of his answer.

Having obtained the result associated with the digit 3 in 32, Rodrigo repeated the same procedure with the 2, moving the sign of 2 on his left hand (Fig. 8.3), to three locations in spaces while counting, in Sign, with his other hand, 3, 4 then 5, 6. He then registered his result on paper (Fig. 8.4).

[4]This research was funded by the Fundação de Amparo à Pesquisa do Estado da Bahia (FAPESB 2008–2010).

[5]The names of the students whose work is presented in the paper have been changed. All of them participated voluntarily in the respective studies.

Fig. 8.2 Signing 3 in different locations while counting

Fig. 8.3 Signing 2 and moving the sign in space

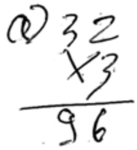

Fig. 8.4 Rodrigo's written expression

This strategy illustrates a difference between signed and spoken languages, in that using Sign, Rodrigo is able to say more than one word at the same time. What is possible, though, in spoken language is that the speech be accompanied by spatial gestures, indicating that the visual–gestural–somatic modality can bring simultaneity to expressions in both cases. Indeed, some of Rodrigo's hand movements are also better described as gestures rather than signs: the beating of the 3 on the table, for example, and the positioning of signs in space to keep track of the number of repetitions.

Although the difference in the meanings of the 3 and the 2 in 32 are not made explicit in Rodrigo's expressions, his negotiation of the calculation 65×6 suggests he was aware of this difference. This time, he started with the digit 5 related to the unit value. Once again, he combined a variety of expressions, including signs, gestures and written inscriptions. He began by signing 5 on his right hand and simultaneously

Fig. 8.5 Simultaneously signing 5 on one hand and 1, 2, 3, 4, 5, 6 on the other

Fig. 8.6 Joining his fingers 2 by 2

signing from 1 to 6 with his left, suggesting that he was thinking 5 once, twice, thrice up until 6 times (Fig. 8.5).[6] He then calculated this by holding out 6 fingers, each of which to represent 5 and then joining his fingers in pairs, with each pair representing 5 + 5 (Fig. 8.6). He repeated the strategy for the digit 6, this time also recording his method on paper (Fig. 8.7). Finally he completes the written multiplication correctly (Fig. 8.7).

In both the calculations that Rodrigo performed, he capitalized on the fact that when using a visual–gestural–somatic language it is possible to say more than one word at the same time. Nunes (2004) also observed this practice in her work with British deaf learners, describing a spontaneously developed strategy to arrive at the sum of two whole numbers by counting up with one hand while simultaneous counting

[6] We note here that the sign in Libras for the number 5 does not involve holding out five fingers.

In fact this is the case only for the numbers from one to four, as shown alongside.	

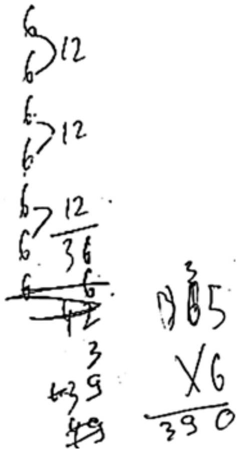

Fig. 8.7 Rodrigo's written calculations

down with the other. Considering these strategies through a Vygotskian lens, it seems that the substitution of spoken language by Sign altered the possibilities through which the calculations might be negotiated. At the very least, we argue that the use of visual–gestural–somatic communication modes brings a simultaneity to linguistic practices not present in the serial-auditory utterances of spoken language. In the next example, we consider another consequence of this substitution, and compare the dynamism of Sign to the more static nature of spoken language.

8.4.2 *Example 2: Bringing Dynamism to Mathematical Discourse*

The second example is drawn from a study carried out in a Mexican school for deaf students. The school adopts a bilingual educational model (written Spanish and MSL), but the particular aim of this study was to design and investigate mathematics learning scenarios in which the dynamics of the class are mediated entirely through Sign and through visual representations: that is, with no recourse to written (or oral) language. The scenario we describe here involves Pythagoras' Theorem and interactions with visual proofs of the relationships between the sides of right-angled triangles that it specifies. Four deaf students participated in the activities, two 19 year olds and two 16 year olds. Both the older students were competent users of MSL, while the younger students were still in the process of learning MSL.

To begin, the four students were presented with a series of figures (Fig. 8.8). They were asked to describe the figures, in accordance with characteristics of their choice, which might include form, area, size, or even color. They were then asked specifically to compare the areas of the different polygons in the figures.

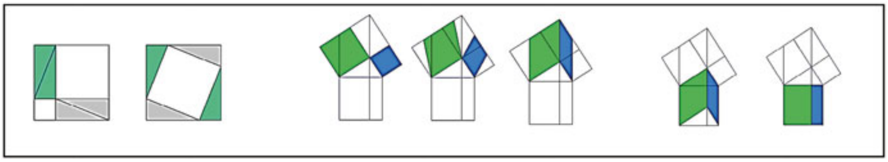

Fig. 8.8 Figures for Pythagoras' Theorem (from the book "Proofs without words," Nelsen, 1993)

Fig. 8.9 "Square" in MSL

In turn, they were shown a series of dynamic geometry applets also illustrating, this time with movement, demonstrations of Pythagoras' Theory. Following the consideration of static and dynamically presented visual proofs, the group worked on a series of examples in which they were asked to relate diagrams with the algebraic representation of the theory and to calculate the values of the sides of the triangle and the area of their squares in a number of specific cases.

A considerable difficulty of researching the interactions in scenarios involving Sign-speakers is the registering of evidence. Video-recordings help to capture the dialogue, but without several cameras filming the same scene, it is easy for signs to occur off camera, or for only part of a sign, say hand-form and movement to be captured, without the accompanying facial expression. Even where the video data is complete, literal transcriptions cannot be made as signed languages are written in space, and there is a danger that, if we rely on written interpretations of the dialogues, we may lose some of the particularities associated with doing mathematics in Sign.

Perhaps the best way to illustrate this difficulty is to present examples of the signs that emerged during the sequence of activities. Some are totally iconic, their form corresponding pictorially to the object being signed. Examples include the sign for squares, rhombi, and for some kinds of triangle. Figure 8.9 presents the MSL sign that was used in the study to represent squares.

Other signs have an iconic component, but are accompanied by some sort of movement. Examples include the signs for rectangles, isosceles, and right-angled triangles, and some of the signs used to make comparisons. In the case of the geometrical figures, the movement conveys a certain level of generality, indicating a class of objects rather than one a specific case, as well as making explicit particular properties of the shape in question. Figure 8.10, for example, presents the MSL sign for rectangle. In its initial appearance, it is the same as the square, but whereas the square sign does not involve movement, to sign "rectangle" involves keeping the hand showing three sides stationary while the finger representing the fourth side is translated in a horizontal way, as if the two horizontal sides were being gradually extended.

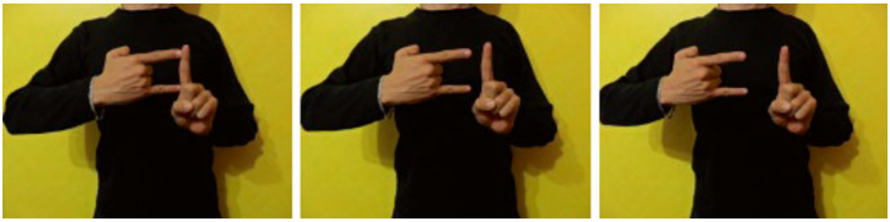

Fig. 8.10 Signing "rectangle" in MSL

Fig. 8.11 Signing "isosceles triangle" in MSL

Fig. 8.12 Signing "more than…"

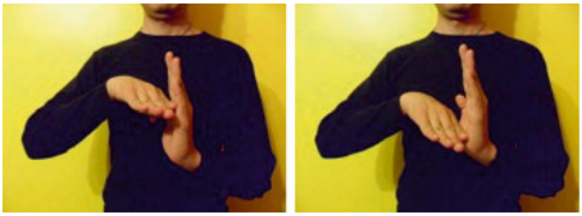

Fig. 8.13 :… and "less than"

The sign for "isosceles triangle" is similarly dynamic and involves a movement which implies maintaining two sides congruent (Fig. 8.11).

These signs are interesting in that they differ both from the spoken words and from the visual drawings that are usually used to represent geometrical objects. In the dynamic sign, the generic nature of the object being denoted is much more explicit than in a paper and pencil drawing or perhaps even in a spoken word. MSL, then, brings a dynamism to mathematical discourse. This dynamism is not only evident in the signs representing objects, but also in those which denote relationships. Figures 8.12 and 8.13 show the movements associated with the MSL signs for comparing "more than" and "less than."

Through the use of signs like these, the four students were able to argue that the sums of the area of the smaller squares in each of the diagrams shown in Fig. 8.8 were equal to the area of the larger square, and to describe the transformations by which they determined this. They were also able to successfully relate the visual representation of the Pythagorean Theorem with its algebraic representation $c^2 = a^2 + b^2$. This indicates that MSL allows for the abstraction of the algebraic representation from a graphic one. Thus, in spite of the fact that the way of expressing their ideas is very iconic, and the ideas themselves based also on the visual material at their disposal, the students could make sense of a less visual means of expressing the mathematical ideas they worked with.

We might even speculate that the ways in which movement is incorporated into the signs expressed by deaf mathematics learners embodies variation in a way that is different to algebraic symbolism but perhaps serves as an effective means of enacting meanings for it: the dynamic movement of, say, the opposite sides of a rectangle in its sign already indicating that the measure of these sides is a value that varies, while their relationship to each other does not. Of course, it may also be the case that the signs constrain as well as afford the interpretations that come to be associated with the objects they are intended to represent. The signs for both "rectangle" and "isosceles triangle" privilege the prototypical orientations of the objects, presenting them horizontally. Based on these observations, we would suggest a need for more research investigating the signs for mathematical terms in different signed languages, so we might better understand how the signs were created and the properties that they appear to privilege or constrain in use.

8.4.3 Traces of Enactments in the Signs and Gestures of Deaf Mathematics Learners

Although the mathematical content, demands and structuring of the activities were rather different in these two examples, there is evidence in both that language resources of a visual–gestural–somatic nature not only enable the expression of mathematical objects and properties, but also shape all the aspects of the learners' activities with them.

The combination of signs and gestures used by Rodrigo as he multiplied were made not so much in order to communicate his ideas to others, they were integral to the processes of thinking that occurred. We might say that the gestures and signs served as visually expressed enactments of imagined activity: Rodrigo's holding up of six fingers, each one of which represented five objects, suggests he was imagining a physical process of combining objects. In a similar way, the signs used to refer to mathematical objects in Mexican Sign Language might also be seen as enactments or re-enactments of the activities involved in producing and exploring such objects. Yet, although firmly connected to enactments of physical doings, the visual–gestural–somatic expressions also bring evidence of processes of abstraction, at least if we define abstraction as some conscious appreciation by learners of the generalized relationships implied in their expressions (as do Mason, 1989; Noss & Hoyles, 1996).

This is especially so in relation to the signs described in the second example: creating a set of isosceles triangles by moving the hand in a way that preserves the congruency of two sides, for instance, a form of explicitly articulating this general property.

In the context of the visual–gestural–somatic expressions of generality made by the deaf learners, abstraction does not seem to involve the detachment of objects, properties, and relationships from the settings in which they were encountered: that is, their mathematics learning does not seem to involve a process of de-contextualization. They continue to sense, to feel, the multimodal experiences involved in the process of identifying generalities: thinking, speaking or signing and gesturing, are accompanied by feeling. Indeed, we offer the gestures and signs described in the examples as one source of evidence to support a premise of embodied cognition: that thinking involves re-enacting and hence re-feeling previously experienced activities. Re-enactment, though, should not necessarily be seen simply as a replay. It is a new performance, that Nemirovsky et al. (2013) describe as a "social-interactive experience of bringing to presence something which is absent in the current surroundings of the participants" (p. 3). In its virtual form, this something can be acted upon in new ways, providing new forms of experiencing its potential in new kinds of activity.

8.5 Visual–Gestural Expressions of Blind Mathematics Students

While it is not surprising that the visual–gestural–somatic modality is central in the case of deaf learners, what would we expect in the language activities of blind mathematics learners? The following example suggests that this modality represents as integral a part of the language activities of the blind as it does for those who see with their eyes, with even students who have never seen (visually) the gestures of others spontaneously producing gestures in the course of their mathematical explorations and explanations.

8.5.1 Example 3: Embodied Abstractions in the Gestures of a Blind Student

This last example comes from an ongoing programme of research[7] in which we are exploring relationships between sensory experience and mathematical knowledge. Our research activities have included attending to how the use of hands to substitute eyes, and touch to substitute sight, in the sense proposed by Vygotsky, impacts upon the mathematical practices of blind learners. Focusing on the hands of these learners as they explore material–tactile–representations of geometrical objects, is

[7] We are grateful to the funding we have received from FAPESP (Project no. 2004/15109-9) and CAPES (Project no. 23038.019444/2009-33) in the course of this programme of research.

illuminating both the extensive use of visual–gestural–somatic language resources by blind students, as well as the embodied nature of the mathematical interpretations that these expressions imply.

We have chosen an episode from a sequence of learning situations undertaken with a group of four learners who attended a mainstream school in São Paulo, Brazil. The learners, whose ages were between 14 and 18 years, were first-year high-school students. Marcos (18 years old), Fabio (16 years old), and Caio (17 years old) were all born with different kinds of congenital blindness, while Leandro (14 years old) lost his sight at the age of 2 in an accident that resulted in the severing of his optic nerve. The sequence involved a series of tasks associated with the study of volume, area, and perimeter, implemented over four research sessions, each of approximately 90 min. In the first two sessions, the activities centered on the area and perimeter of plane figures, the third initiated work on volume and in the final session, the students worked on a task which involved determining the most economical amongst a range of boxes and other rectangular prisms. All the research sessions were videotaped and transcribed for analysis. Two researchers and the schools' special needs teacher also participated in the sessions.

In the first session, the students worked with a wooden board containing the impressions of four different rectangles, which could be filled either with wooden unit cubes or with rectangular and triangular shapes in foam rubber. In this session, the students explored the perimeters and areas of rectangles and right-angled triangles. In particular, they experienced how the area of a triangle could be perceived as half the area of a rectangle with the same height and base (for a more detailed analysis of the activities during this session, see Fernandes & Healy, 2010).

The second session was dedicated to determining the areas of the plane figures represented in foldable cardboard. During this session the students worked with adapted rulers in which the number marks were raised so they could be read tactilely. The episode we present here occurred during the explorations of one of the first cardboard figures analyzed by the students, a right-angled triangle with sides of 5, 12, and 13 cm. Each of the students received a cardboard representation of the same triangle, the idea being they would first determine its area and perimeter individually and then share and agree upon their results.

This example centers upon the strategies of one of the students, Leandro, as he attempted to determine the triangle's area and perimeter. Reminded by Caio and Fabio that its area would be half the area of a rectangle of the same height and width, he calculated it to be 30 cm, and explained his thinking to one of the researchers.

Leandro: The area, the area, I understand it how they said. The rectangle would be 60 and dividing would give 30.

Researcher 1: And why would the area of the rectangle be 60?

Leandro: Because it has one side of 12 (*he traces the side of the triangle which measured 12 cm, Fig.* 8.14a), and the other would be 12 (*traces an imaginary segment in space parallel to the side of triangle he had previously indicated, as shown in Fig.* 8.14b). And then 5 (*again tracing the side of measure 5 cm, Fig.* 8.14c) and 5 (*and an imaginary parallel, Fig.* 8.14d).

Fig. 8.14 (**a**) "one side of 12" (L*eandro traces the 13 cm side*…). (**b**) "the other would be 12" (*… and indicates a parallel*). (**c**) "And then 5" (*He traces the 5 cm side*…). (**d**) "and 5" (*… and indicates its parallel*)

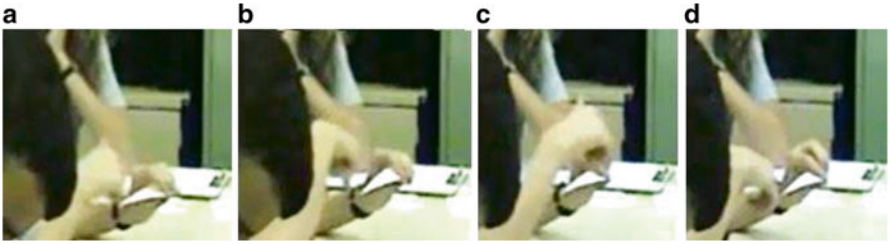

Fig. 8.15 (**a**)"I thought 12," (**b**) "plus 12 on the other side," (**c**) "5," (**d**) "plus 5"

Researcher 1: OK
Leandro: So it would be 30.

The gestures shown in Figs. 8.14a–d were an integral part of Leandro's explanation. We cannot be sure whether they were part of an intentional act to communicate to the researcher (although Leandro could not see his gestures, he knew that the researcher could) or whether they were for himself, as much a part of his thinking as the spoken words. Our conjecture is that, in practice, they served both these functions. Moreover, just as the sign for "rectangle" in MSL emphasizes defining properties of the shape, so too did Leandro's gesturing. Gestures continued to be fundamental as Leandro attempted to determine the perimeter. His first conjecture involved first calculating the perimeter of a 12 by 5 rectangle and then halving the value.

Leandro: I think the perimeter is 17.
Researcher 1: The perimeter of this is 17? Explain why.
Leandro: I thought 12 plus 12 on the other side (*once again he traces two sides of an imaginary rectangle, Fig.* 8.15a, b). 5 plus 5 (*tracing the other two opposite sides Fig.* 8.15c, d). That would give 34 and I divided by 2.17.

This time evoking, through the same set of gestures, the rectangle that had enabled a correct calculation of the triangle's area, led Leandro to overgeneralize the strategy of dividing by 2. Without commenting on the strategy, the researcher suggested that Leandro share his thinking with the other students.

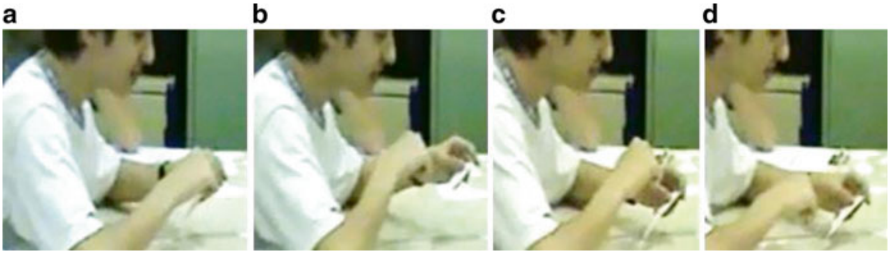

Fig. 8.16 (**a**) "12," (**b**) "plus 12," (**c**) "5," (**d**) "plus 5"

Researcher 1: Leandro has another conjecture about the perimeter. He says that it is 17.

Caio: The right-angled triangle? (*Caio picks up the triangles*)

Leandro: Yes, this here (*holding up his triangle*)

Caio: I don't think so, because if you add the 12 that is here (*runs his finger on the side measuring 12*), the 5 (*traces the 5 cm side*) and the hypoteneuse, you get 30.

This exchange is interesting because both Leandro and Caio use gestures and the deictic reference "here," which the other cannot see. In order to determine that they are discussing the same shape, they hold up in turn the triangle and then, as Caio explains how he calculated the perimeter by adding the measures of the three sides, he traces his finger along each side as he mentions it. In this case, we imagine the gestures were more for themselves than for the other, since, as the dialogue continues, it seems that Leandro has not yet appropriated Caio's method and wants to explain his own,

Leandro: What I did was this Caio. I calculated as if I had the rectangle in my hand. So I went 12 (Fig. 8.16a) plus 12 (Fig. 8.16b). 24. 5 (Fig. 8.16c) plus 5 (Fig. 8.16d). 34. And then I divided.

As he explained to Caio, Leandro repeated, for a third time, the same four gestures in which he traces the two parallel sides of measure 12 cm (one along a side of the triangle and the other in the air) then the two parallel sides of measure 5 cm (again one along a side of the triangle and the other in the air). Caio could not see these gestures, so perhaps it was the verbal reference to the rectangle or the way in which the measures of its sides were recited in congruent pairs that enabled him, and Fabio who had also begun to pay attention to Leandro's explanation, to understand that Leandro's method was incorrect.

Caio: Ah, but the sides of the triangle are not equal (traces the perimeter with his fingers).

Fabio: Not to mention that the hypotenuse is irregular and longer than the others.

In the light of these comments, Leandro revised his method, with Caio's approval.

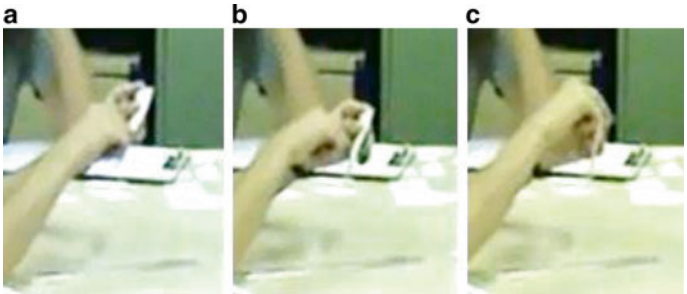

Fig. 8.17 (**a**) "Perimeter is this here," (**b**) "plus this here," (**c**) "Plus this here"

Leandro: Now I get it, So, if you measure the hypotenuse (*he measures the hypotenuse*), the hypotenuse is 13. To find the perimeter of this (*hits the triangle twice*), I calculate 12 plus 13 plus 5?

Caio: Yes, yes.

To check that Leandro had really understood, the second researcher asks Leandro what a perimeter is: his reply, completely correct, once again relied as much on the visual–gestural–somatic modality as on the verbal.

Researcher 2: What is perimeter Leandro?

Leandro: Perimeter is this here (Fig. 8.17a) plus this here (Fig. 8.17b) and this here (Fig. 8.17c).

Leandro's gesture communicates unambiguously to the researcher that he knows what a perimeter is. Because he has a particular figure in hand, it could be argued that his answer is specific rather than general, but in the context of the complete set of gestures that accompanied the dialogues in this episode, we are convinced that he had appropriated a general sense of perimeter and the fact that he chose to define perimeter using a visual–gestural–somatic expression provides evidence for our claim that this modality serves as a language resource as much for those who feel gestures as for those who see them.

The data from this episode also strengthens the claim that gestures emerge as a consequence of imagined re-enactments: in this case as Leandro imagines a rectangle that is not physically there, his hand movements indicating it is almost as if he is feeling the nonpresent shape. The gestures were far from arbitrary hand-waving, with the repeated use of the exact same sequence (Figs. 8.14, 8.15, and 8.16) suggesting that they acted as a kind of embodied abstraction, and a representation of rectangle incorporating aspects of its meaning for Leandro. A striking feature of his gestures, and one that makes them, like the signs of the deaf learners, difficult to capture using the paper and pencil medium, is their dynamic nature. Leandro's fingers were constantly moving as he traced out line segments in ways which preserved their relationship to others in space. Such dynamic gestures in which blind students abstract and express mathematical relationships appear to be characteristic of their interactions with geometrical objects (other examples are available in Healy & Fernandes, 2011 and Healy, 2012).

Just as in the case of deaf mathematics learners, then, it seems that for blind learners too the visual–gestural–somatic modality brings dynamism to mathematical discussions and practices and the ubiquity of such expressions, even amongst those who do not see them, suggest that they are an integral part of the process of learning and doing mathematics.

8.6 Mathematics in Our Hands

The examples presented in this chapter have attempted to explore the role of visual–gestural–somatic expressions in the mathematical practices of deaf learners and blind learners. We began by outlining how, for a considerable period in history, such communication forms were devalued in relation to spoken languages, with, in the extreme case of educating deaf learners, their use prohibited in the classrooms of schools in many countries across the world. One result is that it is only relatively recently that attempts to investigate the role of visual–gestural–somatic expressions in doing and learning mathematics are beginning to emerge. There is, we believe, a factor that we have not yet mentioned that is also contributing to this growing interest. Previously, much research into language use, in mathematics education and beyond, relied on analyses of written transcripts. It is now much more common for video data to form the bases of such analyses. Video-recordings of visual–gestural–somatic expressions, whether in conjunction with spoken language or not, provide a way of recording such utterances for future reference, in a way that writing has traditionally served for recording spoken words.

Not only does this technology make it possible for detailed analyses of Sign and gesture to be undertaken, it also offers the possibility of bringing new language resources to the teaching of mathematics. Neither Libras nor MSL has a widely-used official written representation. This has meant that to access most teaching materials, deaf learners have had to work mainly in their second language. This no longer needs necessarily to be the case, as it would be possible to develop digital resources in which activities are presented in Sign and even in which students' solutions are also recorded in the visual–gestural–somatic modality in which they are produced. Such a scenario is still distant from the realities of the mathematics classrooms in our countries, but we see it as one promising area for future research.

Returning to the present, and to the examples presented in this paper, we have made three main claims. The first is that the visual–gestural–somatic modality is amply used by both deaf students and blind students, in both cases bringing a dynamism and a simultaneity to mathematical discourse less easily expressed in spoken or written representations. These dynamic expressions incorporate properties of the mathematical objects, relations, and operations that they represent and are hence central constituents of the conceptual meanings that the students use in practice.

The second claim concerns the relationships between gestures and Sign and between gestures and speech. Looking at gestures in multiple sensory modalities is central to understanding signed languages, since it is not always easy, even with a competent management of Sign, to distinguish between conventional signs that have a priori meaning and natural gestures that emerge in the discourse. Our data suggest that understanding the meanings intended in the speech and gestures of our blind participants similarly involved considering their utterances as a combination of words and gestures. The deaf student, Rodrigo, and the blind student, Leandro, both used gestures in coordination with the other language resources as tools for thinking. In fact, for Rodrigo both gestures and signs seemed to be directed to organize his own strategies, rather than to communicate with an external interlocutor, while Leandro's words and gestures simultaneously served both roles. Vygotsky (1962) posited word meaning "as a unit of both generalizing thought and social interchange" (p. 9). Our examples suggest that gestures can be similarly conceived as a union of generalization and communication. Words and gestures or signs and gestures were used as simultaneous mediational resources throughout the learners activities, and, as co-temporal simultaneous productions (Goldin-Meadow, 2003), their roles in thinking and communicating are difficult to separate.

It is in this sense that we are attracted to Rotman's view of speech (and by analogy also Sign) as a species of gesture (Rotman, 2009), rather than a position that treats gesture as an appendage or add-on to the "official" language. Indeed, our examples suggest that gestures emerge when no word or sign is available that would communicate the meaning that the students wish to stress. Looking at the relationship between gestures and official languages hence offers a form of reflecting upon the origin and formation of languages.[8]

Our third claim is that the visual–gestural–somatic expressions that emerged in all three examples evince the embodied nature of mathematical cognition perhaps more clearly than the verbal-auditory mode. That is not to say that Sign and gestures are bodily things while spoken and written languages are not; it was this very thinking that led us to neglect the visual–gestural–somatic modality for so long. Our view is that words, signs, and gestures, all forms of what Vygotsky termed symbolic language, are constructions with their roots in the sensory experiences of the learners who produce them. To understand their meanings, we should not try to strip them of the connections with the senses and with feelings, instead we should seek to illuminate these connections so we can better feel, hear, and express the mathematics of all our students.

[8] It has even been argued that the origin of language itself, be it aural or manual, can be plausibly traced back to gesturing (Armstrong & Wilcox, 2003).

References

Armstrong, D. F., & Wilcox, S. (2003). Origins of signed languages. In M. Marschark & P. Spencer (Eds.), *Oxford handbook of deaf studies, language, and education* (pp. 305–318). Oxford, England: Oxford University Press.

Brentari, D. (Ed.). (2010). *Sign languages: A Cambridge language survey*. Cambridge, England: Cambridge University Press.

Brito, L. F. (1995). *Por uma gramática de Língua de Sinais*. Rio de Janeiro, Brazil: Tempo Brasileiro.

Bull, S. (2008). Deafness, numerical cognition and mathematics. In M. Marschark & P. C. Hauser (Eds.), *Deaf cognition: Foundations and outcomes* (pp. 170–200). New York: Oxford University Press.

Cruz, M. (2008). *Gramática de la lengua de señas mexicana*. Doctoral dissertation, Colegio de México, México.

Edwards, L., Ferrera, F., & Russo-Moore, D. (Eds.). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, NC: Information Age.

Fernandes, S. H. A. A., & Healy, L. (2010). A inclusão de alunos cegos nas aulas de matemática: Explorando área, perímetro e volume através do tato. *Bolema: Boletim de Educação Matemática, Rio Claro (SP), 23*(37), 1111–1135.

Gee, J. P. (2014). *An introduction to discourse analysis: Theory and method* (4th ed.). Abingdon, England: Routledge.

Goldin-Meadow, S. (2003). *Hearing gesture: How our hands help us think*. Cambridge, MA: Harvard University Press.

Healy, L. (2012). Mathematical opportunities for students with disabilities. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 100–106). Taipei, Taiwan: PME.

Healy, L., & Fernandes, S. H. A. A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics, 77*, 157–174.

Healy, L., & Fernandes, S. H. A. A. (2014). The gestures of blind mathematics learners. In L. Edwards, F. Ferrera, & D. Russo-Moore (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 125–150). Charlotte, NC: Information Age.

Hewes, G. W. (1996). A history of the study of language origins and the gestural primacy hypothesis. In A. Lock & C. R. Peters (Eds.), *Handbook of human symbolic evolution* (pp. 571–595). New York: Oxford University Press.

Iverson, J. M., & Goldin-Meadow, S. (1998). Why people gesture when they speak. *Nature, 396*(6708), 228.

Kelly, R. R. (2008). Deaf learners and mathematical problem solving. In M. Marschark & P. C. Hauser (Eds.), *Deaf cognition: Foundations and outcomes* (pp. 226–249). New York: Oxford University Press.

Kendon, A. (2008). Some reflections on the relationship between 'gesture' and 'sign'. *Gesture, 8*(3), 348–366.

Lane, H. (1984). *When the mind hears: A history of the deaf*. New York: Random House.

Lang, H. G. (2003). Perspectives on the history of deaf education. In M. Marschark & P. E. Spencer (Eds.), *Oxford handbook of deaf studies, language, and education* (pp. 9–20). New York: Oxford University Press.

LeBaron, C., & Streeck, J. (2000). Gestures, knowledge, and the world. In D. McNeill (Ed.), *Language and gesture* (pp. 118–138). Cambridge, England: Cambridge University Press.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2–8.

McNeill, D. (1985). So you think gestures are nonverbal? *Psychological Review, 92*, 350–371.

Monteiro, J. K., & Andrade, C. G. (2005). Avaliação do raciocínio abstrato, numérico e espacial em adolescentes surdos. *Aletheia, 21*, 93–99.

Moura, M. C. (2000). *O Surdo, caminhos para uma nova identidade*. Rio de Janeiro, Brazil: Editora Revinter.

Nelsen, R. (1993). *Proofs without words: Exercises in visual thinking*. Washington, DC: The Mathematical Association of America.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meaning: Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer.

Nunes, T. (2004). *Teaching mathematics to deaf children*. London: Whurr.

Nunes, T., & Moreno, C. (2002). An intervention program for promoting deaf pupils' achievement in mathematics. *Journal of Deaf Studies and Deaf Education, 7*(2), 120–133.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111–126.

Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics, 70*(2), 91–95.

Roth, W. M. (2010). Incarnation: Radicalizing the embodiment of mathematics. *For the Learning of Mathematics, 30*(2), 8–17.

Roth, W. M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: From classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics, 70*(2), 175–189.

Rotman, B. (2009). Gesture, or the body without organs of speech. *Semiotix. A Global Information Bulletin*, (15). Retrieved April 28, 2011, from http://www.semioticon.com/semiotix/semiotix15/sem-15-02.html.

Sacks, O. (2000). *Seeing voices*. New York: Vintage Books.

Stokoe, W. C. (1960/2005). Sign language structure: An outline of the visual communication system of the American deaf. *Journal of Deaf Studies and Deaf Education, 10*, 3–37.

Vygotsky, L. S. (1962). *Thought and language*. Cambridge, MA: MIT Press.

Vygotsky, L. S. (1981). The instrumental method in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 134–143). Armonk: M.E. Sharpe.

Vygotsky, L.S. (1997). *Obras escogidas V–Fundamentos da defectología* [The fundamentals of defectology] (Julio Guillermo Blank, Trans.). Madrid, Spain: Visor.

Wilcox, S., & Morford, J. P. (2007). Empirical methods in signed language research. In M. Gonzalez-Marquez, I. Mittelberg, S. Coulson, & M. J. Spivey (Eds.), *Methods in cognitive linguistics* (pp. 173–202). Amsterdam, The Netherlands: John Benjamins.