A Variational Principle for Cyclic Polygons with Prescribed Edge Lengths

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Abstract We provide a new proof of the elementary geometric theorem on the existence and uniqueness of cyclic polygons with prescribed side lengths. The proof is based on a variational principle involving the central angles of the polygon as variables. The uniqueness follows from the concavity of the target function. The existence proof relies on a fundamental inequality of information theory. We also provide proofs for the corresponding theorems of spherical and hyperbolic geometry (and, as a byproduct, in $1 + 1$ spacetime). The spherical theorem is reduced to the Euclidean one. The proof of the hyperbolic theorem treats three cases separately: Only the case of polygons inscribed in compact circles can be reduced to the Euclidean theorem. For the other two cases, polygons inscribed in horocycles and hypercycles, we provide separate arguments. The hypercycle case also proves the theorem for "cyclic" polygons in $1 + 1$ spacetime.

1 Introduction

This article is concerned with cyclic polygons, i.e., convex polygons inscribed in a circle. We will provide a new proof of the following elementary theorem in Sect. 2.

Theorem 1.1 *There exists a Euclidean cyclic polygon with $n \geq 3$ sides of lengths $\ell_1, \ldots, \ell_n \in \mathbb{R}_{>0}$ if and only if they satisfy the polygon inequalities*

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$$\ell_k < \sum_{\substack{i=1\\ i\neq k}}^{n} \ell_i, \tag{1}$$

and this cyclic polygon is unique.

Our proof involves a variational principle with the central angles as variables. The variational principle has a geometric interpretation in terms of volume in 3-dimensional hyperbolic space (see Remark 2.6). Another striking feature of our proof is the use of a fundamental inequality of information theory:

Theorem (Information Inequality) *Let* $p = (p_1, \dots, p_m)$ *and* $q = (q_1, \dots, q_m)$ *be discrete probability distributions, then*

$$\sum_{k=1}^{m} p_k \log \frac{p_k}{q_k} \geq 0, \tag{2}$$

and equality holds if and only if $p = q$.

The left hand side of inequality (2) is called the *Kullback–Leibler divergence* or *information gain* of q from p, also the *relative entropy* of p with respect to q. The inequality follows from the strict concavity of the logarithm function (see, e.g., Cover and Thomas [3]).

In Sects. 3 and 4 we provide proofs for non-Euclidean versions of Theorem 1.1. The spherical version requires an extra inequality:

Theorem 1.2 *There exists a spherical cyclic polygon with* $n \geq 3$ *sides of lengths* $\ell_1, \dots, \ell_n \in \mathbb{R}_{>0}$ *if and only if they satisfy the polygon inequalities* (1) *and*

$$\sum_{i=1}^{n} \ell_i < 2\pi, \tag{3}$$

and this cyclic spherical polygon is unique.

Inequality (3) is necessary because the perimeter of a circle in the unit sphere cannot be greater than 2π, and the perimeter of the inscribed polygon is a lower bound. We require strict inequality to exclude polygons that degenerate to great circles (with all interior angles equal to π).

In Sect. 3, we prove Theorem 1.2 by a straightforward reduction to Theorem 1.1: connecting the vertices of a spherical cyclic polygon by straight line segments in the ambient Euclidean $\mathbb{R}^3$, one obtains a Euclidean cyclic polygon.

In the case of hyperbolic geometry, the notion of "cyclic polygon" requires additional explanation. We call a convex hyperbolic polygon *cyclic* if its vertices lie on a curve of constant non-zero curvature. Such a curve is either

- a hyperbolic circle if the curvature is greater than 1,
- a horocycle if the curvature is equal to 1,
- a hypercycle, i.e., a curve at constant distance from a geodesic if the curvature is strictly between 0 and 1.

Theorem 1.3 *There exists a hyperbolic cyclic polygon with $n \geq 3$ sides of lengths $\ell_1, \ldots, \ell_n \in \mathbb{R}$ if and only if they satisfy the polygon inequalities* (1)*, and this cyclic hyperbolic polygon is unique.*

We prove this theorem in Sect. 4. The case of hyperbolic polygons inscribed in circles can be reduced to Theorem 1.1 by considering the hyperboloid model of the hyperbolic plane: Connecting the vertices of a hyperbolic polygon inscribed in a circle by straight line segments in the ambient $\mathbb{R}^{2,1}$, one obtains a Euclidean cyclic polygon.

The cases of polygons inscribed in horocycles and hypercycles cannot be reduced to the Euclidean case because the intrinsic geometry of the affine plane of the polygon is not Euclidean: In the horocycle case, the scalar product is degenerate with a 1-dimensional kernel. Hence, this case reduces to the case of degenerate polygons inscribed in a straight line. It is easy to deal with. In the hypercycle case, the scalar product is indefinite. This case reduces to polygons inscribed in hyperbolas in flat $1 + 1$ spacetime. The variational principle of Sect. 2 can be adapted for this case (see Sect. 5), but the corresponding target function fails to be concave or convex. It may be possible to base a proof of existence and uniqueness on this variational principle, perhaps using a min-max-argument, but we do not pursue this route in this article. Instead, we deal with polygons inscribed in hypercycles using a straightforward analytic argument.

Some history, from ancient to recent. Theorems 1.1–1.3 belong to the circle of results connected with the classical isoperimetric problem. As the subject is ancient and the body of literature is vast, we can only attempt to provide a rough historical perspective and ask for leniency regarding any essential work that we fail to mention.

The early history of the relevant results about polygons is briefly discussed by Steinitz [13, Sect. 16]. Steinitz goes on to discuss analogous results for polyhedra, a topic into which we will not go. A more recent and comprehensive survey of proofs of the isoperimetric property of the circle was given by Blåsjö [2].

It was known to Pappus that the regular n-gon had the largest area among n-gons with the same perimeter, and that the area grew with the number of sides. This was used to argue for the isoperimetric property of the circle:

Theorem 1.4 (Isoperimetric Theorem) *Among all closed planar curves with given length, only the circle encloses the largest area.*

It is not clear who first stated the following theorem about polygons:

Theorem 1.5 (Secant Polygon) *Among all n-gons with given side lengths, only the one inscribed in a circle has the largest area.*

This was proved by Moula [8], by L'Huilier [5] (who cites Moula), and by Steiner [12] (who cites L'Huilier). L'Huilier also proved the following theorem:

Theorem 1.6 (Tangent Polygon) *Among all convex n-gons with given angles, only the one circumscribed to a circle has the largest area when the perimeter is fixed and and smallest perimeter when the area is fixed.*

Steiner also proves versions of Theorems 1.5 and 1.6 for spherical polygons. None of these authors deemed it necessary to prove the existence of a maximizer, an issue that became generally recognized only after Weierstrass [14]. For polygons, the existence of a maximizer follows by a standard compactness argument.

Blaschke [1, Sect. 12], notes that the quadrilateral case ($n = 4$) of Theorem 1.5 can easily be deduced from the Isoperimetric Theorem 1.4 using Steiner's four-hinge method. Conversely, one can similarly deduce Theorem 1.4 and the general Theorem 1.5 from the quadrilateral case of Theorem 1.5. He remarks that the quadrilateral case of Theorem 1.5 can be proved directly by deriving the following equation for the area A of a quadrilateral with sides ℓ_k:

$$A^2 = (s - \ell_1)(s - \ell_2)(s - \ell_3)(s - \ell_4) - \ell_1\ell_2\ell_3\ell_4 \cos^2\theta, \tag{4}$$

where $s = (\ell_1 + \ell_2 + \ell_3 + \ell_4)/2$ is half the perimeter, and θ is the arithmetic mean of two opposite angles.

Neither Blaschke, nor Steiner, L'Huilier, or Moula provide an argument for the uniqueness of the maximizer in Theorem 1.5 or 1.6. It seems that even after Weierstrass, the fact that the sides determine a cyclic polygon uniquely was considered too obvious to deserve a proof.

Penner [9, Theorem 6.2] gives a complete proof of Theorem 1.1. He proceeds by showing that there is one and only one circumcircle radius that allows the construction of a Euclidean cyclic polygon with given sides (provided they satisfy the polygon inequalities).

Schlenker [11] proves Theorems 1.2 and 1.3, and also the isoperimetric property of non-Euclidean cyclic polygons, i.e., the spherical and hyperbolic versions of Theorem 1.5. His proofs of the isoperimetric property are based on the remarkable equation

$$\sum \dot{\alpha}_i v_i = 0 \tag{5}$$

characterizing the change of angles α_i of a spherical or hyperbolic polygon under infinitesimal deformations with fixed side lengths. Here, $v_i \in \mathbb{R}^3$ are the position vectors of the polygon's vertices in the sphere or in the hyperboloid, respectively. To prove the uniqueness of spherical and hyperbolic cyclic polygons with given sides he uses separate arguments similar to Penner's.

2 Euclidean Polygons. Proof of Theorem 1.1

To construct an inscribed polygon with given side lengths $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n_{>0}$ (see Fig. 1) is equivalent to finding a point $(\alpha_1, \ldots \alpha_n)$ in the set

$$D_n = \left\{ \alpha \in \mathbb{R}^n_{>0} \;\Big|\; \sum_{k=1}^{n} \alpha_k = 2\pi \right\} \subset \mathbb{R}^n \tag{6}$$

satisfying, for some $R \in \mathbb{R}$ and for all $k \in \{1, \ldots, n\}$,

$$\frac{\ell_k}{2} = R \sin \frac{\alpha_k}{2}. \tag{7}$$

This problem admits the following variational formulation. Define the function $f_\ell : \mathbb{R}^n \to \mathbb{R}$ by

$$f_\ell(\alpha) = \sum_{k=1}^{n} \big(\mathrm{Cl}_2(\alpha_k) + \log(\ell_k)\, \alpha_k \big) \tag{8}$$

where Cl_2 denotes Clausen's integral [4]:

$$\mathrm{Cl}_2(x) = -\int_0^x \log \left| 2 \sin \frac{t}{2} \right| dt. \tag{9}$$

Clausen's integral is closely related to Milnor's Lobachevsky function [6]:

$$Л(x) = \frac{1}{2}\, \mathrm{Cl}_2(2x).$$

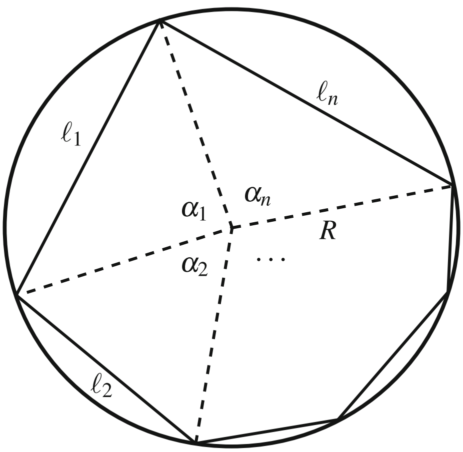

Fig. 1 Euclidean polygon inscribed in a circle

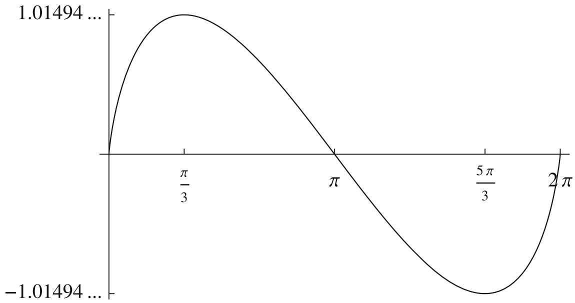

Fig. 2 Graph of Clausen's integral $\mathrm{Cl}_2(x)$

The function $\mathrm{Cl}_2 : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic, and odd. It is differentiable except at integer multiples of 2π where the graph has vertical tangents (see Fig. 2).

Proposition 2.1 (Variational Principle) *A point $\alpha \in D_n$ is a critical point of f_ℓ restricted to D_n if and only if there exists an $R \in \mathbb{R}$ satisfying equations* (7).

Proof A point $\alpha \in D_n$ is a critical point of f_ℓ restricted to D_n if and only if there exists a Lagrange multiplier $\log R$ such that $\nabla f_\ell(\alpha) = (\log R)\nabla g(\alpha)$ for the constraint function $g(\alpha) = \sum \alpha_k$, i.e.,

$$\begin{pmatrix} -\log\left|2\sin\frac{\alpha_1}{2}\right| + \log \ell_1 \\ \vdots \\ -\log\left|2\sin\frac{\alpha_n}{2}\right| + \log \ell_n \end{pmatrix} = \log R \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Since $0 < \alpha_k < 2\pi$ we may omit the absolute value signs, obtaining equations (7). □

Thus, to prove Theorem 1.1, we need to show that f_ℓ has a critical point in D_n if and only if the polygon inequalities (1) are satisfied, and that this critical point is then unique. The following proposition and corollary deal with the uniqueness claim.

Proposition 2.2 *The function f_ℓ is strictly concave on D_n.*

Corollary 2.3 *If f_ℓ has a critical point in D_n, it is the unique maximizer of f_ℓ in the closure $\bar{D}_n = \{\alpha \in \mathbb{R}^n_{\geq 0} \mid \sum \alpha_k = 2\pi\}$.*

This proves the uniqueness claim of Theorem 1.1.

Proof (of Proposition 2.2) We will show that

$$V_n(\alpha) = \sum_{k=1}^{n} \mathrm{Cl}_2(\alpha_k) \tag{10}$$

is strictly concave on D_n. Since V_n differs from f_ℓ by a linear function, this is equivalent to the claim.

Rivin [10, Theorem 2.1] showed that V_3 is strictly concave on D_3. For $n > 3$ we proceed by induction on n by "cutting off a triangle": first, note the obvious identity

$$\begin{aligned} V_n(\alpha_1, \ldots, \alpha_n) = V_{n-1}(\alpha_1, \ldots, \alpha_{n-1} + \alpha_n) - \mathrm{Cl}_2(\alpha_{n-1} + \alpha_n) \\ + \mathrm{Cl}_2(\alpha_{n-1}) + \mathrm{Cl}_2(\alpha_n). \end{aligned}$$

Since Clausen's integral is 2π-periodic and odd,

$$-\mathrm{Cl}_2(\alpha_{n-1} + \alpha_n) = \mathrm{Cl}_2(2\pi - \alpha_{n-1} - \alpha_n) = \mathrm{Cl}_2\left(\sum_{k=1}^{n-2} \alpha_k\right),$$

so

$$V_n(\alpha_1, \ldots, \alpha_n) = V_{n-1}(\alpha_1, \ldots, \alpha_{n-1} + \alpha_n) + V_3\left(\sum_{k=1}^{n-2} \alpha_k, \alpha_{n-1}, \alpha_n\right).$$

Hence, if V_{n-1} and V_3 are strictly concave on D_{n-1} and D_3, respectively, the claim for V_n follows. □

Since f_ℓ attains its maximum on the compact set $\bar{D}_n$, it remains to show that the maximum is attained in D_n if and only if the polygon inequalities (1) are satisfied. This is achieved by the following Propositions 2.4 and 2.5.

Note that $\bar{D}_n$ is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$. Its vertices are the points $2\pi e_1, \ldots, 2\pi e_n$, where e_k are the canonical basis vectors of $\mathbb{R}^n$. The relative boundary of the simplex $\bar{D}_n$ is

$$\partial \bar{D}_n = \{\alpha \in \bar{D}_n \mid \alpha_k = 0 \text{ for at least one } k\}. \tag{11}$$

Proposition 2.4 *If the function f_ℓ attains its maximum on the simplex $\bar{D}_n$ at a boundary point $\alpha \in \partial \bar{D}_n$, then α is a vertex.*

Proof Suppose $\alpha \in \partial \bar{D}_n$ is not a vertex. We need to show that f_ℓ does not attain its maximum at α. This follows from the fact that the derivative of f_ℓ in a direction pointing towards D_n is $+\infty$.

Indeed, suppose $v \in \mathbb{R}^n_{\geq 0}$, $\sum_k v_k = 0$ and $v_k > 0$ if $\alpha_k = 0$. Then $\alpha + tv \in D_n$ for small enough $t > 0$, and because $\lim_{x \to 0} \mathrm{Cl}_2'(x) = +\infty$,

$$\lim_{t \to 0} \frac{d}{dt} f_\ell(\alpha + tv) = +\infty. \tag{12}$$

Hence $f_\ell(\alpha + tv) > f_\ell(\alpha)$ for small enough $t > 0$. □

Proposition 2.5 *The function f_ℓ attains its maximum on $\bar{D}_n$ at a vertex $2\pi e_k$ if and only if*

$$\ell_k \geq \sum_{\substack{i=1 \\ i \neq k}}^{n} \ell_i. \tag{13}$$

Proof By symmetry, it is enough to consider the case $k = n$, i.e., to show that the function f_ℓ attains its maximum on $\bar{D}_n$ at the vertex $(0, \ldots, 0, 2\pi)$ if and only if $\ell_n \geq \sum_{k=1}^{n-1} \ell_k$. To this end, we will calculate the directional derivative of f_ℓ in directions $v \in \mathbb{R}^n$ pointing inside D_n, i.e., satisfying

$$v_k \geq 0 \quad \text{for} \quad k \in \{1, \ldots, n-1\}, \qquad v_n = -\sum_{k=1}^{n-1} v_k < 0.$$

Since we are only interested in the sign, we may assume v to be scaled so that

$$\sum_{k=1}^{n-1} v_k = 1, \qquad v_n = -1.$$

Clausen's integral has the asymptotic behavior

$$\mathrm{Cl}_2(x) = -x \log |x| + x + o(x) \quad \text{as} \quad x \to 0. \tag{14}$$

This can be seen by considering

$$\mathrm{Cl}_2(x) = -\int_0^x \log \left|(2 \sin \tfrac{t}{2})/t\right| \, dt - \int_0^x \log |t| \, dt.$$

Using (14) and the 2π-periodicity of Clausen's integral, one obtains

$$\begin{aligned} f_\ell(2\pi e_n + tv) - f_\ell(2\pi e_n) &= \sum_{k=1}^{n} \big(-t v_k \log |v_k| + t v_k \log \ell_k\big) + o(t) \\ &= -\sum_{k=1}^{n-1} t v_k \log \frac{v_k}{\ell_n} - t \log \ell_n + o(t), \end{aligned}$$

and hence

$$\frac{d}{dt}\Big|_{t=0} f(2\pi e_n + tv) = -\sum_{k=1}^{n-1} v_k \log \frac{v_k}{\ell_k} - \log \ell_n.$$

Now we invoke the information inequality (2) for the discrete probability distributions $(v_1, \ldots, v_{n-1})$ and $(\ell_1, \ldots, \ell_{n-1})/\sum_{k=1}^{n-1} \ell_k$. Thus,

$$\frac{d}{dt}\Big|_{t=0} f(2\pi e_n + tv) = \underbrace{-\sum_{k=1}^{n-1} v_k \log\left(\frac{v_k}{\ell_n/\sum_{m=1}^{n-1}\ell_m}\right)}_{\leq 0} + \log\left(\frac{\sum_{k=1}^{n-1}\ell_k}{\ell_n}\right).$$

If $\ell_n \geq \sum_{k=1}^{n-1}\ell_k$, then

$$\frac{d}{dt}\Big|_{t=0} f(2\pi e_n + tv) \leq 0.$$

With the concavity of f_ℓ (Proposition 2.2), this implies that f_ℓ attains its maximum on $\bar{D}_n$ at $(0, \dots, 0, 2\pi)$.

If, on the other hand, $\ell_n < \sum_{k=1}^{n-1}\ell_k$, then we obtain, for $v_k = \ell_k/\sum_{m=1}^{n-1}\ell_m$,

$$\frac{d}{dt}\Big|_{t=0} f(2\pi e_n + tv) > 0.$$

This implies that f_ℓ does not attain its maximum at $(0, \dots, 0, 2\pi)$. □

This completes the proof of Theorem 1.1.

Remark 2.6 The function V_n has the following interpretation in terms of hyperbolic volume [6]. Consider a Euclidean cyclic n-gon with central angles $\alpha_1, \dots, \alpha_n$. Imagine the Euclidean plane of the polygon to be the ideal boundary of hyperbolic 3-space in the Poincaré upper half-space model. Then the vertical planes through the edges of the polygon and the hemisphere above its circumcircle bound a hyperbolic pyramid with vertices at infinity. Its volume is $\frac{1}{2}V_n(\alpha_1, \dots, \alpha_n)$. Together with Schläfli's differential volume equation (rather, Milnor's generalization that allows for ideal vertices [7]), this provides another way to prove Proposition 2.1.

3 Spherical Polygons. Proof of Theorem 1.2

The polygon inequalities (1) are clearly necessary for the existence of a spherical cyclic polygon because every side is a shortest geodesic. That inequality (3) is also necessary was already noted in the introduction. It remains to show that these inequalities are also sufficient, and that the polygon is unique.

We reduce the spherical case to the Euclidean one as shown in Fig. 3. Connecting the vertices of a spherical cyclic polygon with line segments in the ambient Euclidean space, one obtains a Euclidean cyclic polygon whose circumradius is smaller than 1. Conversely, every Euclidean polygon inscribed in a circle of radius less than 1 corresponds to a unique spherical cyclic polygon. The spherical side lengths ℓ are related to the Euclidean lengths $\bar{\ell}$ by

$$\bar{\ell} = 2\sin\frac{\ell}{2}. \tag{15}$$

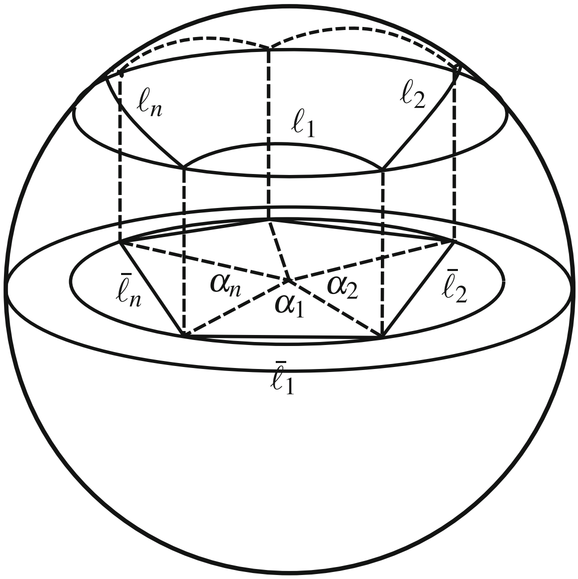

Fig. 3 Spherical and Euclidean polygons

It remains to show the following two propositions:

Proposition 3.1 *If the spherical lengths $\ell \in \mathbb{R}^n_{>0}$ satisfy the inequalities* (1) *and* (3)*, then the Euclidean lengths $\bar{\ell}$ defined by* (15) *satisfy the inequalities* (1) *as well. By Theorem 1.1 there is then a unique Euclidean cyclic polygon $P_{\bar{\ell}}$ with side lengths $\bar{\ell}$.*

Proposition 3.2 *The circumradius $\bar{R}$ of the polygon $P_{\bar{\ell}}$ of Proposition 3.1 is strictly less than* 1.

We will use the following estimate in the proof of Proposition 3.1:

Lemma 3.3 (Sum of Sines Estimate) If $\beta_1, \ldots, \beta_n \in \mathbb{R}_{\geq 0}$ satisfy $\sum_{k=1}^n \beta_k \leq \pi$, then

$$\sin\left(\sum_{k=1}^{n} \beta_k\right) \leq \sum_{k=1}^{n} \sin \beta_k. \tag{16}$$

Proof (of Lemma 3.3) By induction on n, the base case $n = 1$ being trivial. For the inductive step, use the addition theorem,

$$\sin\left(\sum_{k=1}^{n+1} \beta_k\right) = \sin\left(\sum_{k=1}^{n} \beta_k\right) \cos \beta_{n+1} + \cos\left(\sum_{k=1}^{n} \beta_k\right) \sin \beta_{n+1},$$

and note that the cosines are ≤ 1. □

Remark 3.4 The statement of Lemma 3.3 can be strengthened. Equality holds in (16) if and only if at most one β_k is greater than zero. This is easy to see, but we do not need this stronger statement in the following proof.

Proof (of Proposition 3.1) Suppose $\ell_1, \dots, \ell_n \in \mathbb{R}_{>0}$ satisfy the polygon inequalities (1) and (3). We need to show that $\bar{\ell}_1, \dots, \bar{\ell}_n$ defined by (15) satisfy

$$\bar{\ell}_k < \sum_{i \neq k} \bar{\ell}_i. \tag{17}$$

To this end, we will show that

$$\sin \frac{\ell_k}{2} < \sin \Bigl(\sum_{i \neq k} \frac{\ell_i}{2} \Bigr), \tag{18}$$

from which inequality (17) follows by Lemma 3.3. To prove inequality (18), we consider two cases separately.

- $\sum_{i \neq k} \ell_i \leq \pi$. Inequality (18) simply follows from the polygon inequality $\ell_k < \sum_{i \neq k} \ell_i$ and the monotonicity of the sine function on the closed interval $[0, \frac{\pi}{2}]$.
- $\sum_{i \neq k} \ell_i \geq \pi$. Note that $2\pi > \sum_i \ell_i$ implies $2\pi - \ell_k > \sum_{i \neq k} \ell_i$, and hence

$$2\pi > 2\pi - \ell_k > \sum_{i \neq k} \ell_i \geq \pi. \tag{19}$$

  Inequality (18) follows from $\sin \frac{\ell_k}{2} = \sin(\pi - \frac{\ell_k}{2})$ and the monotonicity of the sine function on the closed interval $[\frac{\pi}{2}, \pi]$.

This completes the proof of (18) and hence the proof of Proposition 3.1. □

Proof (of Proposition 3.2) Let α_k be the central angles of the Euclidean cyclic polygon $P_{\bar{\ell}}$. Then

$$\sin \frac{\ell_k}{2} = \frac{\bar{\ell}_k}{2} = \bar{R} \sin \frac{\alpha_k}{2}, \tag{20}$$

by (7) and (15). Note that α_k are the central angles of both the Euclidean and the spherical polygon (provided it exists). We consider two cases separately.

First, suppose that $\alpha_k \leq \pi$ for all k. Since $\sum_k \ell_k < 2\pi = \sum_k \alpha_k$, there is some k such that $\ell_k < \alpha_k$. Then $\sin \frac{\ell_k}{2} < \sin \frac{\alpha_k}{2}$, and equation (20) implies that $\bar{R} < 1$.

Otherwise, since $\sum_k \alpha_k = 2\pi$, there is exactly one i such that $\alpha_i > \pi$, and $\alpha_k < \pi$ for all $k \neq i$. By symmetry, it is enough to consider the case

$$\alpha_1 > \pi, \quad \alpha_k < \pi \quad \text{for} \quad k \in \{2, \dots, n\}.$$

For future reference, we note that $\alpha_1 > \pi$ implies that $\bar{\ell}_1$ is the longest side of $P_{\bar{\ell}}$. (Use (20) and the monotonicity of the sine function.)

We will show $\bar{R} < 1$ by induction on n. First, assume $n = 3$. Then (18) says

$$\sin \frac{\ell_1}{2} < \sin \frac{\ell_2 + \ell_3}{2}.$$

By (20) and using $2\pi - \alpha_1 = \alpha_2 + \alpha_3$, we have

$$\sin\frac{\ell_1}{2} = \bar{R}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2} + \bar{R}\cos\frac{\alpha_2}{2}\sin\frac{\alpha_3}{2}, \tag{21}$$

and

$$\begin{aligned}\sin\frac{\ell_2+\ell_3}{2} &= \sin\frac{\ell_2}{2}\cos\frac{\ell_3}{2} + \cos\frac{\ell_2}{2}\sin\frac{\ell_3}{2} \\ &= \bar{R}\sin\frac{\alpha_2}{2}\cos\frac{\ell_3}{2} + \bar{R}\cos\frac{\ell_2}{2}\sin\frac{\alpha_3}{2}\ .\end{aligned} \tag{22}$$

For at least one $k \in \{2, 3\}$, $\cos\frac{\alpha_k}{2} < \cos\frac{\ell_k}{2}$ and hence $\sin\frac{\alpha_k}{2} > \sin\frac{\ell_k}{2}$. Equation (20) implies $\bar{R} < 1$.

Now assume that $\bar{R} < 1$ has already been shown if $P_{\bar{\ell}}$ has at most n sides. Suppose $P_{\bar{\ell}}$ has $n+1$ sides. The idea of the following argument is to cut off a triangle with sides $\bar{\ell}_n$, $\bar{\ell}_{n+1}$, and $\bar{\lambda} = 2\bar{R}\sin\frac{\alpha_n+\alpha_{n+1}}{2}$. Since $\bar{\lambda} \leq \bar{\ell}_1$ (the longest side), and $\bar{\ell}_1 \leq 2$ by (15), we may define $\lambda = 2\arcsin\frac{\bar{\lambda}}{2}$. Now assume $\bar{R} \geq 1$. Then, by the inductive hypothesis, the polygon inequalities (1) or (3) are violated for the cut-off triangle and the remaining n-gon. Inequality (3) cannot be violated because it was assumed to hold for $\ell_1, \ldots, \ell_{n+1}$. Hence,

$$\ell_1 \geq \ell_2 + \cdots + \ell_{n-1} + \lambda \quad \text{and} \quad \lambda \geq \ell_n + \ell_{n+1}.$$

This implies $\ell_1 \geq \ell_2 + \cdots + \ell_{n+1}$. Conversely, if (1) and (3) hold, then $\bar{R} < 1$. This completes the proof of Proposition 3.2. □

4 Hyperbolic Polygons. Proof of Theorem 1.3

The polygon inequalities (1) are clearly necessary for the existence of a hyperbolic cyclic polygon, because every side is a shortest geodesic. It remains to show that they are also sufficient, and that the polygon is unique, i.e., Proposition 4.2. First, we review some basic facts from hyperbolic geometry.

As in the spherical case (Sect. 3), we will connect vertices by straight line segments in the ambient vector space. But instead of the sphere, we consider the hyperbolic plane in the hyperboloid model,

$$\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} \mid \langle x, x\rangle = -1,\ x_3 > 0\},$$

where $\mathbb{R}^{2,1}$ denotes the vector space $\mathbb{R}^3$ equipped with the scalar product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$