Chapter 5
Challenges and Opportunities for Second Language Learners in Undergraduate Mathematics

Viviane Durand-Guerrier, Mercy Kazima, Paul Libbrecht, Judith Njomgang Ngansop, Leila Salekhova, Nail Tuktamyshov, and Carl Winsløw

5.1 Introduction

Multilingual classrooms are nowadays common due to migration and student mobility. In many countries, university mathematics courses include students who are not native speakers of the language of instruction. We refer to such students as second language learners. The contexts are very different in each country, depending on history, policy,

V. Durand-Guerrier (✉)
University of Montpellier, Case Courrier 051, Montpellier Cedex 5 34095, France
e-mail: vdurand@math.univ-montp2.fr

M. Kazima
University of Malawi, Zomba, Malawi
e-mail: mkazima@cc.ac.mw

P. Libbrecht
Informatics, Weingarten University of Education Kirchplatz 2, Weingarten 88250, Germany
e-mail: paul@cermat.org

J.N. Ngansop
Université de Yaoundé 1, Yaoundé, Cameroun
e-mail: judithnjomg@yahoo.fr

L. Salekhova
Kazan Federal University, Kazan, Republic of Tatarstan, Russia
e-mail: salekhova2009@rambler.ru

N. Tuktamyshov
Kazan University of Architecture and Engineering, Kazan, Republic of Tatarstan, Russia
e-mail: nail1954@gmail.com

C. Winsløw
Institut for Naturfagenes Didaktik, University of Copenhagen, Øster Voldgade 3, Copenhagen 1350, Denmark
e-mail: winslow@ind.ku.dk

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and so on. This diversity is evident in the five cases authors from different countries present in this chapter. Although it is a common assumption that language is not a crucial issue in the learning of advanced mathematics, there is research evidence that even in monolingual contexts, linguistic issues do appear with implications for equity (e.g., Zevenbergen, 2001). We also know that linguistic issues may be different, in nature and scope, for second language learners.

In this chapter, we first describe specific challenges related to mastery of the language of instruction that second language learners might face in learning undergraduate mathematics. We then describe five multilingual contexts, share insights from research, and present emergent proposals for supporting students to overcome difficulties.

5.2 Linguistic Difficulties in Advanced Mathematics for Multilingual Contexts

5.2.1 Lexical and Notational Issues

One important issue is the challenge in translating mathematical terms. It is easy to see that there are lexical differences across languages. These differences are sometimes augmented when we consider specialized uses of the languages, such as the vocabulary of academic mathematics. For example when considering "everyday language" the English word "field" could be translated into French as "domaine," "champs" or "terrain," but none of these translations are valid when considering the use of the term "field" in algebra, where the correct French translation is "corps." More delicate examples of these differences are pairs of expressions in two languages that look or sound similar but differ significantly in meaning. Such differences may have deeper consequences for understanding mathematical texts. For example, in French, a "nombre décimal" has a finite number of non-zero digits on the right hand side of the decimal separator, while in English, a "decimal number" can have infinitely many non-zero digits. As a consequence, even an Anglophone student with a good mastery of "everyday" French might translate a mathematically accurate and precise phrase in French into something mathematically confusing, such as "the set of decimal numbers is not a complete subset of the real number set."

For example, Kazima (2006) reports an experiment in which English words from the vocabulary of probability theory were presented with no context to a mathematics class of secondary level students whose first language was Chichewa. The findings support the claim that even when students used the same words as the teachers, the meaning was different. We know from research in mathematics education that this phenomenon is not specific to second language learners (Mathé, 2012), but as the study shows, it is complicated by the use of two languages. In this example, students moved from English to Chichewa (a language that does not have an established mathematics register), did their thinking in Chichewa, and then translated back into English (p. 187). This study raises another issue, the creation of mathematical terms in a language of instruction that does not have an established mathematics register, as is the case in Chichewa or Tatar. As a general phenomenon, this issue is not specific to mathematics, but as we shall see in this chapter, there are important implications for university-level mathematics.

The case of language-based differences in notation is specific to mathematics, and is also a widely underestimated issue. In his book *History of mathematical notations*, Cajori (1928) aimed "to give not only the first appearance of a symbol (whenever possible), but also the competition encountered and the spread of the symbols among writers in different countries" (p. 1), assuming that such a development could help with notational confusion in contemporary mathematics. Nevertheless, the belief that mathematical notation is "universal" and independent of language or location needs to be carefully considered.

Differences in notation appear even between countries sharing the same mathematical tradition. Libbrecht, Droujkova, and Melis (2011) gave two examples: the notation for half-open intervals in English, [a, b), French or German, [a, b[, and Dutch [a, b>; and the various notations for the greater common divisor, that correspond to an abbreviation of three words, and hence depend on language. Other issues arise when mathematical notations from one tradition are incorporated into a language with a different orthography or textual organization, such as Greek–Roman languages versus Arabic languages (i.e., Bebbouchi, 2011, pp. 529–530). In general, mathematical notations contain a lot of information, so that being able to decode or unpack such information is an important part of mathematical proficiency. For example, the notation $\lim_{x \to a} f(x) = L$ does not express simply an equality between elements of a space; the left part implies that f is a mapping that is defined in a neighborhood of a, and admitting in a, a unique limit.

Another issue concerns notations for which there is no general consensus, even within one country. For example, in most Western countries, if we are given a numerical function f, the expression f^n could refer to the product of function (multiplication), or to the functional power of function (composition). Such ambiguous notation is routinely used in calculus courses. If students are not aware of such subtle differences in the notation statements expressed in these courses, they may misunderstand exercises with this notation. Sometimes the mathematical context (e.g., trigonometry) can implicitly determine which of the two meanings are intended—but this can be confusing to students who (rightly) expect mathematical notation to be unambiguous and consistent across courses. Some of the specificity of mathematical notations arises because they are frequently abbreviations of words. For example, using either the beginning of the word (i.e., *sin* for "sinus of an angle" in many western languages, but *sen* for "seno de un ángulo" in Spanish), or the initials of the words in a group of words such as gcd (greater common divisor), pgcd (plus grand commun diviseur); or kgV (kleinstes gemeinsames Vielfache).

As the number of undergraduates in mathematics courses who use two or more languages increases, the need also increases for both student and instructor awareness of how crucial the context of production of a notation is for its adequate understanding and use. Such awareness would benefit instruction in both multilingual and monolingual contexts. An example of work that supports such awareness is a report (Libbrecht et al., 2011) on an initiative to collaboratively observe the diverging mathematical notations across cultures and languages, create a census of mathematical notations, and develop a collaborative database (Libbrecht et al., 2011, p. 166).

In addition, the undergraduate education phase of students coincides with a phase of independence where fetching knowledge from arbitrary sources is becoming a common practice and, often, a requirement. The most important sources of knowledge for undergraduate studies are the local library and the World Wide Web. The latter has increased largely in usage in recent years for its ease of access, its ubiquity, and the wealth of knowledge it allows access to.

However Wikipedia and most websites present to students a large diversity in ways of expressing mathematics. The issue with using these sources includes, but is not limited to, the dominance of a few languages in the most used (or simply available) online scientific resources for a given mathematical topic. It concerns also the conventional use of vocabulary and notation. In Wikipedia, often considered as a single coherent resource of knowledge as a normal Encyclopaedia would be, pages are often created or verified by multiple persons which leads to inconsistencies which are hard to remove and which a student needs to manage. This problem is similar to reconciling the language and notation used in multiple books. For example, in 2009, one could see the notation for the binomial coefficient written both as C_k^n and as C_n^k depending on the page.[1] Thus students using the web would need to reconstruct the context of the mathematical reasoning when reading this text; this was not necessary generally before university or with the use of paper books.

5.2.2 Logical Issues

Research has shown that tertiary students experience difficulties using logic, such as unpacking the logic of statements, interpreting formal statements, conducting argumentation or engaging in the proving process. For example, there is research evidence that undergraduate students experience difficulties with these aspects of logic that can impede their competencies in proof and proving (e.g., Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Epp, 2011; Selden & Selden, 1995), and that these difficulties are aggravated for second language learners (e.g., Barton, Chan, King, Neville-Barton, & Sneddon, 2005 and this chapter).

Predicate Logic can be used to unpack the logic of a given statement by identifying the logical categories, connectors, quantifiers, and their respective scopes. The fundamental categories are *properties* modeled by one-place predicates such as "to be a primary number"; relationships modeled by two (or more)-place predicates

[1] The versions of these pages at these dates can be seen at http://en.wikipedia.org/w/index.php?title=Binomial_coefficient&oldid=309172219 and at http://en.wikipedia.org/w/index.php?title=Combination&oldid=278280354

such as “to be greater than” (binary relationship). In a given interpretation, arguments of various types can be assigned to properties and relationships: free variables (place-holders); singular elements; generic elements and according to the choices we get either singular propositions or open sentences. For example: “x is a prime number”, where x is a free variable, is an open sentence; it is satisfied by some elements, and not satisfied by others: “29 is a primary number” is a singular sentence which is true, so 29 satisfies the open sentence, while “27 is a prime number” is false, so 27 does not satisfy the open sentence. In a relationship, it is possible to assign singular or generic elements to some places, and free variables to other places. It is also possible to close an open sentence by means of quantifiers, either universal or existential.

Moreover, while in propositional calculus the logical connectors are used to build complex propositions from elementary propositions and are defined by their truth tables, in Predicate calculus, the logical connectors can also be used to build complex open sentences, such as for example “if an integer is even, then its successor is prime,” that could also be closed by quantifiers. As a matter of fact, these quantifiers often remain implicit, or could be hidden through linguistic means depending on the language, and it is common for ambiguities to arise due to the difficulty in identifying the respective scopes of quantifiers and connectors. A well-known difficulty is related to the meaning of the article “*a*” that can either refer to an individual, a generic element, or an implicit universal quantifier. Another difficulty concerns binary relations in universal statements: for example, in French, the statement “les faces du solide sont congrues deux à deux” (the faces of a given solid are pairwise congruent) can mean either “each time you take two faces, they are congruent,” or “each time you take a face, there is another face which is congruent.”

Another aspect concerns the distinction between open sentences and closed sentences, which plays a crucial role in proof and proving (Durand-Guerrier, 2003). In particular, in a given theory, a mathematical definition corresponds to an open sentence, satisfied exactly by the elements that it defines. A theorem is essentially a closed sentence that has been proved in the considered theory, either singular such as “π is transcendental,” or general such as “For any positive integer k, there is a natural number N such that for all $n > N$, there are at least k primes between n and $2n$” (Erdös theorem). A conjecture, however, is a closed sentence that is thought to be true but has not been proved, such as Legendre’s conjecture (which remains unproved): “For every $n > 1$, there exists a prime p, such that $n^2 < p < (n+1)^2$. In many cases, quantification remains implicit or is expressed by linguistic means that depend on each language, so that in some cases it may be difficult for students to identify the logical status of such statements.

5.3 Examples of Various Multilingual Contexts at University

In this section, we describe issues experienced by the authors in a number of countries, to show the variety of contexts.

5.3.1 A Multilingual Situation in Cameroonian Universities

Cameroon is a country in Central Africa; French and English are the two official languages. Teaching occurs in one of the two languages in primary and secondary school. To this end, the educational system is divided into two main subsystems: the *Francophone* subsystem and the *Anglophone* one. The two subsystems are so different that a student who started his education in one of those subsystems cannot easily continue studies in the other subsystem mainly because of the language barrier. This is because, even though the language of the other subsystem is taught in each subsystem, it is only with a very modest level of proficiency achievement. In higher education institutions, Cameroon has Anglo-Saxon style universities such as the universities of Bamenda (in the North-West region) and Buéa (in the South-West region). The other universities are bilingual; courses are given in French and English, depending on the teacher's dominant language.

In addition to the two official languages, there are about 230 ethnic groups, each with its own language (National Institute of Cartography, Cameroon). The current classification, based primarily on linguistic attributes, identifies six major groups. In the French-speaking area of Cameroon, the local languages are widely used in the family and they may strongly influence the way French is used. For instance, as part of her doctoral thesis, Tsoungui (1980) studied the interferences with French and the Ewondo language, spoken by the tribe of the same name. These interferences observed with the French language do not have much influence on English. On the other hand, the use of English by native English speakers is influenced by so called Pidgin-English, a composite language based on the English lexicon, but different than standard English. This is the main language in English-speaking areas of the country. It is also spoken in the contiguous zones of the Littoral and West Provinces, as well as in urban centers. We can say, in accordance with Tsoungui (1980), that "All Cameroons find themselves at least bilingual and often trilingual, even quadrilingual, French and English are never first languages" (p. 27). Consequently, we can find university students who must study in one or two languages—French or English, even though they are not fluent in either one.

5.3.2 The Case of Denmark

Over the past decades, internationalization has become an important priority for Danish higher education, and a priority that seems to have broad political consensus. As a member of the European Union, Denmark takes part in the student exchange program *Erasmus*, with about 2,416 outgoing and 6,186 incoming students in the academic year 2009/2010[2]—numbers which are continuously increasing. The total number of exchange students is about 16,000 (incoming) and 8,000

[2] Source: http://www.iu.dk/nyheder/kort-nyt/erasmus-fortsat-i-fremgang/

(outgoing) every year.[3] However, compared to the total number of students in higher education in Denmark (about 200,000), the numbers are not very high. The government aims to facilitate access to Danish universities for foreign students[4] as well as to encourage Danish students to go abroad.[5] We are mainly concerned with the first of these efforts here.

A main obstacle for foreign students can be the Danish language, which very few students have had the opportunity to learn abroad. However, within the last decades, more graduate courses are offered in English—particularly within scientific and technical fields. There is no common policy in this regard, even within a given university, and the question of the extent to which English should become a common language of instruction in advanced courses has been the subject of fierce debates, involving both practicalities and principles. On the one hand, it is hard to require short-term exchange students in a mathematics program to learn Danish well enough for academic instruction in this language. On the other hand, even in mathematics it is not straightforward for all students and teachers to "switch" languages. The current practices are mixed, with a tendency towards offering all texts in English and at least some of the teaching in English at the graduate level. From experience, this can work reasonably well, but with two major obstacles:

- Not all exchange students are fluent in English and in fact many are much less fluent in English than the average Danish student.
- In pure mathematics, we have a significant number of second-generation immigrant students (informal estimate: at least 10 % of the population), and this group seems to be particularly challenged with English—even when they are fluent in Danish.

In short, immigrant students have struggled to learn a second language (here, Danish) well, (and may have chosen mathematics or other technical subjects in part because they had difficulties with other subjects) and may now be asked to learn a third language, English. It would be a strange paradox if they were to lose out on Denmark's efforts to internationalize her educational systems.

5.3.3 *Increasing Linguistic Diversity in France*

In France, there is only one official language, French, which is used both in everyday life and for instruction in schools. For historical reasons, French is also an official language in some other countries in Europe, in numerous countries in Africa, in a few countries in the Americas or Oceania, alone or together with other languages (often English). There are also countries, mainly in Northern Africa, where French is not an official language but is still used as the language of instruction in

[3] Source: http://www.iu.dk/publikationer/2010-1/mobilitetsstatistik-for-de-videregaaende-uddannelser-2008-09/

[4] See, for example, the website http://studyindenmark.dk/

[5] See, for example, http://www.udiverden.dk/

mathematics from a certain grade on (e.g., secondary school in Tunisia). For this reason, France has historically attracted university students from a variety of countries. Recently, due to student mobility, the number of countries of origin of foreign students is increasing, as is the number of foreign students.

In a note of information,[6] the French Ministry of Tertiary Education and Research provides a synthesis in French of the results of a conjoint inquiry of the UNESCO, OECD, and Eurostat (OED). The results are given for 2007–2008. In France, there were 243,400 foreign tertiary students, representing 11.2 % of the total amount. The increase from 1999 to 2000 is about 75 %. Students from Africa are a majority (43.5 %), far more numerous than those from Europe (21.3 %) or from Asia (21.0 %). Students from China constituted in 2007–2008 the second most important contingent of foreign students in France. These foreign students are studying in France in a great variety of domains (e.g., Social Sciences, Trade and Law, Health and Social Sector, Engineering). If we consider the fact that mathematics is taught in a large spectrum of tertiary institutions, and that even some second language learners coming from countries where French is the official language may experience difficulties using French in academic contexts, it seems clear that many tertiary mathematics educators are likely to face linguistic diversity in their classrooms.

5.3.4 The Case of Malawi

Malawi is a southern African country with 17 local languages of which 4 are major languages in terms of number of speakers (Center for Language Studies, 2010). The most widely spoken language is the national language, Chichewa. English is the official language and the language of instruction from the fifth year of primary school through secondary school, at university, and in other further education. The Malawi school system has 8 years of primary school, 4 years of secondary school, and 4 or more years of university education. The first 4 years of primary school are taught in Chichewa or other local languages. Both English and Chichewa are also taught as subjects of study in schools. The situation in Malawi schools therefore is one where the classrooms are multilingual and students learn mathematics in English, a language that they are also learning and not yet fluent in.

The situation at university is not much different from elementary and secondary schools, in that all classrooms are multilingual and students often share one or more common local languages. University classrooms are more diverse in terms of the languages represented because students come from all parts of the country, unlike in elementary and secondary schools where students are local. Another difference is that at university there are cases where the lecturers or instructors are expatriates who may not share any of the students' local languages, while in elementary and secondary schools teachers are local and share at least one of the languages of the learners.

[6] Source MESR, 2011, Note d'information du 27 juillet 2011 http://media.enseignementsuprecherche.gouv.fr/file/2011/19/6/NIMESR1111_186196.pdf

Although by the time students come to university they would have had at least 8 years of English as the medium of instruction, from experience, it seems that problems with mathematical vocabulary, comprehension, and communication persist.

5.3.5 Bilingualism in Russia: The Case of Tatarstan

Tatar and Russian are the main languages used in the Tatarstan Republic. The Tatar language is used in family and daily life; it is taught as a subject in kindergartens, schools, and universities; newspapers and magazines are published in this language, and it is the language of instruction at all levels. At the same time, Russian is used in all spheres of society, including administration of the Tatar Republic, the courts, the highest authorities, etc., not to mention schools, family, and communication. Various legislative acts created the judicial base for creating a new language situation in Tatarstan and made it possible to design fundamental laws and values in the educational sphere.

There are currently 1,256 schools with one language of instruction (Tatar), and 410 bilingual secondary schools in Tatarstan. In 1990, 24 % of Tatars studied in their native language, mainly in rural areas. At present the rate is around 47 %. Beginning with Perestroika in the 1990s, Tatars had opportunities to acquire education in their native Tatar language not only in lower secondary schools, but also in high schools. At present a system of high-quality bilingual education using Russian and Tatar languages is developing. This process is based on the following principles: the integrity of the educational space of Russia; regional and ethnic needs connected with the social, economic, and political development of the society as a whole and the region in particular; the equal rights of the residents in acquiring higher education and freedom to choose the language of education. Bilingual educational programs exist at present in the Kazan State University of Architecture and Engineering, in the Kazan Federal University, in the Kazan State Technological University, the Kamskiy polytechnic institute, and in other universities and institutions.

One of the most pressing issues now in Tatar society is the creation of mathematical terminology in Tatar. The Mathematical Terminological Commission has developed more than 1,000 terms (Salimov & Tuktamyshov, 2000). There are many borrowed words from Arabic, Farsi, Russian, and West European languages used in Tatar in scientific contexts. Many scholars believe that if a borrowed scientific term is already widely used, it should be preserved.

5.4 Results of Research Studies

In this section, we present results of three selected studies in three linguistic contexts: the first one in New Zealand, where English is the language of instruction, the second in Cameroon where French is the language of instruction, and the third in Tatarstan where both Russian and Tatar are languages for instruction. The first two illustrate the difficulties second language learners experience and the third one presents a tentative approach to overcoming some of these difficulties.

5.4.1 The Case of Students in New Zealand Who Have English as an Additional Language

Like many similar research-intensive universities in the Western world, the University of Auckland in New Zealand receives an increasing number of students whose first language is not the language of instruction in the university. There are many immigrants from China and other non-Anglophone countries. All university instruction is in English, the first language of most New Zealanders. At this particular university, a series of studies were conducted by Barton and Neville-Barton (2003, 2004; Barton et al., 2005). The results, when coupled with the explanatory analysis, reveal phenomena of a more general nature. The studies aimed at comparing the performance of students with English as a first language (L1 students) to that of students with English as an Additional language (called “EAL students”), at various levels of the mathematics program (first and third year).

The first studies (Barton & Neville-Barton, 2003, 2004) considered students in their first year of undergraduate mathematics study. The main outcome was that EAL students at this level received better marks than L1 students, with an extra advantage for recently arrived students (more specifically, with less than 6 years of English learning experience). This was explained by the more solid technical knowledge of students who received primary and secondary mathematics instruction abroad (in countries such as China), and the advantage of recently arrived EAL students who did not have to struggle with the English language at the secondary level. The results are further explained by supplementary tests with mathematical questions phrased with more or less substantial use of natural language. As could be expected, L1 students understand, and perform better, on questions that are heavily based on the natural language (English), while the EAL students were inhibited in this context by the tendency to switch to symbolic modes of work. An interesting further result was the high level of self-reported understanding of textually rich questions by EAL learners, which sharply contrasts with their weaker performance on these questions. On the other hand, their data show that, in typical first-year courses (calculus and linear algebra with a focus on technical work) the EAL students appear to be less disadvantaged by their natural language capabilities, and in fact, any possible disadvantage is outweighed by their stronger capabilities in the symbolic mode.

However, the apparent success of EAL students in the first year does not continue into the second and third year. Barton et al. (2005, p. 728) note that in the second year, there was a significantly higher proportion of students with weak capacities in English than in the first and third year. The difference between the first and second year was explained by a mixture of factors, including a change of admission policies. The difference from second to third year seems to be mainly due to the students with weaker English fluency dropping out. This is confirmed by in-depth analyses of the language requirements for the more advanced courses in mathematics, and by a test of students' comprehension of linguistically and logically complex propositions, and their capacity to transform them (for example, given a mathematical phrase with the structure "If A then B", produce an equivalent phrase with structure "If not B then not A"). Here, a clear disadvantage for EAL learners is apparent, and it was demonstrated using a variety of methods that these difficulties constitute severe obstacles to the work proposed in more advanced courses. This also explains the difference with the results found with first-year students (Barton et al., 2005):

> …the language used in lectures and texts at first-year level is repetitive, confirmatory, and predictable, and is mostly used to describe paradigmatic examples similar to those that students will be required to repeat. At third year level it changes to one-off explanations and logical trains of reasoning with examples that are drawn from a variety of unpredictable areas. In the third year examples are used to illustrate single aspects of a theorem, and are not designed to be copied or reproduced. Rather, students are required to reproduce the logical trains adapted to new situations. (p. 722)

Thus, the work required from third year students in advanced mathematics courses is significantly *more* dependent on discursive fluency in English that includes discernment of logical subtlety in phrases, rather than understanding statements that are mere frames around symbolic representations of algorithms and other computational techniques. So, while the first-year experience seems to confirm the folklore belief that success in mathematics is relatively less dependent on capacities related to the use of natural language, students who are not fluent in the language of instruction faced new difficulties in advanced mathematics in a more theoretical form.

Barton et al. (2005, p. 729) suggest that more effort needs to be put into supporting EAL students with "first language tutorials" or "specific mathematical English additional courses." To the extent that their results are generalizable, in particular if it is a universal trend that the specific need for such measures is more important at more advanced levels, then there is something paradoxical about the shift made to English at these levels in some contexts (such as the Danish one described above, where English is a second language to most learners). More research is needed to investigate the viability of alternatives such as earlier or more partial shifts of instructional language. As an example of such alternatives, it is common in Danish universities to base semi-advanced mathematics courses on textbooks in English, while the instruction is still in Danish. Such practices are usually motivated by the lack of specialized textbooks in a language like Danish, but it may also represent advantages for those students who will later on study or work with mathematics in English.

5.4.2 Logical Issues: The Case of Negation of Quantified Statements

At first glance, the concept of negation could appear as a very simple one, met and used early by children. This simplicity, however, holds only for singular statements, such as "π is a rational number", whose negation is "π is not a rational number." Indeed, as soon as quantifiers are involved, negation becomes more complex, as is well-known by logicians, from Aristotle, who insisted on the distinction between contradiction (logical negation in modern terms) and contradictory, to modern logicians such as Russell. Another serious difficulty concerns pairs of statements of type "Some *A* is *B*" (3) and "Some *A* is not *B*" (4). As already realized by Aristotle, while sentence (3) is affirmative and sentence (4) negative, there is no opposition between the two. In French, a standard way to build negation is the use of the locution "ne … pas". Applied to a quantified statement of the type "Tous les *A* sont *B*" (All *A* are *B*), it provides "Tous les *A* ne sont pas *B*" (All *A* are not *B*). However, as held by Fuchs (1996), such sentences are sometimes used to express "No *A* is *B*". It is important to notice that the standard interpretation in France of such sentences is not congruent with the interpretation suggested by the underlying logical structure which in a word-to-word formalization would lead to "for all *A*, not *B*", while the standard interpretation leads to "There exists *A* which is not *B*". The work of Ben Kilani in the Tunisian context (Ben Kilani, 2005; Durand-Guerrier & Ben Kilani, 2004) showed the difficulties students experienced in understanding *mathematical* negation (see also Edmonds-Wathen et al., this volume).

While at secondary school, dealing explicitly with the negation of quantified statements in mathematical activity is not so common, it is very common at university where students are likely to face negation of quantified statement in various mathematical practices, in particular in indirect proof such as proof by contradiction or proof by contraposition. In this respect, this topic is a specific challenge for second language learners in undergraduate mathematics.

In an ongoing research study in the francophone educational context in Cameroon, Judith Njomgang Ngansop identified the influence of a native language on the learning of mathematics in French, particularly on questions of logic (Njomgang Ngansop & Durand-Guerrier, 2012). The chosen language is Ewondo, one of the main languages in the area of Yaoundé. The work of Tsoungui (1980) who carried out a comparative study between French and Ewondo grammars served as a resource (see paragraph 2.1). As in French, the construction of negation in Ewondo uses a discontinuous morpheme "à … kig" in singular, and "be … kig" in plural; kig is used mainly to reinforce the negation and can be placed after the verb or at the end of the sentence. Other morphemes can be used according to the context (depending on the form of the statement and on the verb tense).

An interview with a fluent Ewondo speaker completed this description, showing significant differences with French: the form of the verb "to be" depends on the form affirmative/negative of the sentence (*bene/bèsé*); for statements of the type "all *A* are *B*", the morpheme "kig" is used after the predicate. For example: the word-to-word translation in French of its negation in Ewondo "be ndabe bese *bèsé* kig vié" is "les boules toutes ne sont pas rouges" that means "toutes les boules ne sont pas rouges" [all the balls are not red]. In standard Ewondo as in standard French, and unlike in standard Arabic, this means "some balls are red, not all." However, while there is ambiguity in French (such sentences are often used to express "No balls are red"), there is no ambiguity in Ewondo: the translation of "no ball is red" is "ndabe *zing* be se kig vié."

A preliminary experiment was carried out with three students who speak Ewondo fluently (but none of them were literate in Ewondo). These students had answered a questionnaire addressed to 80 first-year students comprising the 3 items involving negation and had follow-up sessions for volunteer students to evaluate responses to the test and to clarify some concepts of logic. After this, the students were interviewed by the researcher, who was assisted by a teacher of French language whose mother tongue is Ewondo and who speaks this language fluently. During the interview, students were given three statements including an everyday-life conditional statement; they were asked to translate each statement in Ewondo, then to give the negation in Ewondo, and translate in French (all questions and answers were oral).

The first results show that for the three students, the negative form (e.g., the syntax) is prominent. They translated "toutes les boules sont rouges" (1) correctly in Ewondo, then gave the correct negation in Ewondo, and finally provided "toutes les boules ne sont pas rouges" (2) (the french standard form) as the negation of (1). The three students negated "certains entiers sont pairs" [some integers are even] (3) by "certains nombres entiers ne sont pas pairs" (4) [some integers are not even], which, as already said, is not the negation.

In addition two university mathematics teachers were interviewed. The results with the university teachers support our hypothesis of an effect of Ewondo on French concerning logical statements involving negation. Following Ben Kilani (2005), we use the underlying word-to-word logical structure of the involved statements as an indicator of this complexity both a priori and a posteriori. These preliminary results show the complexity of negation in French for Ewondo speakers and illustrate the importance of logical issues in the dialectics between syntax and semantics (Durand-Guerrier, 2008).

This research study focused on the logical structure of negative sentences, but there are also lexical aspects to this issue. For example, in another study Kazima (2006) points out that some negatives in Chichewa are reverse of negatives in English: for example, in Chichewa, the word for *likely* is the negative of the word for *unlikely*, such that *likely* is literally interpreted as *not unlikely* (p. 172); the empirical results from a study in Malawi show that this affected students' understanding of probability terminology in English and their use in sentences, in particular when negation was involved (p. 187).

Theses results encourage research concerning the use of negation in mathematics throughout the curriculum in multilingual contexts, taking into account mother languages, languages of instruction, and mathematical and logical discourses, in order to identify opportunity for overcoming these difficulties from secondary level. At tertiary level, in addition, it would be worthwhile to study the impact of any difficulties with negation on indirect reasoning such as reasoning by contradiction or by contraposition on the one hand, and on mathematical conceptualization on the other hand.

5.4.3 An Ongoing Experiment for Teacher Training in Tatarstan

We now turn to discussing preliminary results from a teaching experiment in bilingual education using Russian and Tatar in a preservice mathematics teacher preparation program. The experiment was carried out at the Kazan Federal University in

Russia. Tatar is the students' mother tongue and Russian is used as the second language of instruction. Kazan is the capital of Tatarstan—one of the republics of Russia, 53 % of its population are Tatars.

Mathematical language in Tatar has its own peculiarities, especially at the lexical and grammatical level that is reflected in the special terminology, phraseology, and syntax, in terms of lexical and grammatical structure, and in the genre of mathematical texts. For example, there are general syntactic differences between word order in sentences in Russian and Tatar. The main difference lies in the fact that the verb in a Tatar sentence is always written at the end. Another example more specific to mathematics is the difference in the names of fractions in Russian and Tatar. In Russian the value of the numerator is said before the denominator, while in Tatar it is the opposite. For example, the fraction 1/2 in Russian is read as "odna vtoraia" (1 over 2), and in Tatar, "ikeden ber" (2 under 1).

Our research is based on the proposition that language and thought interact in the process of learning and teaching mathematics. Accordingly, it can be argued that cognitive mathematical activity is connected to speech activity, and that the learning of content occurs at the same time with mastering the means of expression in the second language. Based on the ideas of Vygotsky (1934) that thought is accomplished in the word, we propose to develop thinking in a second language with the help of speech-intellectual tasks, which can be divided into conceptual-lexical and mathematical tasks (Salekhova, 2007; Salekhova & Tuktamyshov, 2011). A conceptual-lexical task in mathematics is a task correlating a new mathematical term in the second language with a known mathematical concept and corresponding mathematical term in a student's native language. As a result, two equivalent mathematical terms (one in the native language and the second in the target language) are fixed in student's memory to denote one mathematical concept (Salekhova, 2007).

Conceptual-lexical tasks, which are given to students in the second language, provide an opportunity to introduce new concepts with the help of semantization, without translation. To achieve this goal, it is necessary to introduce the new mathematical concept through the logical development of familiar concepts, which are related with the unknown one, and through the context of the discussed topic. Mathematical tasks on a defined topic that are solved in the second language (Russian) make it possible to develop mental operations in the second language. It is rather difficult to control this process for the teacher. Nevertheless, the method of question-and-response in the second language during the solution to a problem makes it possible to retrace the steps in the reasoning of the student. Thus, solutions of these problems develop mental and verbal activity of students in a second language. This method of training directs students to the object of thought—elements of the problem and their interaction, rather than on the linguistic form of unfamiliar words in a foreign language.

The process of preparing future teachers of mathematics for bilingual schools is divided into three stages. At each stage a certain level of bilingualism is formed. Levels of bilingualism in the field of mathematical discourse are defined from considerations of speech as a tool for forming thought by means of native and second

Table 5.1 Model of bilingual education at the university

Stage of bilingual education	Level of bilingualism
I (1 year)	*The lowest level*
	Forming and wording of a thought with the help of the native language with its further translation into the second language
II (2 year)	*The intermediate level*
	Forming and wording of a thought by means of the native language and then with the help of the second language
III (3–4 years)	*The highest level*
	Forming and wording of a thought by means of the second language

languages. Solving various types of speech-intellectual problems is used as one of the bilingual teaching methods (Table 5.1).

We now analyze the solution of mathematical problems by students in the process of bilingual mathematics teaching. Ninety-one third-year students from the Faculty of Mathematics, of Kazan Federal University were involved in the pedagogical experiment. This group included students who studied mathematics in Tatar at school, and are studying higher mathematics in the university on a bilingual basis (by means of Tatar and Russian).

We used Bloom's Taxonomy (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956) of educational objectives in the cognitive domain to evaluate the results of the teaching experiment (knowledge, comprehension, application, analysis, synthesis, and evaluation). Based on observations of how students discussed and solved the ordinary first-order separable differential equations in Russian, we obtained the following results. The results show that 13 of the students did not master the necessary mathematical concepts and terminology in Russian, as they did not complete the first and second steps of solution. The other 35 students understood the problem, but they made some mistakes in the mathematical transformations. Forty-five students solved the ordinary differential equation correctly. Analysis of students' written works showed that about 50 % of the students mastered the 5 levels of Bloom's Taxonomy. Thus, this approach to bilingual instruction in mathematics, based on the solution of speech-intellectual tasks, seems to have had a positive effect. These results are encouraging and point to the need for future research on the cognitive costs and advantages of bilingual mathematics learning when the language of instruction and the language of application differ.

5.5 Conclusion

In this chapter, we have discussed several examples of the challenges that second language learners might face in advanced mathematics courses. In the first part of the chapter, we provided some a priori considerations to describe the specific forms of lexical challenges in university mathematics, described how notation and logical issues are crucial in advanced mathematics, and discussed access to web-based resources that are mainly in dominant languages. The cases we presented in the second part do not provide an exhaustive picture of the phenomena as they appear in practice. However, they do provide concrete examples of the variety of multilingual contexts at the university level: monolingual teaching with a significant number of second language learners, as in France or in Denmark; bilingual or multilingual systems of teaching (as in Malawi or Tatarstan schools, and university in Cameroon).

With the three case studies presented in the third section, we provided some evidence that second language learners face difficulties. The last example presented in Sect. 5.4 describes an attempt to overcome some of those difficulties. The second example, which focuses on negation, shows the necessity of studies across secondary/tertiary levels. Indeed, university level teaching should draw on what is known from research at the secondary level, and conversely, research at university level should enlighten what aspects to take into consideration at secondary school.

Among the open questions that should be considered in further research, a crucial issue concerns the effects of teaching mathematics in a "dominant language." Could such choices lead to unexpected exclusion phenomena? Which paths can be taken to avoid them? In which respect could multiple languages provide resources in the teaching and learning of mathematics for undergraduates? Due to the diversity of contexts that we described in Sect. 5.3, it is likely that possible answers will be strongly dependent on the linguistic context.

To sum up, contrary to popular belief, the study of advanced mathematics is indeed sensitive to language matters, and language diversity can impact the learning of a large number of mathematics students across the world. We hope that the mathematics education community involved in advanced mathematics, as well teachers and researchers, will become aware of this international issue in learning and teaching mathematics.

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